Grassmann Algebra

Exploring applications of extended vector algebra with *Mathematica*

Incomplete draft   January 2001

John Browne

Swinburne University of Technology, Melbourne, Australia

Table of Contents

2 The Exterior Product

6 The Interior Product

7 Exploring Screw Algebra

8 Exploring Mechanics

9 Exploring Grassmann Algebra

To be completed

10 Exploring the Generalized Product

To be completed

11 Exploring Hypercomplex Algebra

To be completed

12 Exploring Clifford Algebra

To be completed

13 Exploring Grassmann Matrix Algebra

To be completed

Guide to *GrassmannAlgebra*

To be completed

A Brief Biography of Grassmann

To be completed

Notation

To be completed

Glossary

To be completed

Bibliography

To be completed

Index

To be completed

Preface

The origins of this book

This book has grown out of an interest in Grassmann's work over the past three decades. There is something fascinating about the beauty with which the mathematical structures Grassmann discovered (invented, if you will) describe the physical world, and something also fascinating about how these beautiful structures have been largely lost to the mainstreams of mathematics and science.

Who was Grassmann?

Hermann Günther Grassmann was born in 1809 in Stettin, near the border of Germany and Poland. He was only 23 when he discovered the method of adding and multiplying points and vectors which was to become the foundation of his *Ausdehnungslehre*. In 1839 he composed a work on the study of tides entitled *Theorie der Ebbe und Flut*, which was the first work ever to use vectorial methods. In 1844 Grassmann published his first *Ausdehnungslehre* (*Die lineale Ausdehnungslehre ein neuer Zweig der Mathematik*) and in the same year won a prize for an essay which expounded a system satisfying an earlier search by Leibniz for an 'algebra of geometry'. Despite these achievements, Grassmann received virtually no recognition.

In 1862 Grassmann re-expounded his ideas from a different viewpoint in a second *Ausdehnungslehre (Die Ausdehnungslehre. Vollständig und in strenger Form*). Again the work was met with resounding silence from the mathematical community, and it was not until the latter part of his life that he received any significant recognition from his contemporaries. Of these, most significant were J. Willard Gibbs who discovered his works in 1877 (the year of Grassmann's death), and William Kingdon Clifford who discovered them in depth about the same time. Both became quite enthusiastic about this new mathematics.

A more detailed biography of Grassmann may be found at the end of the book.

From the *Ausdehnungslehre* to *GrassmannAlgebra*

The term '*Ausdehnungslehre*' is variously translated as 'extension theory', 'theory of extension', or 'calculus of extension'. In this book we will use these terms to refer to Grassmann's original work and to other early work in the same notational and conceptual tradition (particularly that of Edward Wyllys Hyde, Henry James Forder and Alfred North Whitehead).

The term 'Exterior Calculus' will be reserved for the calculus of *exterior differential forms*, originally developed by Elie Cartan from the *Ausdehnungslehre*. This is an area in which there are many excellent texts, and which is outside the scope of this book.

The term 'Grassmann algebra' will be used to describe that body of algebraic theory and results based on the *Ausdehnungslehre*, but extended to include more recent results and viewpoints. This will be the basic focus of this book.

Finally, the term '*GrassmannAlgebra*' will be used to refer to the *Mathematica* based software package which accompanies it.

■ What is Grassmann algebra?

The intrinsic power of Grassmann algebra arises from its fundamental product operation, the exterior product. The exterior product codifies the property of linear dependence, so essential for modern applied mathematics, directly into the algebra. Simple non-zero elements of the algebra may be viewed as representing constructs of linearly independent elements. For example, a simple bivector is the exterior product of two vectors; a line is the exterior product of two points and a plane is the exterior product of three points.

■ The focus of this book

The primary focus of this book is to provide a readable account in modern notation of Grassmann's major algebraic contributions to mathematics and science. I would like it to be accessible to scientists and engineers, students and professionals alike. Consequently I have tried to avoid all mathematical terminology which does not make an essential contribution to understanding the basic concepts. The only assumptions I have made as to the reader's background is that they have some familiarity with basic linear algebra.

The secondary concern of this book is to provide an environment for exploring applications of the Grassmann algebra. For general applications in higher dimensional spaces, computations by hand in any algebra become tedious, indeed limiting, thus restricting the hypotheses that can be explored. For this reason the book includes a package for exploring Grassmann algebra, called *GrassmannAlgebra*, which may be found on the CD at the back of the book. *GrassmannAlgebra* has been developed in *Mathematica*. You can read the book without using the package, or you can use the package to extend the examples in the text, experiment with hypotheses, or explore your own interests.

■ The power of *Mathematica*

Mathematica is a powerful system for doing mathematics on a computer. It has an inbuilt programming language ideal for extending its capabilities to other mathematical systems like Grassmann algebra. It also has a sophisticated mathematical typesetting capability. This book uses both. The chapters are typeset by *Mathematica* in its standard notebook format, making the book interactive in its electronic version with the *GrassmannAlgebra* package. *GrassmannAlgebra* can turn an exploration requiring days by hand into one requiring just minutes of computing time.

■ How you can use this book

Chapter 1: *Introduction* provides a brief preparatory overview of the book, introducing the seminal concepts of each chapter, and solidifying them with simple examples. This chapter is designed to give you a global appreciation with which better to understand the detail of the chapters which follow. However, it is independent of those chapters, and may be read as far as your interest takes you.

Chapters 2 to 6: *The Exterior Product*, *The Regressive Product*, *Geometric Interpretations*, *The Complement*, and *The Interior Product* develop the basic theory. These form the essential core for a working knowledge of Grassmann algebra and for the explorations in subsequent chapters. They should be read sequentially.

Chapters 7 to 13: *Exploring Screw Algebra*, *Mechanics*, *Grassmann Algebra*, *The Generalized Product*, *Hypercomplex Algebra*, *Clifford Algebra*, and *Grassmann Matrix Algebra*, may be read independently of each other, with the proviso that the chapters on hypercomplex and Clifford algebra depend on the notion of the generalized product.

Sections within the text which use *GrassmannAlgebra* or *Mathematica* are signified with the *Mathematica* icon. Wherever possible, explanation is provided so that the results are still intelligible without a knowledge of *Mathematica*. *Mathematica* input/output dialogue is in indented Courier font, with the input in bold. For example:

```
Expand[(x + y)^3]
```

$$x^3 + 3\,x^2\,y + 3\,x\,y^2 + y^3$$

The *Guide to GrassmannAlgebra* is a summary of the most important functions of the *GrassmannAlgebra* package. This may be used as an overview of the package capabilities or as a 'Quick Start' guide.

The *GrassmannAlgebra* website

As with most books these days, this book has its own website: www.GrassmannAlgebra.**xxx.zz**. On this site you will find, amongst other things, errata, additional material, upgrades to the *GrassmannAlgebra* package, links to other Grassmann algebra sites, notification of new or related editions, reviews and my email address.

Acknowledgements

Finally I would like to acknowledge those who were instrumental in the evolution of this book: Cecil Pengilley who originally encouraged me to look at applying Grassmann's work to engineering; the University of Melbourne, University of Rochester and Swinburne University of Technology which sheltered me while I pursued this interest; Janet Blagg who peerlessly edited the text; my family who had to put up with my writing, and finally to Stephen Wolfram who created *Mathematica* and provided me with a visiting scholar's grant to work at Wolfram Research Institute in Champaigne where I began developing the *GrassmannAlgebra* package.

Above all however, I must acknowledge Hermann Grassmann. His contribution to mathematics and science puts him among the great thinkers of the nineteenth century.

I hope you enjoy exploring this beautiful mathematical system.

> For I have every confidence that the effort I have applied to the science reported upon here, which has occupied a considerable span of my lifetime and demanded the most intense exertions of my powers, is not to be lost. … a time will come when it will be drawn forth from the dust of oblivion and the ideas laid down here will bear fruit. … some day these ideas, even if in an altered form, will reappear and with the passage of time will participate in a lively intellectual exchange. For truth is eternal, it is divine; and no phase in the development of truth, however small the domain it embraces, can pass away without a trace. It remains even if the garments in which feeble men clothe it fall into dust.

Hermann Grassmann
in the foreword to the Ausdehnungslehre of 1862
translated by Lloyd Kannenberg

1. Introduction

1.1 Background

The mathematical representation of physical entities

Two of the more important mathematical systems for representing the entities of contemporary engineering and physical science are the (three-dimensional) vector algebra and the more general tensor algebra. Grassmann algebra is more general than vector algebra, overlaps aspects of the tensor algebra, and predates both of them. In this book we will show that it is only via Grassmann algebra that many of the geometric and physical entities commonly used in the engineering and physical sciences may be represented mathematically in a way which correctly models their pertinent properties and leads straightforwardly to principal results.

As a case in point we may take the concept of *force*. It is well known that a force is not satisfactorily represented by a (free) vector, yet contemporary practice is still to use a (free) vector calculus for this task. The deficiency may be made up for by verbal appendages to the mathematical statements: for example 'where the force **f** acts along the line through the point **P**'. Such verbal appendages, being necessary, and yet not part of the calculus being used, indicate that the calculus itself is not adequate to model force satisfactorily. In practice this inadequacy is coped with in terms of a (free) vector calculus by the introduction of the concept of *moment*. The conditions of equilibrium of a rigid body include a condition on the sum of the moments of the forces about any point. The justification for this condition is not well treated in contemporary texts. It will be shown later however that by representing a force correctly in terms of an element of the Grassmann algebra, both force-vector and moment conditions for the equilibrium of a rigid body are natural consequences of the algebraic processes alone.

Since the application of Grassmann algebra to mechanics was known during the nineteenth century one might wonder why, with the 'progress of science', it is not currently used. Indeed the same question might be asked with respect to its application in many other fields. To attempt to answer these questions, a brief biography of Grasssmann is included as an appendix. In brief, the scientific world was probably not ready in the nineteenth century for the new ideas that Grassmann proposed, and now, in the twenty-first century, seems only just becoming aware of their potential.

The central concept of the *Ausdehnungslehre*

Grassmann's principal contribution to the physical sciences was his discovery of a natural language of geometry from which he derived a geometric calculus of significant power. For a mathematical representation of a physical phenomenon to be 'correct' it must be of a tensorial nature and since many 'physical' tensors have direct geometric counterparts, a calculus applicable to geometry may be expected to find application in the physical sciences.

The word '*Ausdehnungslehre*' is most commonly translated as '*theory of extension*', the fundamental product operation of the theory then becoming known as the *exterior product*. The notion of *extension* has its roots in the interpretation of the algebra in geometric terms: an element of the algebra may be 'extended' to form a higher order element by its (exterior) product with another, in the way that a point may be extended to a line, or a line to a plane by a point exterior to it. The notion of *exteriorness* is equivalent algebraically to that of linear independence. If the exterior product of elements of grade 1 (for example, points or vectors) is non-zero, then they are independent.

A line may be defined by the exterior product of *any* two distinct points on it. Similarly, a plane may be defined by the exterior product of *any* three points in it, and so on for higher dimensions. This independence with respect to the specific points chosen is an important and fundamental property of the exterior product. Each time a higher dimensional object is required it is simply created out of a lower dimensional one by multiplying by a new element in a new dimension. Intersections of elements are also obtainable as products.

Simple elements of the Grassmann algebra may be interpreted as defining subspaces of a linear space. The exterior product then becomes the operation for building higher dimensional subspaces (higher order elements) from a set of lower dimensional independent subspaces. A second product operation called the *regressive product* may then be defined for determining the common lower dimensional subspaces of a set of higher dimensional non-independent subspaces.

Comparison with the vector and tensor algebras

The Grassmann algebra is a tensorial algebra, that is, it concerns itself with the types of mathematical entities and operations necessary to describe physical quantities in an invariant manner. In fact, it has much in common with the algebra of anti-symmetric tensors – the exterior product being equivalent to the anti-symmetric tensor product. Nevertheless, there are conceptual and notational differences which make the Grassmann algebra richer and easier to use.

Rather than a sub-algebra of the tensor calculus, it is perhaps more meaningful to view the Grassmann algebra as a super-algebra of the three-dimensional vector algebra since both commonly use invariant (component free) notations. The principal differences are that the Grassmann algebra has a dual axiomatic structure, it can treat of higher order elements than vectors and can differentiate between points and vectors, it generalizes the notion of 'cross product', is independent of dimension, and possesses the structure of a true algebra.

Algebraicizing the notion of linear dependence

Another way of viewing Grassmann algebra is as linear or vector algebra onto which has been introduced a product operation which algebraicizes the notion of linear dependence. This product operation is called the *exterior product* and is symbolized with a wedge $\wedge$.

If vectors x_1, x_2, x_3, … are linearly dependent, then it turns out that their exterior product is zero: $x_1 \wedge x_2 \wedge x_3 \wedge \ldots == 0$. If they are independent, their exterior product is non-zero.

Conversely, if the exterior product of vectors x_1, x_2, x_3, … is zero, then the vectors are linearly dependent. Thus the exterior product brings the critical notion of linear dependence into the realm of direct algebraic manipulation.

Although this might appear to be a relatively minor addition to linear algebra, we expect to demonstrate in this book that nothing could be further from the truth: the consequences of being able to model linear dependence with a product operation are far reaching, both in facilitating an understanding of current results, and in the generation of new results for many of the algebras and their entities used in science and engineering today. These include of course linear and multilinear algebra, but also the complex numbers, quaternions, octonions, Clifford algebras, Pauli and Dirac algebras, screw algebra, vector algebra and parts of the tensor algebra.

Grassmann algebra as a geometric calculus

Most importantly however, Grassmann's contribution has enabled the operations and entities of all of these algebras *to be interpretable geometrically*, thus enabling us to bring to bear the power of geometric visualization and intuition into our algebraic manipulations.

It is well known that a vector x_1 may be interpreted geometrically as representing a *direction* in space. If the space has a metric, then the *magnitude* of x_1 is interpreted as its *length*. The introduction of the exterior product enables us to *extend* the entities of the space to higher dimensions. The exterior product of two vectors $x_1 \wedge x_2$, called a *bivector*, may be visualized as the two-dimensional analogue of a direction, that is, a *planar direction*. If the space has a metric, then the *magnitude* of $x_1 \wedge x_2$ is interpreted as its *area*, and similarly for higher order products.

Graphic showing a bivector as a parallelogram parallel to two vectors, with its area as its magnitude.

For applications to the physical world, however, the Grassmann algebra possesses a critical capability that no other algebra possesses: it can distinguish between *points* and *vectors* and treat them as separate tensorial entities. Lines and planes are examples of higher order constructs from points and vectors, which have both position and direction. A *line* can be represented by the exterior product of any two points on it, or by any point on it with a vector parallel to it.

Graphic showing two views of a line:
1) Through two points
2) Through one point parallel to a vector

A *plane* can be represented by the exterior product of any three points on it, any two points with a vector parallel to it, or any point on it with a bivector parallel to it.

Graphic showing three views of a plane:
1) Through three points
2) Through two points parallel to a vector
3) Through one point parallel to two vectors

Finally, it should be noted that the Grassmann algebra subsumes all of real algebra, the exterior product reducing in this case to the usual product operation amongst real numbers.

Here then is a geometric calculus *par excellence*. We hope you enjoy exploring it.

1.2 The Exterior Product

The anti-symmetry of the exterior product

The exterior product of two vectors $\mathbf{x}$ and $\mathbf{y}$ of a linear space yields the bivector $\mathbf{x} \wedge \mathbf{y}$. The bivector is *not* a vector, and so does not belong to the original linear space. In fact the bivectors form their own linear space.

The fundamental defining characteristic of the exterior product is its *anti-symmetry*. That is, the product changes sign if the order of the factors is reversed.

$$\boxed{\mathbf{x} \wedge \mathbf{y} == -\mathbf{y} \wedge \mathbf{x}} \quad 1.1$$

From this we can easily show the equivalent relation, that the exterior product of a vector with itself is zero.

$$\boxed{\mathbf{x} \wedge \mathbf{x} == 0} \quad 1.2$$

This is as expected because $\mathbf{x}$ is linearly dependent on itself.

The exterior product is associative, distributive, and behaves as expected with scalars.

Exterior products of vectors in a three-dimensional space

By way of example, suppose we are working in a three-dimensional space, with basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. Then we can express vectors $\mathbf{x}$ and $\mathbf{y}$ as a linear combination of these basis vectors:

$$\mathbf{x} == a_1\, \mathbf{e}_1 + a_2\, \mathbf{e}_2 + a_3\, \mathbf{e}_3$$

$$\mathbf{y} == b_1\, \mathbf{e}_1 + b_2\, \mathbf{e}_2 + b_3\, \mathbf{e}_3$$

Graphic showing two separate vectors x and y.

Here, the $\mathbf{a_i}$ and $\mathbf{b_i}$ are of course scalars. Taking the exterior product of $\mathbf{x}$ and $\mathbf{y}$ and multiplying out the product expresses the *bivector* $\mathbf{x \wedge y}$ as a linear combination of *basis bivectors*.

$$x \wedge y == (a_1\, e_1 + a_2\, e_2 + a_3\, e_3) \wedge (b_1\, e_1 + b_2\, e_2 + b_3\, e_3)$$

$$x \wedge y == a_1\, b_1\, e_1 \wedge e_1 + a_1\, b_2\, e_1 \wedge e_2 + a_1\, b_3\, e_1 \wedge e_3 + a_2\, b_1\, e_2 \wedge e_1 + a_2\, b_2\, e_2 \wedge e_2 + a_2\, b_3\, e_2 \wedge e_3 + a_3\, b_1\, e_3 \wedge e_1 + a_3\, b_2\, e_3 \wedge e_2 + a_3\, b_3\, e_3 \wedge e_3$$

Graphic showing these two joined as a bivector.

The first simplification we can make is to put all basis bivectors of the form $\mathbf{e_i \wedge e_i}$ to zero. The second simplification is to use the anti-symmetry of the product and collect the terms of the bivectors which are not *essentially* different (that is, those that may differ only in the order of their factors, and hence differ only by a sign). The product $\mathbf{x \wedge y}$ can then be written:

$$x \wedge y == (a_1\, b_2 - a_2\, b_1)\, e_1 \wedge e_2 + (a_2\, b_3 - a_3\, b_2)\, e_2 \wedge e_3 + (a_3\, b_1 - a_1\, b_3)\, e_3 \wedge e_1$$

The scalar factors appearing here are just those which would have appeared in the usual vector *cross product* of $\mathbf{x}$ and $\mathbf{y}$. However, there is an important difference. The exterior product expression does not require the vector space to have a metric, while the cross product, because it generates a vector *orthogonal* to $\mathbf{x}$ and $\mathbf{y}$, necessarily assumes a metric. Furthermore, the exterior product is valid for any number of vectors in spaces of arbitrary dimension, while the cross product is necessarily confined to products of two vectors in a space of three dimensions.

For example, we may continue the product by multiplying $\mathbf{x \wedge y}$ by a third vector $\mathbf{z}$.

$$z == c_1\, e_1 + c_2\, e_2 + c_3\, e_3$$

$$x \wedge y \wedge z == ((a_1\, b_2 - a_2\, b_1)\, e_1 \wedge e_2 + (a_2\, b_3 - a_3\, b_2)\, e_2 \wedge e_3 + (a_3\, b_1 - a_1\, b_3)\, e_3 \wedge e_1) \wedge (c_1\, e_1 + c_2\, e_2 + c_3\, e_3)$$

Adopting the same simplification procedures as before we obtain the *trivector* $\mathbf{x \wedge y \wedge z}$ expressed in basis form.

$$x \wedge y \wedge z == (a_1\, b_2\, c_3 - a_3\, b_2\, c_1 + a_2\, b_3\, c_1 + a_3\, b_1\, c_2 - a_1\, b_3\, c_2 - a_2\, b_1\, c_3)\, e_1 \wedge e_2 \wedge e_3$$

Graphic showing a third vector forming a parallelepiped.

A trivector in a space of three dimensions has just one component. Its coefficient is the *determinant* of the coefficients of the original three vectors. Clearly, if these three vectors had been linearly dependent, this determinant would have been zero. In a metric space, this coefficient would be proportional to the *volume* of the parallelepiped formed by the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Hence the geometric interpretation of the algebraic result: if $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are lying in a planar direction, that is, they are dependent, then the volume of the parallelepiped defined is zero.

We see here also that the exterior product begins to give geometric meaning to the often inscrutable operations of the algebra of determinants. In fact we shall see that *all* the operations of determinants are straightforward consequences of the properties of the exterior product.

In three-dimensional *metric* vector algebra, the vanishing of the scalar triple product of three vectors is often used as a criterion of their linear dependence, whereas in fact the vanishing of their exterior product (valid in a *non-metric* space) would suffice. It is interesting to note that the notation for the scalar triple product, or 'box' product, is Grassmann's original notation, viz **[x y z]**.

Finally, we can see that the exterior product of more than three vectors in a three-dimensional space will always be zero, since they must be dependent.

Terminology: elements and entities

To this point we have been referring to the elements of the space under discussion as *vectors*, and their higher order constructs in three dimensions as *bivectors* and *trivectors*. In the general case we will refer to the exterior product of an unspecified number of vectors as a *multivector*, and the exterior product of *m* vectors as an *m-vector*.

The word 'vector' however, is in current practice used in two distinct ways. The first and traditional use endows the vector with its well-known geometric properties of direction and (possibly) magnitude. In the second and more recent, the term vector may refer to any element of a linear space, even though the space is devoid of geometric context.

In this book, we adopt the traditional practice and use the term vector only when we intend it to have its traditional geometric interpretation. When referring to an element of a linear space which we are not specifically interpreting geometrically, we simply use the term *element*. The exterior product of *m* elements of a linear space will thus be referred to as an *m-element*.

Science and engineering make use of mathematics by endowing its constructs with geometric or physical interpretations. We will use the term *entity* to refer to an element which we specifically wish to endow with a geometric or physical interpretation. For example we would say that *points* and *positions* are 1-entities because they are represented by 1-elements, while *lines* and *forces* are 2-entities because they are represented by 2-elements. Points, vectors, lines and planes are examples of *geometric* entities. Positions, directions, forces and moments are examples of *physical* entities. We *interpret* elements of a linear space geometrically or physically, while we *represent* geometric or physical entities by elements of a linear space.

Insert a TABLE here of elements and entities with their terminology.

The grade of an element

The exterior product of *m* 1-elements is called an *m*-element. The value *m* is called the *grade* of the *m*-element. For example the element $x \wedge y \wedge u \wedge v$ is of grade 4.

An *m*-element may be *denoted* by a symbol underscripted with the value *m*. For example: $\underset{4}{\alpha} == x \wedge y \wedge u \wedge v$.

For simplicity, however, we do not generally denote 1-elements with an underscripted '1'.

The grade of a scalar is 0. We shall see that this is a natural consequence of the exterior product axioms formulated for elements of general grade.

The *dimension* of the underlying linear space of 1-elements is denoted by *n*.

The *complementary grade* of an *m*-element in an *n*-space is *n–m*.

Interchanging the order of the factors in an exterior product

The exterior product is defined to be associative. Hence interchanging the order of any two adjacent factors will change the sign of the product:

$$\ldots \wedge \mathbf{x} \wedge \mathbf{y} \wedge \ldots == -(\ldots \wedge \mathbf{y} \wedge \mathbf{x} \wedge \ldots)$$

In fact, interchanging the order of any two *non-adjacent* factors will also change the sign of the product.

$$\ldots \wedge \mathbf{x} \wedge \ldots \wedge \mathbf{y} \wedge \ldots == -(\ldots \wedge \mathbf{y} \wedge \ldots \wedge \mathbf{x} \wedge \ldots)$$

To see why this is so, suppose the number of factors between $\mathbf{x}$ and $\mathbf{y}$ is *m*. First move $\mathbf{y}$ to the immediate left of $\mathbf{x}$. This will cause *m*+1 changes of sign. Then move $\mathbf{x}$ to the position that $\mathbf{y}$ vacated. This will cause *m* changes of sign. In all there will be 2*m*+1 changes of sign, equivalent to just one sign change.

Note that it is only elements of odd grade that anti-commute. If at least one of the elements of a product is of even grade, then they commute. For example, 2-elements commute with all other elements.

$$(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z} == \mathbf{z} \wedge (\mathbf{x} \wedge \mathbf{y})$$

A brief summary of the properties of the exterior product

In this section we summarize a few of the more important properties of the exterior product that we have already introduced informally. In Chapter 2: The Exterior Product the complete set of axioms are discussed.

- The exterior product of an *m*-element and a *k*-element is an (*m*+*k*)-element.
- The exterior product is associative.

$$\boxed{\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \wedge \underset{r}{\gamma} == \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \wedge \underset{r}{\gamma}\right)} \quad \mathbf{1.3}$$

- The unit scalar acts as an identity under the exterior product.

$$\boxed{\underset{m}{\alpha} == 1 \wedge \underset{m}{\alpha} == \underset{m}{\alpha} \wedge 1} \quad \mathbf{1.4}$$

- Scalars factor out of products.

$$\left(a \underset{m}{\alpha}\right) \wedge \underset{k}{\beta} == \underset{m}{\alpha} \wedge \left(a \underset{k}{\beta}\right) == a \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \qquad 1.5$$

- An exterior product is anti-commutative whenever the grades of the factors are both odd.

$$\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{m\,k}\, \underset{k}{\beta} \wedge \underset{m}{\alpha} \qquad 1.6$$

- The exterior product is both left and right distributive under addition.

$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \wedge \underset{r}{\gamma} == \underset{m}{\alpha} \wedge \underset{r}{\gamma} + \underset{m}{\beta} \wedge \underset{r}{\gamma} \qquad \underset{m}{\alpha} \wedge \left(\underset{r}{\beta} + \underset{r}{\gamma}\right) == \underset{m}{\alpha} \wedge \underset{r}{\beta} + \underset{m}{\alpha} \wedge \underset{r}{\gamma} \qquad 1.7$$

1.3 The Regressive Product

The regressive product as a dual product to the exterior product

One of Grassmann's major contributions, which appears to be all but lost to current mathematics, is the *regressive product*. The regressive product is the real foundation for the theory of the inner and scalar products (and their generalization, the interior product). Yet the regressive product is often ignored and the inner product defined as a new construct independent of the regressive product. This approach not only has potential for inconsistencies, but also fails to capitalize on the wealth of results available from the natural duality between the exterior and regressive products. The approach adopted in this book follows Grassmann's original concept. The regressive product is a simple dual operation to the exterior product and an enticing and powerful symmetry is lost by ignoring it, particularly in the development of metric results involving complements and interior products.

The underlying beauty of the *Ausdehnungslehre* is due to this symmetry, which in turn is due to the fact that linear spaces of m-elements and linear spaces of (n–m)-elements have the same dimension. This too is the key to the duality of the exterior and regressive products. For example, the exterior product of m 1-elements is an m-element. The dual to this is that the regressive product of m (n–1)-elements is an (n–m)-element. This duality has the same form as that in a Boolean algebra: if the exterior product corresponds to a type of 'union' then the regressive product corresponds to a type of 'intersection'.

It is this duality that permits the definition of complement, and hence interior, inner and scalar products defined in later chapters. To underscore this duality it is proposed to adopt here the ∨ ('vee') for the regressive product operation.

Unions and intersections of spaces

Consider a (non-zero) 2-element $x \wedge y$. We can test to see if any given 1-element z is in the subspace spanned by x and y by taking the exterior product of $x \wedge y$ with z and seeing if the result is zero. From this point of view, $x \wedge y$ is an element which can be used to *define* the subspace instead of the individual 1-elements x and y.

Thus we can define the *space* of $x \wedge y$ as the set of all 1-elements z such that $x \wedge y \wedge z == 0$. We extend this to more general elements by defining the space of a simple element $\underset{m}{\alpha}$ as the set of all 1-elements β such that $\underset{m}{\alpha} \wedge \beta == 0$.

We will also need the notion of congruence. Two elements are *congruent* if one is a scalar factor times the other. For example x and $2x$ are congruent; $x \wedge y$ and $-x \wedge y$ are congruent. Congruent elements define the same subspace. We denote congruence by the symbol ≡. For two elements, congruence is equivalent to linear dependence. For more than two elements however, although the elements may be linearly dependent, they are not necessarily congruent. The following concepts of union and intersection only make sense up to congruence.

A *union of elements* is an element defining the subspace they *together* span.

The dual concept to union of elements is *intersection of elements*. An *intersection of elements* is an element defining the subspace they span *in common*.

Suppose we have three independent 1-elements: x, y, and z. A union of $x \wedge y$ and $y \wedge z$ is any element congruent to $x \wedge y \wedge z$. An intersection of $x \wedge y$ and $y \wedge z$ is any element congruent to y.

The regressive product enables us to calculate intersections of elements. Thus it is the product operation correctly dual to the exterior product. In fact, in Chapter 3: The Regressive Product we will develop the axiom set for the regressive product from that of the exterior product, and also show that we can reverse the procedure.

A brief summary of the properties of the regressive product

In this section we summarize a few of the more important properties of the regressive product that we have already introduced informally. In Chapter 3: The Regressive Product we discuss the complete set of axioms.

- The regressive product of an *m*-element and a *k*-element in an *n*-space is an (*m*+*k*–*n*)-element.

- The regressive product is associative.

$$\left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \vee \underset{r}{\gamma} == \underset{m}{\alpha} \vee \left(\underset{k}{\beta} \vee \underset{r}{\gamma}\right) \qquad 1.8$$

- The unit *n*-element $\underset{n}{1}$ acts as an identity under the regressive product.

$$\underset{m}{\alpha} == \underset{n}{1} \vee \underset{m}{\alpha} == \underset{m}{\alpha} \vee \underset{n}{1} \qquad 1.9$$

- Scalars factor out of products.

$$\left(a\,\alpha_m\right)\vee\beta_k == \alpha_m\vee\left(a\,\beta_k\right) == a\left(\alpha_m\vee\beta_k\right) \qquad 1.10$$

- A regressive product is anti-commutative whenever the complementary grades of the factors are both odd.

$$\alpha_m\vee\beta_k == (-1)^{(n-m)\,(n-k)}\,\beta_k\vee\alpha_m \qquad 1.11$$

- The regressive product is both left and right distributive under addition.

$$\left(\alpha_m+\beta_m\right)\vee\gamma_r == \alpha_m\vee\gamma_r+\beta_m\vee\gamma_r \qquad \alpha_m\vee\left(\beta_r+\gamma_r\right) == \alpha_m\vee\beta_r+\alpha_m\vee\gamma_r \qquad 1.12$$

The Common Factor Axiom

Up to this point we have no way of connecting the dual axiom structures of the exterior and regressive products. That is, given a regressive product of an *m*-element and a *k*-element, how do we find the (*m*+*k*–*n*)-element to which it is equivalent, expressed only in terms of exterior products?

To make this connection we need to introduce a further axiom which we call the Common Factor Axiom. The form of the Common Factor Axiom may seem somewhat arbitrary, but it is in fact one of the simplest forms which enable intersections to be calculated. This can be seen in the following application of the axiom to a vector 3-space.

Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are three independent vectors in a vector 3-space. The Common Factor Axiom says that the regressive product of the two bivectors $x\wedge z$ and $y\wedge z$ may also be expressed as the regressive product of a 3-element $x\wedge y\wedge z$ with $\mathbf{z}$.

$$(x\wedge z)\vee(y\wedge z) == (x\wedge y\wedge z)\vee z$$

Since the space is 3-dimensional, we can write any 3-element such as $x\wedge y\wedge z$ as a scalar factor ($\mathbf{a}$, say) times the unit 3-element.

$$(x\wedge y\wedge z)\vee z == \left(a\,\underset{3}{1}\right)\vee z == a\,z$$

This then gives us the axiomatic structure to say that the regressive product of two elements possessing an element in common is congruent to that element. Another way of saying this is: the regressive product of two elements *defines* their intersection.

$$(x\wedge z)\vee(y\wedge z) \equiv z$$

Of course this is just a simple case. More generally, let α_m, β_k, and γ_p be *simple* elements with $m+k+p = n$, where n is the dimension of the space. Then the *Common Factor Axiom* states that

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma} \qquad m+k+p == n \tag{1.13}$$

There are many rearrangements and special cases of this formula which we will encounter in later chapters. For example, when p is zero, the Common Factor Axiom shows that the regressive product of an m-element with an $(n-m)$-element is a scalar which can be expressed in the alternative form of a regressive product with the unit **1**.

$$\underset{m}{\alpha}\vee\underset{n-m}{\beta} == \left(\underset{m}{\alpha}\wedge\underset{n-m}{\beta}\right)\vee 1$$

The Common Factor Axiom allows us to prove a particularly useful result: the Common Factor Theorem. The Common Factor Theorem expresses *any* regressive product in terms of exterior products alone. This of course enables us to calculate intersections of arbitrary elements.

Most importantly however we will see later that the Common Factor Theorem has a counterpart expressed in terms of exterior and interior products, called the Interior Common Factor Theorem. This forms the principal expansion theorem for interior products and from which we can derive all the important theorems relating exterior and interior products.

The Interior Common Factor Theorem, and the Common Factor Theorem upon which it is based, are possibly the most important theorems in the Grassmann algebra.

In the next section we informally apply the Common Factor Theorem to obtain the intersection of two bivectors in a three-dimensional space.

The intersection of two bivectors in a three-dimensional space

Suppose that $x\wedge y$ and $u\wedge v$ are non-congruent bivectors in a three dimensional space. Since the space has only three dimensions, the bivectors must have an intersection. We denote the regressive product of $x\wedge y$ and $u\wedge v$ by z:

$$z == (x\wedge y)\vee(u\wedge v)$$

We will see in Chapter 3: The Regressive Product that this can be expanded by the *Common Factor Theorem* to give

$$(x\wedge y)\vee(u\wedge v) == (x\wedge y\wedge v)\vee u - (x\wedge y\wedge u)\vee v \tag{1.14}$$

But we have already seen in Section 1.2 that in a 3-space, the exterior product of three vectors will, in any given basis, give the basis trivector, multiplied by the determinant of the components of the vectors making up the trivector.

Additionally, we note that the regressive product (intersection) of a vector with an element like the basis trivector completely containing the vector, will just give an element congruent to itself. Thus the regressive product leads us to an explicit expression for an intersection of the two bivectors.

$$z \equiv \mathrm{Det}[x, y, v]\, u - \mathrm{Det}[x, y, u]\, v$$

Here **`Det[x,y,v]`** is the determinant of the components of **x**, **y**, and **v**. We could also have obtained an equivalent formula expressing **z** in terms of **x** and **y** instead of **u** and **v** by simply interchanging the order of the bivector factors in the original regressive product.

Graphic showing the intersection of two bivectors as a vector with components in either bivector.

In Section 1.4: Geometric Interpretations below, we will see how we can apply the Common Factor Theorem to geometry to get the intersections of lines and planes.

1.4 Geometric Interpretations

Points and vectors

In this section we introduce *two* different types of *geometrically interpreted* elements which can be represented by the elements of a linear space: *vectors* and *points*. Then we look at the interpretations of the various higher order elements that we can generate from them by the exterior product. Finally we see how the regressive product can be used to calculate intersections of these higher order elements.

As discussed in Section 1.2, the term 'vector' is often used to refer to an element of a linear space with no intention of implying an interpretation. In this book however, we reserve the term for a particular type of geometric interpretation: that associated with representing *direction*.

But an element of a linear space may also be interpreted as a *point*. Of course vectors may also be used to represent points, but only *relative to another given point* (the information for which is not in the vector). These vectors are properly called *position vectors*. Common practice often omits explicit reference to this other given point, or perhaps may refer to it verbally. Points can be represented satisfactorily in many cases by position vectors alone, but when *both position and direction* are required in the same element we must distinguish mathematically between the two.

To describe true position in a three-dimensional physical space, a linear space of *four* dimensions is required, one for an origin point, and the other three for the three space directions. Since the exterior product is independent of the dimension of the underlying space, it can deal satisfactorily with points and vectors together. The usual three-dimensional vector algebra however cannot.

Suppose **x**, **y**, and **z** are elements of a linear space interpreted as vectors. Vectors always have a direction. But only when the linear space has a metric do they also have a *magnitude*. Since to this stage we have not yet introduced the notion of metric, we will only be discussing interpretations and applications which do not require elements (other than congruent elements) to be commensurable.

As we have seen in the previous sections, multiplying vectors together with the exterior product leads to higher order entities like bivectors and trivectors. These entities are interpreted as representing directions and their higher dimensional analogues.

Sums and differences of points

A *point* is defined as the sum of the *origin point* and a *vector*. If $\mathbb{O}$ is the origin, and $\mathbf{x}$ is a vector, then $\mathbb{O}+\mathbf{x}$ is a point.

$$\mathbf{P} == \mathbb{O} + \mathbf{x} \qquad 1.15$$

The vector $\mathbf{x}$ is called the *position vector* of the point $\mathbf{P}$.

Graphic showing a point as a sum of the origin and a vector.

The *difference* of two points is a vector since the origins cancel.

$$\mathbf{P}_1 - \mathbf{P}_2 == (\mathbb{O} + \mathbf{x}_1) - (\mathbb{O} + \mathbf{x}_2) == \mathbf{x}_1 - \mathbf{x}_2$$

A scalar multiple of a point is called a *weighted point*. For example, if $\mathbf{m}$ is a scalar, $\mathbf{m}\,\mathbf{P}$ is a weighted point with weight $\mathbf{m}$.

The *sum* of two points gives the point halfway between them with a weight of 2.

$$\mathbf{P}_1 + \mathbf{P}_2 == (\mathbb{O} + \mathbf{x}_1) + (\mathbb{O} + \mathbf{x}_2) == 2\left(\mathbb{O} + \frac{\mathbf{x}_1 + \mathbf{x}_2}{2}\right)$$

Graphic showing the weighted point between two points.

Historical Note
The point was originally considered the fundamental geometric entity of interest. However the difference of points was clearly no longer a point, since reference to the origin had been lost. Sir William Rowan Hamilton coined the term 'vector' for this new entity since adding a vector to a point 'carried' the point to a new point.

Determining a mass-centre

Consider a collection of points $\mathbf{P}_i$ weighted with masses $\mathbf{m}_i$. The sum of the weighted points gives the point $\mathbf{P}_G$ at the mass-centre (centre of gravity) weighted with the total mass $\mathbf{M}$. First add the weighted points and collect the terms involving the origin.

$$\begin{aligned}\mathbf{M}\,\mathbf{P}_G &== \mathbf{m}_1\,(\mathbb{O} + \mathbf{x}_1) + \mathbf{m}_2\,(\mathbb{O} + \mathbf{x}_2) + \mathbf{m}_3\,(\mathbb{O} + \mathbf{x}_3) + \ldots \\ &== (\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \ldots)\,\mathbb{O} + (\mathbf{m}_1\,\mathbf{x}_1 + \mathbf{m}_2\,\mathbf{x}_2 + \mathbf{m}_3\,\mathbf{x}_3 + \ldots)\end{aligned}$$

Dividing through by the total mass $\mathbf{M}$ gives the centre of mass.

$$\mathbf{P}_G == \mathbb{O} + \frac{(\mathbf{m}_1\,\mathbf{x}_1 + \mathbf{m}_2\,\mathbf{x}_2 + \mathbf{m}_3\,\mathbf{x}_3 + \ldots)}{(\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 + \ldots)}$$

Graphic showing a set of masses as weighted points.

Lines and planes

The exterior product of a point and a vector gives a *line-bound vector*. Line-bound vectors are the entities we need for mathematically representing lines. A line is the space of a line-bound vector. That is, it consists of all the points on the line and all the vectors in the direction of the line. To avoid convoluted terminology though, we will often refer to line-bound vectors simply as lines.

A line-bound vector can be defined by the exterior product of two points, or of a point and a vector. In the latter case we represent the line $\mathbf{L}$ *through* the point $\mathbf{P}$ in the direction of $\mathbf{x}$ by any entity *congruent* to the exterior product of $\mathbf{P}$ and $\mathbf{x}$.

$$\mathbf{L} \equiv \mathbf{P} \wedge \mathbf{x}$$

Graphic showing line defined by point and vector.

A line is independent of the specific points used to define it. To see this, consider another point $\mathbf{Q}$ on the line. Since $\mathbf{Q}$ is on the line it can be represented by the sum of $\mathbf{P}$ with an arbitrary scalar multiple of the vector $\mathbf{x}$:

$$\mathbf{L} \equiv \mathbf{Q} \wedge \mathbf{x} == (\mathbf{P} + a\,\mathbf{x}) \wedge \mathbf{x} == \mathbf{P} \wedge \mathbf{x}$$

A line may also be represented by the exterior product of *any two points* on it.

$$\mathbf{L} \equiv \mathbf{P} \wedge \mathbf{Q} == \mathbf{P} \wedge (\mathbf{P} + a\,\mathbf{x}) == a\,\mathbf{P} \wedge \mathbf{x}$$

$$\boxed{\mathbf{L} \equiv \mathbf{P}_1 \wedge \mathbf{P}_2 \equiv \mathbf{P} \wedge \mathbf{x}} \qquad 1.16$$

Graphic showing a line defined by either any of several pairs of points or point and vector pairs.

These concepts extend naturally to higher dimensional constructs. For example a plane π may be represented by the exterior product of any three points on it, any two points on it together with a vector in it (not parallel to the line joining the points), or a single point on it together with a bivector in the direction of the plane.

$$\boxed{\pi \equiv \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{P}_3 \equiv \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{x} \equiv \mathbf{P}_1 \wedge \mathbf{x} \wedge \mathbf{y}} \qquad 1.17$$

To build higher dimensional geometric entities from lower dimensional ones, we simply take their exterior product. For example we can build a line by taking the exterior product of a point with any point or vector *exterior* to it. Or we can build a plane by taking the exterior product of a line with any point or vector exterior to it.

Graphic showing a plane constructed out of a line and a point.

Graphic showing a plane constructed out of a point and a bivector.

The intersection of two lines

To find intersections of geometric entities, we use the regressive product. For example, suppose we have two lines in a plane and we want to find the point of intersection $\mathbf{P}$. As we have seen we can represent the lines in a number of ways. For example:

$$L_1 \equiv P_1 \wedge x_1 == (\mathbb{O} + \nu_1) \wedge x_1 == \mathbb{O} \wedge x_1 + \nu_1 \wedge x_1$$

$$L_2 \equiv P_2 \wedge x_2 == (\mathbb{O} + \nu_2) \wedge x_2 == \mathbb{O} \wedge x_2 + \nu_2 \wedge x_2$$

The point of intersection of L_1 and L_2 is the point $\mathbf{P}$ given by the regressive product of the lines L_1 and L_2.

$$P \equiv L_1 \vee L_2 == (\mathbb{O} \wedge x_1 + \nu_1 \wedge x_1) \vee (\mathbb{O} \wedge x_2 + \nu_2 \wedge x_2)$$

Expanding the formula for $\mathbf{P}$ gives four terms:

$$\begin{aligned} P \equiv\ & (\mathbb{O} \wedge x_1) \vee (\mathbb{O} \wedge x_2) + (\nu_1 \wedge x_1) \vee (\mathbb{O} \wedge x_2) + \\ & (\mathbb{O} \wedge x_1) \vee (\nu_2 \wedge x_2) + (\nu_1 \wedge x_1) \vee (\nu_2 \wedge x_2) \end{aligned}$$

The Common Factor Theorem for the regressive product of elements of the form $(x \wedge y) \vee (u \wedge v)$ was introduced as formula 1.14 in Section 1.3 as:

$$(x \wedge y) \vee (u \wedge v) == (x \wedge y \wedge v) \vee u - (x \wedge y \wedge u) \vee v$$

We now use this formula to expand each of the terms in $\mathbf{P}$:

$$(\mathbb{O} \wedge x_1) \vee (\mathbb{O} \wedge x_2) == (\mathbb{O} \wedge x_1 \wedge x_2) \vee \mathbb{O} - (\mathbb{O} \wedge x_1 \wedge \mathbb{O}) \vee x_2$$

$$(\nu_1 \wedge x_1) \vee (\mathbb{O} \wedge x_2) == (\nu_1 \wedge x_1 \wedge x_2) \vee \mathbb{O} - (\nu_1 \wedge x_1 \wedge \mathbb{O}) \vee x_2$$

$$\begin{aligned} & (\mathbb{O} \wedge x_1) \vee (\nu_2 \wedge x_2) == -(\nu_2 \wedge x_2) \vee (\mathbb{O} \wedge x_1) == \\ & -(\nu_2 \wedge x_2 \wedge x_1) \vee \mathbb{O} + (\nu_2 \wedge x_2 \wedge \mathbb{O}) \vee x_1 \end{aligned}$$

$$(\nu_1 \wedge x_1) \vee (\nu_2 \wedge x_2) == (\nu_1 \wedge x_1 \wedge x_2) \vee \nu_2 - (\nu_1 \wedge x_1 \wedge \nu_2) \vee x_2$$

The term $(\mathbb{O} \wedge x_1 \wedge \mathbb{O})$ is zero because of the exterior product of repeated factors. The four terms involving the exterior product of three vectors, for example $(\nu_1 \wedge x_1 \wedge x_2)$, are also zero since any three vectors in a two-dimensional vector space must be dependent. Hence we can express the point of intersection as congruent to the weighted point $\mathbf{P}$:

$$P \equiv (\mathbb{O} \wedge x_1 \wedge x_2) \vee \mathbb{O} + (\mathbb{O} \wedge \nu_2 \wedge x_2) \vee x_1 - (\mathbb{O} \wedge \nu_1 \wedge x_1) \vee x_2$$

If we express the vectors in terms of a basis, e_1 and e_2 say, we can reduce this formula (after some manipulation) to:

$$P \equiv \mathbb{O} + \frac{\mathrm{Det}[\nu_2, x_2]}{\mathrm{Det}[x_1, x_2]} x_1 - \frac{\mathrm{Det}[\nu_1, x_1]}{\mathrm{Det}[x_1, x_2]} x_2$$

Here, the determinants are the determinants of the coefficients of the vectors in the given basis.

Graphic showing the intersection of two lines.

To verify that $\mathbf{P}$ does indeed lie on both the lines $\mathbf{L_1}$ and $\mathbf{L_2}$, we only need to carry out the straightforward verification that the products $\mathbf{P} \wedge \mathbf{L_1}$ and $\mathbf{P} \wedge \mathbf{L_2}$ are both zero.

Although this approach *in this simple case* is certainly more complex than the standard algebraic approach, its interest lies in the fact that it is immediately generalizable to intersections of any geometric objects in spaces of any number of dimensions.

1.5 The Complement

The complement as a correspondence between spaces

The Grassmann algebra has a duality in its structure which not only gives it a certain elegance, but is also the basis of its power. We have already introduced the regressive product as the dual *product operation* to the exterior product. In this section we extend the notion of duality to define the *complement of an element*. The notions of orthogonality and interior, inner and scalar products are all based on the complement.

Consider a linear space of dimension n with basis $e_1, e_2, \ldots, e_n$. The set of all the essentially different m-element products of these basis elements forms the basis of another linear space, but this time of dimension $\binom{n}{m}$. For example, when n is 3, the linear space of 2-elements has three elements in its basis: $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_2 \wedge e_3$.

The anti-symmetric nature of the exterior product means that there are just as many basis elements in the linear space of (n–m)-elements as there are in the linear space of m-elements. Because these linear spaces have the same dimension, we can set up a *correspondence* between m-elements and (n–m)-elements. That is, given any m-element, we can determine its corresponding (n–m)-element. The (n–m)-element is called the *complement* of the m-element. Normally this correspondence is set up between basis elements and extended to all other elements by linearity.

The Euclidean complement

Suppose we have a three-dimensional linear space with basis e_1, e_2, e_3. We *define* the *Euclidean complement* of each of the basis elements as the basis 2-element whose exterior product with the basis element gives the basis 3-element $e_1 \wedge e_2 \wedge e_3$. We denote the complement of an element by placing a 'bar' over it. Thus:

$$\overline{e_1} == e_2 \wedge e_3 \quad \Rightarrow \quad e_1 \wedge \overline{e_1} == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_2} == e_3 \wedge e_1 \quad \Rightarrow \quad e_2 \wedge \overline{e_2} == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_3} == e_1 \wedge e_2 \quad \Rightarrow \quad e_3 \wedge \overline{e_3} == e_1 \wedge e_2 \wedge e_3$$

The Euclidean complement is the simplest type of complement and defines a Euclidean metric, that is, where the basis elements are mutually orthonormal. This was the only type of complement considered by Grassmann. In Chapter 5: The Complement we will show, however, how Grassmann's concept of complement is easily extended to more general metrics. Note carefully that we will be using the notion of complement to *define* the notions of orthogonality and metric, and until we do this, we will not be relying on their existence.

With the definitions above, we can now define the Euclidean complement of a general 1-element $\mathbf{x} == \mathbf{a}\,\mathbf{e_1} + \mathbf{b}\,\mathbf{e_2} + \mathbf{c}\,\mathbf{e_3}$, extending the definition on the basis elements by linearity.

$$\bar{\mathbf{x}} == \overline{a\,e_1 + b\,e_2 + c\,e_3} == a\,\overline{e_1} + b\,\overline{e_2} + c\,\overline{e_3} == a\,e_2 \wedge e_3 + b\,e_3 \wedge e_1 + c\,e_1 \wedge e_2$$

In Section 1.6 we will see that the complement of an element is orthogonal to the element, because we will define the interior product using the complement. We can start to see how the scalar product of a 1-element with itself might arise by expanding the product $\mathbf{x} \wedge \bar{\mathbf{x}}$:

$$\mathbf{x} \wedge \bar{\mathbf{x}} == (a\,e_1 + b\,e_2 + c\,e_3) \wedge (a\,e_2 \wedge e_3 + b\,e_3 \wedge e_1 + c\,e_1 \wedge e_2)$$
$$== (a^2 + b^2 + c^2)\,e_1 \wedge e_2 \wedge e_3$$

The complement of the basis 2-elements can be defined in a manner analogous to that for 1-elements, that is, such that the exterior product of a basis 2-element with its complement is equal to the basis 3-element. The complement of a 2-element in 3-space is therefore a 1-element.

$$\overline{e_2 \wedge e_3} == e_1 \quad \Longrightarrow \quad e_2 \wedge e_3 \wedge \overline{e_2 \wedge e_3} == e_2 \wedge e_3 \wedge e_1 == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_3 \wedge e_1} == e_2 \quad \Longrightarrow \quad e_3 \wedge e_1 \wedge \overline{e_3 \wedge e_1} == e_3 \wedge e_1 \wedge e_2 == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_1 \wedge e_2} == e_3 \quad \Longrightarrow \quad e_1 \wedge e_2 \wedge \overline{e_1 \wedge e_2} == e_1 \wedge e_2 \wedge e_3$$

To complete the definition of Euclidean complement in a 3-space we note that:

$$\bar{1} == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_1 \wedge e_2 \wedge e_3} == 1$$

Summarizing these results for the Euclidean complement of the basis elements of a Grassmann algebra in three dimensions shows the essential symmetry of the complement operation.

BASIS	COMPLEMENT
1	$e_1 \wedge e_2 \wedge e_3$
e_1	$e_2 \wedge e_3$
e_2	$-(e_1 \wedge e_3)$
e_3	$e_1 \wedge e_2$
$e_1 \wedge e_2$	e_3
$e_1 \wedge e_3$	$-e_2$
$e_2 \wedge e_3$	e_1
$e_1 \wedge e_2 \wedge e_3$	1

The complement of a complement

Applying these results to the 1-element $\mathbf{x}$ we can see that the complement of the complement of $\mathbf{x}$ is just $\mathbf{x}$ itself.

$$x == a\, e_1 + b\, e_2 + c\, e_3$$

$$\bar{x} == \overline{a\, e_1 + b\, e_2 + c\, e_3} == a\, \overline{e_1} + b\, \overline{e_2} + c\, \overline{e_3} == a\, e_2 \wedge e_3 + b\, e_3 \wedge e_1 + c\, e_1 \wedge e_2$$

$$\bar{\bar{x}} == \overline{a\, e_2 \wedge e_3 + b\, e_3 \wedge e_1 + c\, e_1 \wedge e_2} ==$$
$$a\, \overline{e_2 \wedge e_3} + b\, \overline{e_3 \wedge e_1} + c\, \overline{e_1 \wedge e_2} == a\, e_1 + b\, e_2 + c\, e_3$$

$$\Longrightarrow \quad \bar{\bar{x}} == x$$

More generally, as we shall see in Chapter 5: The Complement, we can show that the complement of the complement of *any* element is the element itself, apart from a possible sign.

$$\bar{\bar{\underset{m}{\alpha}}} == (-1)^{m\,(n-m)}\, \underset{m}{\alpha} \qquad 1.18$$

This result is independent of the correspondence that we set up between the *m*-elements and (*n*–*m*)-elements of the space, *except that the correspondence must be symmetric*. This is equivalent to the commonly accepted requirement that the metric tensor (and inner product) be symmetric. We call this the *Repeated Complement Theorem*.

The Complement Axiom

From the Common Factor Axiom we can derive a powerful relationship between the Euclidean complements of elements and their exterior and regressive products. The Euclidean complement of the exterior product of two elements is equal to the regressive product of their complements.

$$\overline{\underset{m}{\alpha} \wedge \underset{k}{\beta}} == \bar{\underset{m}{\alpha}} \vee \bar{\underset{k}{\beta}} \qquad 1.19$$

This result, which we call the *Complement Axiom*, is the quintessential formula of the Grassmann algebra. It expresses the duality of its two fundamental operations on elements and their complements. We note the formal similarity to de Morgan's law in Boolean algebra.

We will also see that adopting this formula for general complements will enable us to compute the complement of any element of a space once we have defined the complements of its basis 1-elements.

Graphic showing the complement of a bivector as the intersection of two complements $\overline{\alpha \wedge \beta} == \bar{\alpha} \vee \bar{\beta}$.

With these basic notions, definitions and formulae in hand, we can now proceed to define the interior product.

1.6 The Interior Product

The definition of the interior product

The interior product is a generalization of the inner or scalar product. First we will define the interior product and then show how the inner and scalar products are special cases.

The *interior product* of an element $\underset{m}{\alpha}$ with an element $\underset{k}{\beta}$ is denoted $\underset{m}{\alpha} \ominus \underset{k}{\beta}$ and *defined* to be the *regressive* product of $\underset{m}{\alpha}$ with the complement of $\underset{k}{\beta}$.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \underset{m}{\alpha} \vee \overline{\underset{k}{\beta}} \qquad \mathbf{1.20}$$

The grade of an interior product $\underset{m}{\alpha} \ominus \underset{k}{\beta}$ may be seen from the definition to be $m+(n-k)-n = m-k$.

Note that while the grade a regressive product depends on the dimension of the underlying space, the grade of an interior product is independent of the dimension of the underlying space.

The order of factors in an interior product is important. When the grade of the first element is *less* than that of the second element, the result is necessarily zero.

Inner products and scalar products

The interior product of two elements $\underset{m}{\alpha}$ and $\underset{m}{\beta}$ *of the same grade* is (also) called their *inner product*. Since the grade of an interior product is the difference of the grades of its factors, an inner product is always of grade zero, hence scalar.

A special case of the interior or inner product in which the two factors of the product are grade 1 is called a *scalar product* to conform to the common usage.

We can show from the definition and the Repeated Complement Theorem that *the inner product is symmetric*, that is, the order of the factors is immaterial.

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} \quad == \quad \underset{m}{\beta} \ominus \underset{m}{\alpha} \qquad \mathbf{1.21}$$

When the inner product is between simple elements it can be expressed as the determinant of the array of scalar products according to the following formula:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) == \mathbf{Det}[\alpha_i \ominus \beta_j] \qquad \mathbf{1.22}$$

For example, the inner product of two 2-elements $\alpha_1 \wedge \alpha_2$ and $\beta_1 \wedge \beta_2$ may be written:

$$(\alpha_1 \wedge \alpha_2) \ominus (\beta_1 \wedge \beta_2) == (\alpha_1 \ominus \beta_1)\ (\alpha_2 \ominus \beta_2) - (\alpha_1 \ominus \beta_2)\ (\alpha_2 \ominus \beta_1)$$

Calculating interior products

We can use the definition to calculate interior products. In what follows we calculate the interior products of representative basis elements of a 3-space with Euclidean metric. As expected, the scalar products $e_1 \ominus e_1$ and $e_1 \ominus e_2$ turn out to be 1 and 0 respectively.

$$e_1 \ominus e_1 == e_1 \vee \overline{e_1} == e_1 \vee (e_2 \wedge e_3) == (e_1 \wedge e_2 \wedge e_3) \vee 1 == \bar{1} \vee 1 == 1$$

$$e_1 \ominus e_2 == e_1 \vee \overline{e_2} == e_1 \vee (e_3 \wedge e_1) == (e_1 \wedge e_3 \wedge e_1) \vee 1 == 0 \vee 1 == 0$$

Inner products of identical basis 2-elements are unity:

$$(e_1 \wedge e_2) \ominus (e_1 \wedge e_2) == (e_1 \wedge e_2) \vee \overline{(e_1 \wedge e_2)} == \\ (e_1 \wedge e_2) \vee e_3 == (e_1 \wedge e_2 \wedge e_3) \vee 1 == \bar{1} \vee 1 == 1$$

Inner products of non-identical basis 2-elements are zero:

$$(e_1 \wedge e_2) \ominus (e_2 \wedge e_3) == (e_1 \wedge e_2) \vee \overline{(e_2 \wedge e_3)} == \\ (e_1 \wedge e_2) \vee e_1 == (e_1 \wedge e_2 \wedge e_3) \vee 1 == \bar{1} \vee 1 == 1$$

If a basis 2-element contains a given basis 1-element, then their interior product is not zero:

$$(e_1 \wedge e_2) \ominus e_1 == (e_1 \wedge e_2) \vee \overline{e_1} == \\ (e_1 \wedge e_2) \vee (e_2 \wedge e_3) == (e_1 \wedge e_2 \wedge e_3) \vee e_2 == \bar{1} \vee e_2 == e_2$$

If a basis 2-element does *not* contain a given basis 1-element, then their interior product is zero:

$$(e_1 \wedge e_2) \ominus e_3 == (e_1 \wedge e_2) \vee \overline{e_3} == (e_1 \wedge e_2) \vee (e_1 \wedge e_2) == 0$$

Expanding interior products

To expand interior products, we use the Interior Common Factor Theorem. This theorem shows how an interior product of a simple element $\underset{m}{\alpha}$ with another, not necessarily simple element of equal or lower grade $\underset{k}{\beta}$, may be expressed as a linear combination of the ν (= $\binom{m}{k}$) essentially different factors $\underset{m-k}{\alpha_i}$ (of grade m–k) of the simple element of higher degree.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \sum_{i=1}^{\nu} \left(\underset{k}{\alpha_i} \ominus \underset{k}{\beta}\right) \underset{m-k}{\alpha_i}$$

$$\underset{m}{\alpha} == \underset{k}{\alpha_1} \wedge \underset{m-k}{\alpha_1} == \underset{k}{\alpha_2} \wedge \underset{m-k}{\alpha_2} == \cdots == \underset{k}{\alpha_\nu} \wedge \underset{m-k}{\alpha_\nu} \qquad 1.23$$

For example, the Interior Common Factor Theorem may be used to prove a relationship involving the interior product of a 1-element x with the exterior product of two factors, each of which may not be simple. This relationship and the special cases that derive from it find application throughout the explorations in the rest of this book.

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus x == \left(\underset{m}{\alpha} \ominus x\right) \wedge \underset{k}{\beta} + (-1)^m \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \ominus x\right) \qquad 1.24$$

The interior product of a bivector and a vector

Suppose x is a vector and $x_1 \wedge x_2$ is a simple bivector. The interior product of the bivector $x_1 \wedge x_2$ with the vector x is the vector $(x_1 \wedge x_2) \ominus x$. This can be expanded by the Interior Common Factor Theorem, or the formula above derived from it, to give:

$$(x_1 \wedge x_2) \ominus x == (x \ominus x_1)\, x_2 - (x \ominus x_2)\, x_1$$

Since the result is expressed as a linear combination of x_1 and x_2 it is clearly contained in the bivector $x_1 \wedge x_2$.

The result is also orthogonal to x. We can show this by taking its scalar product with x:

$$((x \ominus x_1)\, x_2 - (x \ominus x_2)\, x_1) \ominus x == (x \ominus x_1)\,(x_2 \ominus x) - (x \ominus x_2)\,(x_1 \ominus x) == 0$$

Graphic showing the interior product of a vector with a bivector:
Shows orthogonal projection and vector in planar direction at right angle to orthogonal projection.

These concepts may easily be extended to geometric entities of arbitrary dimension.

1.7 Exploring Screw Algebra

To be completed

1.8 Exploring Mechanics

To be completed

1.9 Exploring Grassmann Algebras

To be completed

1.10 Exploring The Generalised Product

To be completed

1.11 Exploring Hypercomplex Algebras

To be completed

1.12 ExploringClifford Algebras

To be completed

1.13 ExploringGrassmann Matrix Algebras

To be completed

2 The Exterior Product

2.1 Introduction

The exterior product is the natural fundamental product operation for elements of a linear space. Although it is not a closed operation (that is, the product of two elements is not itself an element of the *same* linear space), the products it generates form a series of new linear spaces, the totality of which can be used to define a true algebra, which is closed.

The exterior product is naturally and fundamentally connected with the notion of linear dependence. Several 1-elements are linearly dependent if and only if their exterior product is zero. All the properties of determinants flow naturally from the simple axioms of the exterior product. The notion of 'exterior' is equivalent to the notion of linear independence, since elements which are truly exterior to one another (that is, not lying in the same space) will have a non-zero product. This product has a straightforward geometric interpretation as the multi-dimensional equivalent to the 1-elements from which it is constructed. And if the space possesses a metric, its measure or magnitude may be interpreted as a length, area, volume, or hyper-volume according to the grade of the product.

However, the exterior product does not require the linear space to possess a metric. This is in direct contrast to the three-dimensional vector calculus in which the vector (or cross) product *does* require a metric, since it is defined as the vector *orthogonal* to its two factors, and orthogonality is a metric concept. Some of the results which use the cross product can equally well be cast in terms of the exterior product, thus avoiding unnecessary assumptions.

We start the chapter with an informal discussion of the exterior product, and then collect the axioms for exterior linear spaces together more formally into a set of 12 axioms which combine those of the underlying field and underlying linear space with the specific properties of the exterior product. In later chapters we will derive equivalent sets for the regressive and interior products, to which we will often refer.

Next, we pin down the linear space notions that we have introduced axiomatically by introducing a basis onto the primary linear space and showing how this can induce a basis onto each of the other linear spaces generated by the exterior product. A constantly useful partner to a basis element of any of these exterior linear spaces is its cobasis element. The exterior product of a basis element with its cobasis element always gives the *n*-element of the algebra. We develop the notion of cobasis following that of basis.

The next three sections look at some standard topics in linear algebra from a Grassmannian viewpoint: determinants, cofactors, and the solution of systems of linear equations. We show that all the well-known properties of determinants, cofactors, and linear equations proceed directly from the properties of the exterior product.

We finish the chapter with a discussion of two concepts dependent on the exterior product that have no counterpart in standard linear algebra: simplicity and exterior division. If an element is simple it can be factorized into 1-elements. If a simple element is divided by another simple element contained in it, the result is not unique. Both these properties will find application later in our explorations of geometry.

2.2 The Exterior Product

Basic properties of the exterior product

Let $\underset{0}{\Lambda}$ denote the field of real numbers. Its elements, called 0-*elements* (or scalars) will be denoted by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, … . Let $\underset{1}{\Lambda}$ denote a linear space of dimension *n* whose field is $\underset{0}{\Lambda}$. Its elements, called 1-*elements*, will be denoted by $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, … .

We call $\underset{1}{\Lambda}$ a linear space rather than a vector space and its elements 1-elements rather than vectors because we will be interpreting its elements as points as well as vectors.

A second linear space denoted $\underset{2}{\Lambda}$ may be constructed as the space of sums of exterior products of 1-elements taken two at a time. The *exterior product* operation is denoted by $\wedge$, and has the following properties:

$$a\,(x \wedge y) == (a\,x) \wedge y \qquad 2.1$$

$$x \wedge (y + z) == x \wedge y + x \wedge z \qquad 2.2$$

$$(y + z) \wedge x == y \wedge x + z \wedge x \qquad 2.3$$

$$x \wedge x == 0 \qquad 2.4$$

An important property of the exterior product (which is sometimes taken as an axiom) is the *anti-symmetry* of the product of two 1-elements.

$$x \wedge y == -y \wedge x \qquad 2.5$$

This may be proved from the *distributivity* and *nilpotency* axioms since:

$$(x + y) \wedge (x + y) == 0$$
$$\Longrightarrow x \wedge x + x \wedge y + y \wedge x + y \wedge y == 0$$
$$\Longrightarrow x \wedge y + y \wedge x == 0$$
$$\Longrightarrow x \wedge y == -y \wedge x$$

An element of $\underset{2}{\Lambda}$ will be called a 2-*element* (of *grade* 2) and is denoted by a kernel letter with a '2' written below. For example:

$$\underset{2}{\alpha} == x \wedge y + z \wedge w + \cdots$$

A *simple* 2-element is the exterior product of two 1-elements.

It is important to note the distinction between a 2-element and a simple 2-element (a distinction of no consequence in $\underset{1}{\Lambda}$, where all elements are simple). A 2-element is in general a sum of simple 2-elements, and is not generally expressible as the product of two 1-elements (except where $\underset{1}{\Lambda}$ is of dimension $n \leq 3$). The structure of $\underset{2}{\Lambda}$, whilst still that of a linear space, is thus richer than that of $\underset{1}{\Lambda}$, that is, it has more properties.

Example: 2-elements in a three-dimensional space are simple

By way of example, the following shows that when $\underset{1}{\Lambda}$ is of dimension 3, every element of $\underset{2}{\Lambda}$ is simple.

Suppose that $\underset{1}{\Lambda}$ has basis e_1, e_2, e_3. Then a basis of $\underset{2}{\Lambda}$ is the set of all essentially different (linearly independent) products of basis elements of $\underset{1}{\Lambda}$ taken two at a time: $e_1 \wedge e_2$, $e_2 \wedge e_3$, $e_3 \wedge e_1$. (The product $e_1 \wedge e_2$ is not considered essentially different from $e_2 \wedge e_1$ in view of the anti-symmetry property).

Let a general 2-element $\underset{2}{\alpha}$ be expressed in terms of this basis as:

$$\underset{2}{\alpha} == a\, e_1 \wedge e_2 + b\, e_2 \wedge e_3 + c\, e_3 \wedge e_1$$

Without loss of generality, suppose $a \neq 0$. Then $\underset{2}{\alpha}$ can be recast in the form below, thus proving the proposition.

$$\underset{2}{\alpha} == a \left(e_1 - \frac{b}{a}\, e_3\right) \wedge \left(e_2 - \frac{c}{a}\, e_3\right)$$

Historical Note

In the *Ausdehnungslehre* of 1844 Grassmann denoted the exterior product of two symbols α and β by a simple concatenation *viz.* $\alpha\,\beta$; whilst in the *Ausdehnungslehre* of 1862 he enclosed them in square brackets *viz.* $[\alpha\,\beta]$. This notation has survived in the three-dimensional vector calculus as the 'box' product $[\alpha\,\beta\,\gamma]$ used for the triple scalar product. Modern usage denotes the exterior product operation by the wedge $\wedge$, thus $\alpha\wedge\beta$. Amongst other writers, Whitehead [ ] used the 1844 version whilst Forder [ ] and Cartan [ ] followed the 1862 version.

Setting up scalars in *GrassmannAlgebra*

GrassmannAlgebra allows the setting up of an environment in which given symbols are declared to be scalars (0-elements). You can always see what your current scalar symbols are by entering `Scalars`.

When first loaded, *GrassmannAlgebra* sets up a default list of scalars. To see what they are type `Scalars` and enter:

```
Scalars
```

GrassmannAlgebra will then return the list:

$\{a, b, c, d, e, f, g, h, \mathbb{k}, \underset{0}{\_}, (\_\ominus\_)\,?\text{InnerProductQ}\}$

This means that the following are taken to be scalars:
1) Any of the symbols {**a**, **b**, **c**, **d**, **e**, **f**, **g**, **h**}.
2) The symbol $\mathbb{k}$.
2) Any symbol underscripted with a zero.
3) Any inner products.

To declare your own list of scalars, enter `DeclareScalars[` *`list`* `]`.

```
DeclareScalars[{a, b, c}]
```

{a, b, c}

Now, if we enter `Scalars` we see the new list:

```
Scalars
```

{a, b, c}

You can always return to the default declaration by entering:

```
DeclareScalars[DefaultScalars]
```

$\{a, b, c, d, e, f, g, h, \mathbb{k}, \underset{0}{\_}, (\_\ominus\_)\,?\text{InnerProductQ}\}$

or simply:

```
DeclareDefaultScalars[]
```

$\{a, b, c, d, e, f, g, h, \mathbb{k}, \underset{0}{\_}, (\_\ominus\_)\,?\text{InnerProductQ}\}$

Declaring which symbols you wish to be interpreted as scalars is important to *GrassmannAlgebra* because all other (undefined) pure symbols will be interpreted as 1-elements. An underscripted symbol will be interpreted as having the grade indicated by its underscript. For example $\underset{2}{\alpha}$ will be interpreted as having a grade of 2.

At any stage you can declare extra scalars and have them added to the list of currently declared scalars. (More precisely, it is the *union* of the current list and the new list that becomes the declared list.) For example:

```
DeclareExtraScalars[{a, aa, aaa}]
```

$\{a, aa, aaa, b, c, d, e, f, g, h, \Bbbk, (\_\ominus\_)\,?\text{InnerProductQ}, \underset{0}{\_}\}$

You can use *patterns* in the list of declared scalars. For example, you may wish to declare all subscripted symbols with the kernel symbol α as scalars.

```
DeclareExtraScalars[{α_}]
```

$\{a, aa, aaa, b, c, d, e, f, g,$
$h, \Bbbk, (\_\ominus\_)\,?\text{InnerProductQ}, \alpha_{\_}, \underset{0}{\_}\}$

We can check if a given element is a scalar, or all the elements in a list are scalars, with `ScalarQ`.

$\mathbf{ScalarQ}[\{\alpha_{\_}, \alpha_2, \alpha_{\mathbf{new}}, \mathbf{x}\ominus\mathbf{y}, \underset{\mathbf{0}}{\mathbf{x}}, \mathbf{x}\}]$

```
{True, True, True, True, True, False}
```

Entering exterior products

The exterior product is symbolized by the wedge '∧'. This may be entered either by typing `\[Wedge]`, by typing the alias ESC^ESC, or by clicking on the ∧ button on the palette.

As an infix operator, **x∧y∧z∧⋯** is interpreted as **Wedge[x,y,z,⋯]**.

To enter **x∧y** type **x**ESC^ESC**y**, or click the ■∧■ button on the palette. The first placeholder will be selected automatically for immediate typing. To move to the next placeholder, press the tab key. To produce a multifactored exterior product, click the palette button as many times as required. To take the exterior product with an existing expression, select the expression and click the ■∧■ button.

For example, to type **(x+y)∧(x+z)**, first type **(x+y)** and select it, click the ■∧■ button, then type **(x+z)**.

2.3 Exterior Linear Spaces

Constructing *m*-elements

In the previous section the exterior product was introduced as an operation on the elements of the linear space $\underset{1}{\Lambda}$ to produce elements belonging to a new space $\underset{2}{\Lambda}$. Just as $\underset{2}{\Lambda}$ was constructed from sums of exterior products of elements of $\underset{1}{\Lambda}$ two at a time, so $\underset{m}{\Lambda}$ may be constructed from sums of exterior products of elements of $\underset{1}{\Lambda}$ m at a time.

A *simple m-element* is a product of the form:

$$x_1 \wedge x_2 \wedge \cdots \wedge x_m \qquad x_i \in \underset{1}{\Lambda}$$

An *m-element* is a linear combination of simple *m*-elements. It is denoted by a kernel letter with an 'm' written below:

$$\underset{m}{\alpha} = a_1\, x_1 \wedge x_2 \wedge \cdots \wedge x_m + a_2\, y_1 \wedge y_2 \wedge \cdots \wedge y_m + \cdots \qquad a_i \in \underset{0}{\Lambda}$$

The number m is called the *grade* of the *m*-element.

The exterior linear space $\underset{0}{\Lambda}$ has essentially only one element in its basis, which we take as the unit 1. All other elements of $\underset{0}{\Lambda}$ are scalar multiples of this basis element.

The exterior linear space $\underset{n}{\Lambda}$ (where n is the dimension of $\underset{1}{\Lambda}$) has essentially only one element in its basis: $e_1 \wedge e_2 \wedge \cdots \wedge e_n$. All other elements of $\underset{n}{\Lambda}$ are scalar multiples of this basis element.

Any m-element, where $m > n$, is zero, since there are no more than n independent elements available from which to construct it, and the nilpotency property causes it to vanish.

Spaces and congruence

The *space* of a simple m-element $\underset{m}{\alpha}$ is the set of all 1-elements α belonging to $\underset{m}{\alpha}$, that is, such that $\alpha \wedge \underset{m}{\alpha} == 0$. $\underset{m}{\alpha}$ is said to *define* its own space. We say that α *is contained in the space of* $\underset{m}{\alpha}$ and by extension sometimes simply say that α *is contained in* $\underset{m}{\alpha}$.

A simple element $\underset{k}{\beta}$ is said to be contained in another simple element $\underset{m}{\alpha}$ if and only if the space of $\underset{k}{\beta}$ is a subset of that of $\underset{m}{\alpha}$.

We say that two elements are *congruent* if one is a scalar multiple of the other, and denote the relationship with the symbol $\equiv$. For example we may write:

$$\underset{m}{\alpha} \equiv a\, \underset{m}{\alpha}$$

We can therefore say that if two elements are congruent, then their spaces are the same.

Note that whereas several congruent elements are linearly dependent, several linearly dependent elements are not necessarily congruent.

When we have one element equal to a scalar multiple of another, $\underset{m}{\alpha} == a\,\underset{m}{\beta}$ say, we may sometimes take the liberty of writing the scalar multiple as a quotient of the two elements:

$$a == \frac{\underset{m}{\alpha}}{\underset{m}{\beta}}$$

These notions will be particularly useful in the discussion of *unions* and *intersections* in the next chapter, and on applications to geometry.

The associativity of the exterior product

The exterior product is associative in all groups of (adjacent) factors. For example:

$$(x_1 \wedge x_2) \wedge x_3 \wedge x_4 == x_1 \wedge (x_2 \wedge x_3) \wedge x_4 == (x_1 \wedge x_2 \wedge x_3) \wedge x_4 == \cdots$$

Hence

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \wedge \underset{p}{\gamma} == \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \wedge \underset{p}{\gamma}\right) \qquad 2.6$$

Thus the brackets may be omitted altogether.

From this associativity together with the anti-symmetric property of 1-elements it may be shown that *the exterior product is anti-symmetric in all* (1-*element*) *factors*. That is, a transposition of any two 1-element factors changes the sign of the product. For example:

$$x_1 \wedge x_2 \wedge x_3 \wedge x_4 == -x_3 \wedge x_2 \wedge x_1 \wedge x_4 == x_3 \wedge x_4 \wedge x_1 \wedge x_2 == \cdots$$

Furthermore, from the nilpotency axiom, *a product with two identical* 1-*element factors is zero*. For example:

$$x_1 \wedge x_2 \wedge x_3 \wedge x_2 == 0$$

■ Example: Non-simple elements are not generally nilpotent

It should be noted that for simple elements $\underset{m}{\alpha}$, $\underset{m}{\alpha} \wedge \underset{m}{\alpha} == 0$, but that non-simple elements do not necessarily possess this property, as the following example shows.

Suppose $\underset{1}{\Lambda}$ is of dimension 4 with basis e_1, e_2, e_3, e_4. Then the following exterior product of identical elements is not zero:

$$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4) == 2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

Calculating exterior products

To expand an exterior product, use the function `ExpandProducts`. For example, to expand the product in the previous example:

ExpandProducts$[(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)]$

$e_1 \wedge e_2 \wedge e_1 \wedge e_2 + e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_3 \wedge e_4 \wedge e_1 \wedge e_2 + e_3 \wedge e_4 \wedge e_3 \wedge e_4$

Note that `ExpandProducts` does not simplify the expression.

To simplify an expression by putting nilpotent terms to zero, use the `NilpotentSimplify` function.

NilpotentSimplify$[e_1 \wedge e_2 \wedge e_1 \wedge e_2 + e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_3 \wedge e_4 \wedge e_1 \wedge e_2 + e_3 \wedge e_4 \wedge e_3 \wedge e_4]$

$e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_3 \wedge e_4 \wedge e_1 \wedge e_2$

There is still a further simplification that can be effected by noting that these two terms are equal. We can make this equality evident by reordering the factors of the terms into a consistent standard ordering. The *GrassmannAlgebra* function for doing this is `ToStandardOrdering`.

ToStandardOrdering$[e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_3 \wedge e_4 \wedge e_1 \wedge e_2]$

$2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$

To perform all these functions together use the `GrassmannSimplify` function. This can also be accessed by its alias $\mathcal{G}$ (ESC**scG**ESC) (script G), so that selecting the original expression to be simplified or evaluated and clicking the $\mathcal{G}$**[■]** button on the palette should give the above result directly. However, we must also make sure the basis is at least four-dimensional by entering `DeclareBasis[4]`, otherwise the `GrassmannSimplify` will see that any 4-elements in a lesser dimensional space will be zero. (We discuss setting up bases in a later section.)

DeclareBasis[4]

$\{e_1, e_2, e_3, e_4\}$

$\mathcal{G}[(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)]$

$2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$

2.4 Axioms for Exterior Linear Spaces

Summary of axioms

The axioms for exterior linear spaces are summarized here for future reference. They are a composite of the requisite field, linear space, and exterior product axioms, and may be used to derive the properties discussed informally in the preceding sections.

◆ ∧ 1: **The sum of *m*-elements is itself an *m*-element.**

$$\left\{\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{m}{\beta} \in \underset{m}{\Lambda}\right\} \Longrightarrow \underset{m}{\alpha} + \underset{m}{\beta} \in \underset{m}{\Lambda} \tag{2.7}$$

◆ ∧ 2: **Addition of *m*-elements is associative.**

$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) + \underset{m}{\gamma} == \underset{m}{\alpha} + \left(\underset{m}{\beta} + \underset{m}{\gamma}\right) \tag{2.8}$$

◆ ∧ 3: ***m*-elements have an additive identity (zero element).**

$$\mathcal{E}xists\left[\underset{m}{\mathbf{0}},\ \underset{m}{\mathbf{0}} \in \underset{m}{\Lambda}\right] : \underset{m}{\alpha} == \underset{m}{\mathbf{0}} + \underset{m}{\alpha} \tag{2.9}$$

◆ ∧ 4: ***m*-elements have an additive inverse.**

$$\mathcal{E}xists\left[-\underset{m}{\alpha},\ -\underset{m}{\alpha} \in \underset{m}{\Lambda}\right] : \underset{m}{\mathbf{0}} == \underset{m}{\alpha} + -\underset{m}{\alpha} \tag{2.10}$$

◆ ∧ 5: **Addition of *m*-elements is commutative.**

$$\underset{m}{\alpha} + \underset{m}{\beta} == \underset{m}{\beta} + \underset{m}{\alpha} \tag{2.11}$$

◆ ∧ 6: **The exterior product of an *m*-element and a *k*-element is an (*m*+*k*)-element.**

$$\left\{\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{k}{\beta} \in \underset{k}{\Lambda}\right\} \Longrightarrow \underset{m}{\alpha} \wedge \underset{k}{\beta} \in \underset{m+k}{\Lambda} \tag{2.12}$$

◆ ∧ 7: **The exterior product is associative.**

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \wedge \underset{r}{\gamma} == \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \wedge \underset{r}{\gamma}\right) \tag{2.13}$$

◆ **∧ 8: There is a unit scalar which acts as an identity under the exterior product.**

$$\mathcal{E}xists\left[1,\ 1 \in \underset{0}{\Lambda}\right] : \underset{m}{\alpha} == 1 \wedge \underset{m}{\alpha} \tag{2.14}$$

◆ **∧ 9: Non-zero scalars have inverses with respect to the exterior product.**

$$\mathcal{E}xists\left[\underset{0}{\alpha}^{-1},\ \underset{0}{\alpha}^{-1} \in \underset{0}{\Lambda}\right] : 1 == \underset{0}{\alpha} \wedge \underset{0}{\alpha}^{-1},\ \mathcal{F}or\mathcal{A}ll\left[\underset{0}{\alpha},\ \underset{0}{\alpha} \neq \underset{0}{0} \in \underset{0}{\Lambda}\right] \tag{2.15}$$

◆ **∧ 10: The exterior product of elements of odd grade is anti-commutative.**

$$\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{m\,k}\, \underset{k}{\beta} \wedge \underset{m}{\alpha} \tag{2.16}$$

◆ **∧ 11: Additive identities act as multiplicative zero elements under the exterior product.**

$$\mathcal{E}xists\left[\underset{k}{0},\ \underset{k}{0} \in \underset{k}{\Lambda}\right] : \underset{k}{0} \wedge \underset{m}{\alpha} == \underset{k+m}{0} \tag{2.17}$$

◆ **∧ 12: The exterior product is both left and right distributive under addition.**

$$\begin{gathered}\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \wedge \underset{r}{\gamma} == \underset{m}{\alpha} \wedge \underset{r}{\gamma} + \underset{m}{\beta} \wedge \underset{r}{\gamma} \\ \underset{m}{\alpha} \wedge \left(\underset{r}{\beta} + \underset{r}{\gamma}\right) == \underset{m}{\alpha} \wedge \underset{r}{\beta} + \underset{m}{\alpha} \wedge \underset{r}{\gamma}\end{gathered} \tag{2.18}$$

◆ **∧ 13: Scalar multiplication commutes with the exterior product.**

$$\underset{m}{\alpha} \wedge \left(a\, \underset{k}{\beta}\right) == \left(a\, \underset{m}{\alpha}\right) \wedge \underset{k}{\beta} == a \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \tag{2.19}$$

◆ **A convention for the zeros**

It can be seen from the above axioms that each of the linear spaces has its own zero. These zeros all have the same properties, so that for simplicity we will denote them all by the one symbol **0**.

Grassmann algebras

Under the foregoing axioms it may be directly shown that:

1. $\underset{0}{\Lambda}$ is a field.
2. The $\underset{m}{\Lambda}$ are linear spaces over $\underset{0}{\Lambda}$.
3. The direct sum of the $\underset{m}{\Lambda}$ is an algebra.

This algebra is called a Grassmann algebra. Its elements are sums of elements from the $\underset{m}{\Lambda}$, thus allowing closure over both addition and exterior multiplication. For example we can evaluate the following product to give another element of the algebra:

$$\mathcal{G}[(1+2\,x+3\,x\wedge y+4\,x\wedge y\wedge z)\wedge(1+2\,x+3\,x\wedge y+4\,x\wedge y\wedge z)]$$

$$1+4\,x+6\,x\wedge y+8\,x\wedge y\wedge z$$

A Grassmann algebra is also a linear space of dimension 2^n, where *n* is the dimension of the underlying linear space $\underset{1}{\Lambda}$. We will refer to a Grassmann algebra whose underlying linear space is of dimension *n* as an *n*-algebra.

Grassmann algebras will be discussed further in Chapter 9.

On the nature of scalar multiplication

The anti-commutativity axiom $\wedge$10 for general elements states that:

$$\underset{m}{\alpha}\wedge\underset{k}{\beta} == (-1)^{m\,k}\,\underset{k}{\beta}\wedge\underset{m}{\alpha}$$

If one of these factors is a scalar ($\underset{k}{\beta} == a$, say; $k = 0$), the axiom reduces to:

$$\underset{m}{\alpha}\wedge a == a\wedge\underset{m}{\alpha}$$

Since by axiom $\wedge$6, each of these terms is an *m*-element, we may permit the exterior product to subsume the normal field multiplication. Thus, if a is a scalar, $a\wedge\underset{m}{\alpha}$ is equivalent to $a\,\underset{m}{\alpha}$. The latter (conventional) notation will usually be adopted.

In the usual definitions of linear spaces no discussion is given to the nature of the product of a scalar and an element of the space. A notation is usually adopted (that is, the omission of the product sign) that leads one to suppose this product to be of the same nature as that between two scalars. From the axioms above it may be seen that both the product of two scalars and the product of a scalar and an element of the linear space may be interpreted as exterior products.

$$\boxed{a\wedge\underset{m}{\alpha} == \underset{m}{\alpha}\wedge a == a\,\underset{m}{\alpha} \qquad a\in\underset{0}{\Lambda}} \qquad 2.20$$

Factoring scalars

GrassmannAlgebra provides the function `FactorScalars` to factor out any scalars in a product so that they are collected at the beginning of the product. For example:

```
FactorScalars[5 (2 x) ∧ (3 y) ∧ (b z)]
```

```
30 b x ∧ y ∧ z
```

If any of the factors in the product are scalars, they will also be collected at the beginning. Here **a** is a scalar since it has been declared as one, either by default or by `DeclareScalars`.

```
FactorScalars[5 (2 x) ∧ (3 a) ∧ (b z)]
```

```
30 a b x ∧ z
```

`FactorScalars` works with any Grassmann expression, or list of Grassmann expressions:

```
FactorScalars[5 (2 x) ∧ (3 y) ∧ (b z) + 5 (2 x) ∧ (3 a) ∧ (b z)]
```

```
30 a b x ∧ z + 30 b x ∧ y ∧ z
```

```
FactorScalars[{5 (2 x) ∧ (3 y) ∧ (b z), 5 (2 x) ∧ (3 a) ∧ (b z)}]
```

```
{30 b x ∧ y ∧ z, 30 a b x ∧ z}
```

Grassmann expressions

A *Grassmann expression* is any valid element of a Grassmann algebra. To check whether an expression is a Grassmann expression, we can use the *GrassmannAlgebra* function `GrassmannExpressionQ`:

```
GrassmannExpressionQ[x ∧ y]
```

```
True
```

`GrassmannExpressionQ` also works on lists of expressions. Below we determine that whereas the product of a scalar **a** and a 1-element **x** is a valid element of the Grassmann algebra, the ordinary multiplication of two 1-elements **x y** is not.

```
GrassmannExpressionQ[{x ∧ y, a x, x y}]
```

```
{True, True, False}
```

Calculating the grade of a Grassmann expression

The grade of an *m*-element is *m*.

At this stage we have not yet begun to use expressions for which the grade is other than obvious by inspection. However, as will be seen later, the grade will not always be obvious, especially when general Grassmann expressions, for example those involving Clifford or hypercomplex numbers, are being considered. Clifford numbers have components of differing grade.

GrassmannAlgebra has the function `Grade` for calculating the grade of a Grassmann expression. `Grade` works with single Grassmann expressions or with lists (to any level) of expressions.

```
Grade[x ∧ y ∧ z]
```

```
3
```

```
Grade[{1, x, x ∧ y, x ∧ y ∧ z}]
```

```
{0, 1, 2, 3}
```

It will also calculate the grades of the elements in a more general expression.

```
Grade[1 + x + x ∧ y + x ∧ y ∧ z]
```

```
{0, 1, 2, 3}
```

Note that if we try to calculate the grade of an element which simplifies to **0**, we will get the flag **Grade0** returned.

```
Grade[x ∧ y ∧ z ∧ w]
```

```
Grade0
```

The expression **x∧y∧z∧w** is zero because the current default basis is three-dimensional. In the next section we will discuss bases and how to change them.

2.5 Bases

Bases for exterior linear spaces

Suppose $\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}$ is a basis for $\underset{1}{\Lambda}$. Then, as is well known, any element of $\underset{1}{\Lambda}$ may be expressed as a linear combination of these basis elements.

A basis for $\underset{2}{\Lambda}$ may be constructed from the $\mathbf{e_i}$ by taking all the essentially different non-zero products $\mathbf{e_{i_1}} \wedge \mathbf{e_{i_2}}$ $(i_1, i_2 : 1, \ldots, n)$. Two products are *essentially different* if they do not involve the same 1-elements. There are obviously $\binom{n}{2}$ such products, making $\binom{n}{2}$ also the dimension of $\underset{2}{\Lambda}$.

In general, a basis for $\underset{m}{\Lambda}$ may be constructed from the $\mathbf{e_i}$ by taking all the essentially different products $\mathbf{e_{i_1}} \wedge \mathbf{e_{i_2}} \wedge \cdots \wedge \mathbf{e_{i_m}}$ $(i_1, i_2, \ldots, i_m : 1, \ldots, n)$. $\underset{m}{\Lambda}$ is thus $\binom{n}{m}$-dimensional.

Essentially different products are of course linearly independent.

Setting up a basis in *GrassmannAlgebra*

GrassmannAlgebra allows the setting up of an environment in which given symbols are declared basis elements of $\underset{1}{\Lambda}$. You can always see what your current basis is by entering `Basis`.

When first loaded, *GrassmannAlgebra* sets up a default basis:

```
Basis
```

$\{e_1, e_2, e_3\}$

This is the basis of a 3-dimensional linear space which may be interpreted as a 3-dimensional vector space if you wish.

To declare your own basis, enter `DeclareBasis[` *`list`* `]`. For example:

```
DeclareBasis[{i, j, k}]
```

{i, j, k}

Now, if we enter `Basis` we confirm that the current basis is the one we declared:

```
Basis
```

{i, j, k}

You can always return to the default basis by entering `DeclareBasis[DefaultBasis]`, or simply `DeclareDefaultBasis[]`.

```
DeclareDefaultBasis[]
```

$\{e_1, e_2, e_3\}$

If you want a basis of higher dimension, include the dimension as an argument to `DeclareBasis`. All default basis elements are of the form $\mathbf{e_i}$.

```
DeclareBasis[8]
```

$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

An even shorter way of declaring such a basis is by entering the double-struck capital V (ESC**dsV**ESC) subscripted with the dimension of the space required. This compact form is useful in explorations involving frequent changes of basis.

$\mathbb{V}_5$

$\{e_1, e_2, e_3, e_4, e_5\}$

Generating bases of exterior linear spaces

The function `Basis⋀[m]` generates a list of basis elements of $\underset{m}{\Lambda}$ arranged in standard order from the declared basis of $\underset{1}{\Lambda}$. For example, we declare the basis to be that of a 3-dimensional vector space, and then compute the bases of $\underset{0}{\Lambda}$, $\underset{1}{\Lambda}$, $\underset{2}{\Lambda}$, and $\underset{3}{\Lambda}$.

$\mathbb{V}_3$

$\{e_1, e_2, e_3\}$

Basis⋀[0]

1

Basis⋀[1]

$\{e_1, e_2, e_3\}$

Basis⋀[2]

$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$

Basis⋀[3]

$e_1 \wedge e_2 \wedge e_3$

`Basis⋀[]` generates a list of the basis elements of all the $\underset{m}{\Lambda}$. This is a basis of the three-dimensional Grassmann algebra. As a linear space, the three-dimensional Grassmann algebra has $2^3 = 8$ dimensions corresponding to its 8 basis elements.

Basis⋀[]

$\{1, e_1, e_2, e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3\}$

Tables and palettes of basis elements

Creating a table of basis elements

If you would like to create a table of the basis elements of all the exterior linear spaces induced by the declared basis, you can use the *GrassmannAlgebra* command `BasisTable[]`. For example if you are working in a three-dimensional vector space and declare the basis to be **{i,j,k}**:

DeclareBasis[{i, j, k}]

{i, j, k}

Entering `BasisTable[]` would then give you the 8 basis elements of the 3-dimensional Grassmann algebra:

```
BasisTable[]
```

Λ	BASIS
$\underset{0}{\Lambda}$	1
$\underset{1}{\Lambda}$	{i, j, k}
$\underset{2}{\Lambda}$	{i ∧ j, i ∧ k, j ∧ k}
$\underset{3}{\Lambda}$	i ∧ j ∧ k

You can edit this table or copy and paste from it.

Creating a palette of basis elements

If you want to create a palette instead, you can use the `BasisPalette[]` command.

```
BasisPalette[]
```

Λ	BASIS
$\underset{0}{\Lambda}$	1
$\underset{1}{\Lambda}$	{i, j, k}
$\underset{2}{\Lambda}$	{i ∧ j, i ∧ k, j ∧ k}
$\underset{3}{\Lambda}$	i ∧ j ∧ k

You can now click on any of the buttons to get its contents pasted into your notebook. For example, clicking on both the elements of the third row gives:

$\underset{2}{\Lambda}$ **{i ∧ j, i ∧ k, j ∧ k}**

Creating a table or palette of basis elements in *n* dimensions

If you would like to create a table or palette of the basis elements of all the exterior linear spaces induced by the default basis of dimension *n*, you can use the command `BasisTable[n]` or `BasisPalette[n]` where *n* is a positive integer. The generation of this table or palette does not depend on the currently declared basis.

```
BasisTable[0]
```

Λ	BASIS
$\underset{0}{\Lambda}$	1

BasisTable[1]

$\bigwedge$	BASIS
$\underset{0}{\bigwedge}$	1
$\underset{1}{\bigwedge}$	$\{e_1\}$

BasisTable[4]

$\bigwedge$	BASIS
$\underset{0}{\bigwedge}$	1
$\underset{1}{\bigwedge}$	$\{e_1, e_2, e_3, e_4\}$
$\underset{2}{\bigwedge}$	$\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\underset{3}{\bigwedge}$	$\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\underset{4}{\bigwedge}$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$

Standard ordering

The *standard ordering* of the basis of $\underset{1}{\bigwedge}$ is defined as the ordering of the elements in the declared list of basis elements. This ordering induces a natural standard ordering on the basis elements of $\underset{m}{\bigwedge}$: if the basis elements of $\underset{1}{\bigwedge}$ were letters of the alphabet arranged alphabetically, then the basis elements of $\underset{m}{\bigwedge}$ would be words arranged alphabetically.

For example, if we take **{A,B,C,D}** as basis, we can see that the basis elements for each of the bases obtained from entering `BasisTable[]` are arranged alphabetically. (Equivalently, if the basis elements were digits, the ordering would be numeric.)

DeclareBasis[{A, B, C, D}]

{A, B, C, D}

BasisTable[]

$\bigwedge$	BASIS
$\underset{0}{\bigwedge}$	1
$\underset{1}{\bigwedge}$	$\{A, B, C, D\}$
$\underset{2}{\bigwedge}$	$\{A \wedge B, A \wedge C, A \wedge D, B \wedge C, B \wedge D, C \wedge D\}$
$\underset{3}{\bigwedge}$	$\{A \wedge B \wedge C, A \wedge B \wedge D, A \wedge C \wedge D, B \wedge C \wedge D\}$
$\underset{4}{\bigwedge}$	$A \wedge B \wedge C \wedge D$

Indexing basis elements of exterior linear spaces

Just as we can denote the basis elements of $\underset{1}{\Lambda}$ by indexed symbols $\mathbf{e_i}$, (i : 1, ⋯, n), so too can we denote the basis elements of $\underset{m}{\Lambda}$ by indexed symbols $\underset{m}{\mathbf{e_i}}$, ($i$: 1, ⋯, $\binom{n}{m}$). This will enable us to denote the basis elements of an exterior linear space without the 'orgy of indices' which would accrue if we denoted the elements explicitly.

For example, suppose we have a basis $\{e_1, e_2, e_3, e_4\}$, then the standard bases are given by:

```
BasisTable[4]
```

Λ	BASIS
$\underset{0}{\Lambda}$	1
$\underset{1}{\Lambda}$	$\{e_1, e_2, e_3, e_4\}$
$\underset{2}{\Lambda}$	$\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$
$\underset{3}{\Lambda}$	$\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$
$\underset{4}{\Lambda}$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$

We can then denote these basis elements more compactly as:

$$\left\{\underset{2}{e_1}, \underset{2}{e_2}, \underset{2}{e_3}, \underset{2}{e_4}, \underset{2}{e_5}, \underset{2}{e_6}\right\} == \{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$$

$$\left\{\underset{3}{e_1}, \underset{3}{e_2}, \underset{3}{e_3}, \underset{3}{e_4}\right\} == \{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\}$$

$$\underset{4}{e} == e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

2.6 Cobases

Definition of a cobasis

The notion of *cobasis* of a basis will be conceptually and notationally useful in our development of later concepts. The cobasis of a basis of $\underset{m}{\Lambda}$ is simply the basis of $\underset{n-m}{\Lambda}$ which has been ordered in a special way relative to the basis of $\underset{m}{\Lambda}$.

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis for $\underset{1}{\Lambda}$. The *cobasis element* associated with a basis element e_i is denoted $\underline{e_i}$ and is defined as the product of the remaining basis elements such that the exterior product of a basis element with its cobasis element is equal to the basis element of $\underset{n}{\Lambda}$.

$$e_i \wedge \underline{e_i} == e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

That is:

$$\underline{e_i} = (-1)^{i-1}\, e_1 \wedge \cdots \wedge \Box_i \wedge \cdots \wedge e_n \qquad 2.21$$

where $\Box_i$ means that e_i is missing from the product.

The choice of the underbar notation to denote the cobasis may be viewed as a mnemonic to indicate that the element $\underline{e_i}$ is the basis element of $\underset{n}{\Lambda}$ with e_i 'struck out' from it.

Cobasis elements have similar properties to Euclidean complements, which are denoted with *overbars*. However, it should be noted that the underbar denotes an element, and is not an operation. For example: $\underline{a\, e_i}$ is *not* defined.

More generally, the cobasis element of the basis *m*-element $\underset{m}{e_i} = e_{i_1} \wedge \cdots \wedge e_{i_m}$ of $\underset{m}{\Lambda}$ is denoted $\underline{\underset{m}{e_i}}$ and is defined as the product of the remaining basis elements such that:

$$\underset{m}{e_i} \wedge \underline{\underset{m}{e_i}} == e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 2.22$$

That is:

$$\underline{\underset{m}{e_i}} == (-1)^{K_m}\, e_1 \wedge \cdots \wedge \Box_{i_1} \wedge \cdots \wedge \Box_{i_m} \wedge \cdots \wedge e_n \qquad 2.23$$

where $K_m = \sum_{\gamma=1}^{m} i_\gamma + \frac{1}{2}\, m\, (m+1)$.

From the above definition it can be seen that the exterior product of a basis element with the cobasis element of another basis element is zero. Hence we can write:

$$\underset{m}{e_i} \wedge \underline{\underset{m}{e_j}} == \delta_{ij}\, e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 2.24$$

The cobasis of unity

The natural basis of $\underset{0}{\Lambda}$ is 1. The cobasis $\underline{1}$ of this basis element is defined by formula 2.24 as the product of the remaining basis elements such that:

$$1 \wedge \underline{1} == e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

Thus:

$$\underline{1} == e_1 \wedge e_2 \wedge \cdots \wedge e_n == \underset{n}{e} \qquad 2.25$$

Any of these three representations will be called the *basis n-element* of the space. $\underline{1}$ may also be described as the *cobasis of unity*. Note carefully that $\underline{1}$ is different in different bases.

Tables and palettes of cobasis elements

We can create tables and palettes of basis elements with their corresponding cobasis elements by the commands `CobasisTable` and `CobasisPalette`.

Entering `CobasisTable[ ]` or `CobasisPalette[ ]` will generate a table or palette of the basis elements of the algebra generated by the currently declared basis. For the default basis $\{e_1, e_2, e_3\}$ we then have:

`CobasisTable[]`

Λ	BASIS	COBASIS
$\underset{0}{\Lambda}$	1	$i \wedge j \wedge k$
$\underset{1}{\Lambda}$	i	$j \wedge k$
$\underset{1}{\Lambda}$	j	$-(i \wedge k)$
$\underset{1}{\Lambda}$	k	$i \wedge j$
$\underset{2}{\Lambda}$	$i \wedge j$	k
$\underset{2}{\Lambda}$	$i \wedge k$	$-j$
$\underset{2}{\Lambda}$	$j \wedge k$	i
$\underset{3}{\Lambda}$	$i \wedge j \wedge k$	1

If you want to generate a default basis table or palette for a space of another dimension, include the dimension as the argument:

`CobasisTable[2]`

Λ	BASIS	COBASIS
$\underset{0}{\Lambda}$	1	$e_1 \wedge e_2$
$\underset{1}{\Lambda}$	e_1	e_2
$\underset{1}{\Lambda}$	e_2	$-e_1$
$\underset{2}{\Lambda}$	$e_1 \wedge e_2$	1

If you want to create a table or palette for your own basis, first declare the basis.

`DeclareBasis[{`$\mathcal{J}$`, `$\mathcal{Q}$`, `$\mathcal{B}$`}]`

$\{\mathcal{J}, \mathcal{Q}, \mathcal{B}\}$

```
CobasisPalette[]
```

Λ	BASIS	COBASIS
$\underset{0}{\Lambda}$	1	$\mathcal{J}\wedge\mathcal{Q}\wedge\mathcal{B}$
$\underset{1}{\Lambda}$	$\mathcal{J}$	$\mathcal{Q}\wedge\mathcal{B}$
$\underset{1}{\Lambda}$	$\mathcal{Q}$	$-(\mathcal{J}\wedge\mathcal{B})$
$\underset{1}{\Lambda}$	$\mathcal{B}$	$\mathcal{J}\wedge\mathcal{Q}$
$\underset{2}{\Lambda}$	$\mathcal{J}\wedge\mathcal{Q}$	$\mathcal{B}$
$\underset{2}{\Lambda}$	$\mathcal{J}\wedge\mathcal{B}$	$-\mathcal{Q}$
$\underset{2}{\Lambda}$	$\mathcal{Q}\wedge\mathcal{B}$	$\mathcal{J}$
$\underset{3}{\Lambda}$	$\mathcal{J}\wedge\mathcal{Q}\wedge\mathcal{B}$	1

We can use the function `Cobasis` to compute the cobasis elements of any basis element or list of basis elements from any of the $\underset{m}{\Lambda}$ of the currently declared basis. For example, suppose you are working in a space of 5 dimensions.

$\mathbb{V}_5$**; Cobasis**$[\{e_1, e_1\wedge e_3\wedge e_4, e_2\wedge e_3\}]$

$\{e_2\wedge e_3\wedge e_4\wedge e_5, e_2\wedge e_5, e_1\wedge e_4\wedge e_5\}$

The cobasis of a cobasis

Let $\underset{m}{e_i}$ be a basis element and $\underline{\underset{m}{e_i}}$ be its cobasis element. Then the cobasis element of this cobasis element is denoted $\underline{\underline{\underset{m}{e_j}}}$ and is defined as expected by the product of the remaining basis elements such that:

$$\underline{\underset{m}{e_i}}\wedge\underline{\underline{\underset{m}{e_j}}} == \delta_{ij}\, e_1\wedge e_2\wedge\cdots\wedge e_n \qquad 2.26$$

The left-hand side may be rearranged to give:

$$\underline{\underset{m}{e_i}}\wedge\underline{\underline{\underset{m}{e_j}}} == (-1)^{m\,(n-m)}\,\underline{\underline{\underset{m}{e_j}}}\wedge\underline{\underset{m}{e_i}}$$

which, by comparison with the definition for the cobasis shows that:

$$\underline{\underline{\underset{m}{e_j}}} == (-1)^{m\,(n-m)}\,\underset{m}{e_i} \qquad 2.27$$

That is, the cobasis element of a cobasis element of an element is (apart from a possible sign) equal to the element itself. We shall find this formula useful as we develop general formulae for the complement and interior products in later chapters.

2.7 Determinants

Determinants from exterior products

All the properties of determinants follow naturally from the properties of the exterior product.

The determinant:

$$D == \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,n} \\ a_{21} & a_{22} & \cdots & a_{2\,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

may be calculated by considering each of its rows (or columns) as a 1-element. Here, we consider rows. Development by columns may be obtained *mutatis mutandis*. Introduce an arbitrary basis e_i and let

$$\alpha_i == a_{i1}\, e_1 + a_{i2}\, e_2 + \cdots + a_{in}\, e_n$$

Then form the exterior product of all the α_i. We can arrange the α_i in rows to portray the effect of an array:

$$\begin{aligned} \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n == & (a_{11}\, e_1 + a_{12}\, e_2 + \cdots + a_{1\,n}\, e_n) \\ & \wedge\, (a_{21}\, e_1 + a_{22}\, e_2 + \cdots + a_{2\,n}\, e_n) \\ & \quad \vdots \qquad\quad \vdots \qquad\quad \cdots \quad \vdots \\ & \wedge\, (a_{n1}\, e_1 + a_{n2}\, e_2 + \cdots + a_{nn}\, e_n) \end{aligned}$$

The determinant D is then the coefficient of the resulting *n*-element:

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n == D\, e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

Because $\underset{n}{\Lambda}$ is of dimension 1, we can express this uniquely as:

$$D = \frac{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n}{e_1 \wedge e_2 \wedge \cdots \wedge e_n} \qquad 2.28$$

Properties of determinants

All the well-known properties of determinants proceed directly from the properties of the exterior product.

◆ **1 The determinant changes sign on the interchange of any two rows.**

The exterior product is anti-symmetric in any two factors.

$$\cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_j \wedge \cdots == - \cdots \wedge \alpha_j \wedge \cdots \wedge \alpha_i \wedge \cdots$$

◆ **2 The determinant is zero if two of its rows are equal.**

The exterior product is nilpotent.

$$\cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_i \wedge \cdots == 0$$

◆ **3 The determinant is multiplied by a factor *k* if any row is multiplied by *k*.**

The exterior product is commutative with respect to scalars.

$$\alpha_1 \wedge \cdots \wedge (k\, \alpha_i) \wedge \cdots == k\, (\alpha_1 \wedge \cdots \wedge \alpha_i \wedge \cdots)$$

◆ **4 The determinant is equal to the sum of *p* determinants if each element of a given row is the sum of *p* terms.**

The exterior product is distributive with respect to addition.

$$\alpha_1 \wedge \cdots \wedge \left(\sum_i \alpha_i\right) \wedge \cdots == \sum_i (\alpha_1 \wedge \cdots \wedge \alpha_i \wedge \cdots)$$

◆ **5 The determinant is unchanged if to any row is added scalar multiples of other rows.**

The exterior product is unchanged if to any factor is added multiples of other factors.

$$\alpha_1 \wedge \cdots \wedge \left(\alpha_i + \sum_{j \neq i} k^j\, \alpha_j\right) \wedge \cdots == \alpha_1 \wedge \cdots \wedge \alpha_i \wedge \cdots$$

The Laplace expansion technique

The Laplace expansion technique is equivalent to the calculation of the exterior product in four stages:
1. Take the exterior product of any *p* of the α_i .
2. Take the exterior product of the remaining *n-p* of the α_i .
3. Take the exterior product of the results of the first two operations.
4. Adjust the sign to ensure the parity of the original ordering of the α_i is preserved.

Each of the first two operations produces an element with $\binom{n}{p} = \binom{n}{n-p}$ terms.

A generalization of the Laplace expansion technique is evident from the fact that the exterior product of the α_i may be effected in *any* grouping and sequence which facilitate the computation.

Calculating determinants

As an example take the following matrix. We can calculate its determinant with *Mathematica*'s inbuilt `Det` function:

$$\mathbf{Det}\left[\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & a \\ 0 & b & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & g & 0 \\ 0 & f & 0 & d & 0 & c & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & d & 0 & b & 0 \end{pmatrix}\right]$$

$$a\,b^2\,c^2\,e^2\,h - a\,b\,c\,e^2\,f^2\,h$$

Alternatively, we can set $\underset{1}{\Lambda}$ to be 8-dimensional.

```
DeclareBasis[8]
```

$$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$$

and then evaluate the exterior product of the 8 rows or columns expressed as 1-elements. Here we use `GrassmannSimplify` (alias $\mathcal{G}$) to expand by columns.

$$\mathcal{G}[((a\,e_1 + g\,e_7) \wedge (b\,e_3 + f\,e_6) \wedge (e\,e_5) \wedge (c\,e_2 + d\,e_6) \wedge (e\,e_4 + d\,e_8) \wedge (f\,e_3 + c\,e_6) \wedge (g\,e_5 + b\,e_8) \wedge (a\,e_2 + h\,e_7))]$$

$$a\,b\,c\,e^2\,(b\,c - f^2)\,h\,e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7 \wedge e_8$$

The determinant is then the coefficient of the basis 8-element.

Speeding up calculations

Because of the large number of rules `GrassmannSimplify` has to check, it is often faster to use the generalized Laplace expansion technique. That is, the order in which the products are calculated is arranged with the objective of minimizing the number of resulting terms produced at each stage. We do this by dividing the product up into groupings of factors where each grouping contains as few basis elements as possible in common with the others. We then calculate the result from each grouping before combining them into the final result. Of course, we must ensure that the sign of the overall regrouping is the same as the original expression.

In the example below, we have rearranged the determinant calculation above into three groupings, which we calculate successively before combining the results. This grouping took an order of magnitude less time than the direct approach above.

$$-\mathcal{G}[\mathcal{G}[(a\,e_1 + g\,e_7) \wedge (c\,e_2 + d\,e_6) \wedge (a\,e_2 + h\,e_7)] \wedge \mathcal{G}[(b\,e_3 + f\,e_6) \wedge (f\,e_3 + c\,e_6)] \wedge \mathcal{G}[(e\,e_5) \wedge (e\,e_4 + d\,e_8) \wedge (g\,e_5 + b\,e_8)]]$$

$$a\,b\,c\,e^2\,(b\,c - f^2)\,h\,e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7 \wedge e_8$$

Historical Note
Grassmann applied his *Ausdehnungslehre* to the theory of determinants and linear equations quite early in his work. Later, Cauchy published his technique of 'algebraic keys' which essentially duplicated Grassmann's results. To claim priority, Grassmann was led to publish his only paper in French, obviously directed at Cauchy: '*Sur les différents genres de multiplication*' ('On different types of multiplication') [26]. For a complete treatise on the theory of determinants from a Grassmannian viewpoint see R. F. Scott [51].

2.8 Cofactors

Cofactors from exterior products

The cofactor is an important concept in the usual approach to determinants. One often calculates a determinant by summing the products of the elements of a row by their corresponding cofactors. Cofactors divided by the determinant form the elements of an inverse matrix. In the subsequent development of the Grassmann algebra, particularly the development of the complement, we will find it useful to see how cofactors arise from exterior products.

Consider the product of *n* 1-elements introduced in the previous section:

$$\begin{aligned}\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n = &(a_{11}\,e_1 + a_{12}\,e_2 + \cdots + a_{1\,n}\,e_n)\\ \wedge\, &(a_{21}\,e_1 + a_{22}\,e_2 + \cdots + a_{2\,n}\,e_n)\\ &\quad\vdots \qquad\quad \vdots \qquad \cdots \quad \vdots\\ \wedge\, &(a_{n1}\,e_1 + a_{n2}\,e_2 + \cdots + a_{nn}\,e_n)\end{aligned}$$

Omitting the first factor α_1 :

$$\begin{aligned}\alpha_2 \wedge \cdots \wedge \alpha_n = &(a_{21}\,e_1 + a_{22}\,e_2 + \cdots + a_{2\,n}\,e_n)\\ &\quad\vdots \qquad\quad \vdots \qquad \cdots \quad \vdots\\ \wedge\, &(a_{n1}\,e_1 + a_{n2}\,e_2 + \cdots + a_{nn}\,e_n)\end{aligned}$$

and multiplying out the remaining *n*–1 factors results in an expression of the form:

$$\alpha_2 \wedge \cdots \wedge \alpha_n = \underline{a_{11}}\,e_2 \wedge e_3 \wedge \cdots \wedge e_n + \underline{a_{12}}\,(-e_1 \wedge e_3 \wedge \cdots \wedge e_n) + \cdots + \underline{a_{1\,n}}\,(-1)^{n-1}\,e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1}$$

Here, the signs of the basis (*n*–1)-elements have been specifically chosen so that they correspond to the cobasis elements of e_1 to e_n. The underscored scalar coefficients have yet to be interpreted. Thus we can write:

$$\alpha_2 \wedge \cdots \wedge \alpha_n == \underline{a_{11}}\ \underline{e_1} + \underline{a_{12}}\ \underline{e_2} + \cdots + \underline{a_{1\,n}}\ \underline{e_n}$$

If we now premultiply by the first factor α_1 we get a particularly symmetric form.

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n == (a_{11}\ e_1 + a_{12}\ e_2 + \cdots + a_{1\,n}\ e_n) \wedge (\underline{a_{11}}\ \underline{e_1} + \underline{a_{12}}\ \underline{e_2} + \cdots + \underline{a_{1\,n}}\ \underline{e_n})$$

Multiplying this out and remembering that the exterior product of a basis element with the cobasis element of another basis element is zero, we get the sum:

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n == (a_{11}\ \underline{a_{11}} + a_{12}\ \underline{a_{12}} + \cdots + a_{1\,n}\ \underline{a_{1\,n}})\ e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

But we have already seen that:

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n == D\ e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

Hence the determinant D can be expressed as the sum of products:

$$D == a_{11}\ \underline{a_{11}} + a_{12}\ \underline{a_{12}} + \cdots + a_{1\,n}\ \underline{a_{1\,n}}$$

showing that the $\underline{a_{ij}}$ are the *cofactors* of the a_{ij} .

Mnemonically we can visualize the $\underline{a_{ij}}$ as the determinant with the row and column containing a_{ij} as 'struck out' by the underscore.

Of course, there is nothing special in the choice of the element α_i about which to expand the determinant. We could have written the expansion in terms of any factor (row or column):

$$D == a_{i1}\ \underline{a_{i1}} + a_{i2}\ \underline{a_{i2}} + \cdots + a_{in}\ \underline{a_{in}}$$

It can be seen immediately that the product of the elements of a row with the cofactors of any *other* row is zero, since in the exterior product formulation the row must have been included in the calculation of the cofactors.

Summarizing these results for both row and column expansions we have:

$$\boxed{\sum_{j=1}^{n} a_{ij}\ \underline{a_{kj}} == \sum_{j=1}^{n} a_{ji}\ \underline{a_{jk}} == D\ \delta_{ik}} \qquad 2.29$$

In matrix terms of course this is equivalent to the standard result $A\ A_c{}^T == D\ I$ or equivalently $A^{-1} == \frac{A_c{}^T}{D}$, where A_c is the matrix of cofactors of the elements of A.

The Laplace expansion in cofactor form

We have already introduced the Laplace expansion technique in our discussion of determinants:
1. Take the exterior product of any *m* of the α_i
2. Take the exterior product of the remaining *n–m* of the α_i
3. Take the exterior product of the results of the first two operations (in the correct order).

In this section we revisit it more specifically in the context of the cofactor. The results will be important in deriving later results.

Let $\underset{m}{e_i}$ be basis *m*-elements and $\underline{\underset{m}{e_i}}$ be their corresponding cobasis (*n*–*m*)-elements (*i*: 1,…, *ν*). To fix ideas, suppose we expand a determinant about the first *m* rows:

$$(\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge (\alpha_{m+1} \wedge \cdots \wedge \alpha_n) == \left(a_1 \underset{m}{e_1} + a_2 \underset{m}{e_2} + \cdots + a_\nu \underset{m}{e_\nu}\right)$$
$$\wedge \left(\underline{a_1}\, \underline{\underset{m}{e_1}} + \underline{a_2}\, \underline{\underset{m}{e_2}} + \cdots + \underline{a_\nu}\, \underline{\underset{m}{e_\nu}}\right)$$

Here, the a_i and $\underline{a_i}$ are simply the coefficients resulting from the partial expansions. As with basis 1-elements, the exterior product of a basis *m*-element with the cobasis element of another basis *m*-element is zero. That is:

$$\underset{m}{e_i} \wedge \underline{\underset{m}{e_j}} == \delta_{ij}\, e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

Expanding the product and applying this simplification yields finally:

$$D == a_1 \underline{a_1} + a_2 \underline{a_2} + \cdots + a_\nu \underline{a_\nu}$$

This is the Laplace expansion. The a_i are minors and the $\underline{a_i}$ are their cofactors.

Calculation of determinants using minors and cofactors

Consider the fourth order determinant:

$$D_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix};$$

We wish to calculate the determinant using the cofactor methods above. We will use *GrassmannAlgebra* to assist with the computations, so we must declare the basis to be 4-dimensional and the a_{ij} to be scalars.

```
𝕍4; DeclareExtraScalars[a_];
```

Case 1: Calculation by expansion about the first row

$$\alpha_1 = a_{11}\, e_1 + a_{12}\, e_2 + a_{13}\, e_3 + a_{14}\, e_4;$$

$$\alpha_2 \wedge \alpha_3 \wedge \alpha_4 = (a_{21}\, e_1 + a_{22}\, e_2 + a_{23}\, e_3 + a_{24}\, e_4)$$
$$\wedge (a_{31}\, e_1 + a_{32}\, e_2 + a_{33}\, e_3 + a_{34}\, e_4)$$
$$\wedge (a_{41}\, e_1 + a_{42}\, e_2 + a_{43}\, e_3 + a_{44}\, e_4);$$

Expanding this product:

$$\mathcal{G}[\alpha_2 \wedge \alpha_3 \wedge \alpha_4]$$

$$(a_{23}\ (-a_{32}\ a_{41} + a_{31}\ a_{42}) + a_{22}\ (a_{33}\ a_{41} - a_{31}\ a_{43}) + a_{21}\ (-a_{33}\ a_{42} + a_{32}\ a_{43}))\ e_1 \wedge e_2 \wedge e_3 + (a_{24}\ (-a_{32}\ a_{41} + a_{31}\ a_{42}) + a_{22}\ (a_{34}\ a_{41} - a_{31}\ a_{44}) + a_{21}\ (-a_{34}\ a_{42} + a_{32}\ a_{44}))\ e_1 \wedge e_2 \wedge e_4 + (a_{24}\ (-a_{33}\ a_{41} + a_{31}\ a_{43}) + a_{23}\ (a_{34}\ a_{41} - a_{31}\ a_{44}) + a_{21}\ (-a_{34}\ a_{43} + a_{33}\ a_{44}))\ e_1 \wedge e_3 \wedge e_4 + (a_{24}\ (-a_{33}\ a_{42} + a_{32}\ a_{43}) + a_{23}\ (a_{34}\ a_{42} - a_{32}\ a_{44}) + a_{22}\ (-a_{34}\ a_{43} + a_{33}\ a_{44}))\ e_2 \wedge e_3 \wedge e_4$$

By inspection of the basis elements in this result we see that both $\mathbf{e_1 \wedge e_2 \wedge e_3}$ and $\mathbf{e_1 \wedge e_3 \wedge e_4}$ require a change of sign to bring them into the required cobasis form.

$$\mathbf{e_1 \wedge e_2 \wedge e_3 == -\underline{e_4}}$$

$$\mathbf{e_1 \wedge e_2 \wedge e_4 == \underline{e_3}}$$

$$\mathbf{e_1 \wedge e_3 \wedge e_4 == -\underline{e_2}}$$

$$\mathbf{e_2 \wedge e_3 \wedge e_4 == \underline{e_1}}$$

Reflecting this change of sign in the coefficients, the corresponding cofactors become:

$$\mathbf{\underline{a_{11}} = (a_{24}\ (-a_{33}\ a_{42} + a_{32}\ a_{43}) + a_{23}\ (a_{34}\ a_{42} - a_{32}\ a_{44}) + a_{22}\ (-a_{34}\ a_{43} + a_{33}\ a_{44}));}$$

$$\mathbf{\underline{a_{12}} = -(a_{24}\ (-a_{33}\ a_{41} + a_{31}\ a_{43}) + a_{23}\ (a_{34}\ a_{41} - a_{31}\ a_{44}) + a_{21}\ (-a_{34}\ a_{43} + a_{33}\ a_{44}));}$$

$$\mathbf{\underline{a_{13}} = (a_{24}\ (-a_{32}\ a_{41} + a_{31}\ a_{42}) + a_{22}\ (a_{34}\ a_{41} - a_{31}\ a_{44}) + a_{21}\ (-a_{34}\ a_{42} + a_{32}\ a_{44}));}$$

$$\mathbf{\underline{a_{14}} = -(a_{23}\ (-a_{32}\ a_{41} + a_{31}\ a_{42}) + a_{22}\ (a_{33}\ a_{41} - a_{31}\ a_{43}) + a_{21}\ (-a_{33}\ a_{42} + a_{32}\ a_{43}));}$$

We can then calculate the determinant from:

$$\mathbf{D_1 = a_{11}\ \underline{a_{11}} + a_{12}\ \underline{a_{12}} + a_{13}\ \underline{a_{13}} + a_{14}\ \underline{a_{14}}}$$

$$a_{14}\ (-a_{23}\ (-a_{32}\ a_{41} + a_{31}\ a_{42}) - a_{22}\ (a_{33}\ a_{41} - a_{31}\ a_{43}) - a_{21}\ (-a_{33}\ a_{42} + a_{32}\ a_{43})) + a_{13}\ (a_{24}\ (-a_{32}\ a_{41} + a_{31}\ a_{42}) + a_{22}\ (a_{34}\ a_{41} - a_{31}\ a_{44}) + a_{21}\ (-a_{34}\ a_{42} + a_{32}\ a_{44})) + a_{12}\ (-a_{24}\ (-a_{33}\ a_{41} + a_{31}\ a_{43}) - a_{23}\ (a_{34}\ a_{41} - a_{31}\ a_{44}) - a_{21}\ (-a_{34}\ a_{43} + a_{33}\ a_{44})) + a_{11}\ (a_{24}\ (-a_{33}\ a_{42} + a_{32}\ a_{43}) + a_{23}\ (a_{34}\ a_{42} - a_{32}\ a_{44}) + a_{22}\ (-a_{34}\ a_{43} + a_{33}\ a_{44}))$$

■ Case 2: Calculation by expansion about the first two rows

Expand the product of the first and second rows.

$$\alpha_1 \wedge \alpha_2 = (a_{11}\, e_1 + a_{12}\, e_2 + a_{13}\, e_3 + a_{14}\, e_4) \wedge (a_{21}\, e_1 + a_{22}\, e_2 + a_{23}\, e_3 + a_{24}\, e_4);$$

$\mathcal{G}[\alpha_1 \wedge \alpha_2]$

$$(-a_{12}\, a_{21} + a_{11}\, a_{22})\, e_1 \wedge e_2 + (-a_{13}\, a_{21} + a_{11}\, a_{23})\, e_1 \wedge e_3 + (-a_{14}\, a_{21} + a_{11}\, a_{24})\, e_1 \wedge e_4 + (-a_{13}\, a_{22} + a_{12}\, a_{23})\, e_2 \wedge e_3 + (-a_{14}\, a_{22} + a_{12}\, a_{24})\, e_2 \wedge e_4 + (-a_{14}\, a_{23} + a_{13}\, a_{24})\, e_3 \wedge e_4$$

Denote the coefficients of these basis 2-elements by a_i.

$$a_1 = -a_{12}\, a_{21} + a_{11}\, a_{22};$$
$$a_2 = -a_{13}\, a_{21} + a_{11}\, a_{23};$$
$$a_3 = -a_{14}\, a_{21} + a_{11}\, a_{24};$$
$$a_4 = -a_{13}\, a_{22} + a_{12}\, a_{23};$$
$$a_5 = -a_{14}\, a_{22} + a_{12}\, a_{24};$$
$$a_6 = -a_{14}\, a_{23} + a_{13}\, a_{24};$$

Expand the product of the third and fourth rows.

$$\alpha_3 \wedge \alpha_4 = (a_{31}\, e_1 + a_{32}\, e_2 + a_{33}\, e_3 + a_{34}\, e_4) \wedge (a_{41}\, e_1 + a_{42}\, e_2 + a_{43}\, e_3 + a_{44}\, e_4);$$

$\mathcal{G}[\alpha_3 \wedge \alpha_4]$

$$(-a_{32}\, a_{41} + a_{31}\, a_{42})\, e_1 \wedge e_2 + (-a_{33}\, a_{41} + a_{31}\, a_{43})\, e_1 \wedge e_3 + (-a_{34}\, a_{41} + a_{31}\, a_{44})\, e_1 \wedge e_4 + (-a_{33}\, a_{42} + a_{32}\, a_{43})\, e_2 \wedge e_3 + (-a_{34}\, a_{42} + a_{32}\, a_{44})\, e_2 \wedge e_4 + (-a_{34}\, a_{43} + a_{33}\, a_{44})\, e_3 \wedge e_4$$

Denote the coefficients of these basis 2-elements by $\underline{a_i}$, making sure to correspond the correct cofactors by corresponding the correct cobasis elements.

$$\underline{a_6} = (-a_{32}\, a_{41} + a_{31}\, a_{42});$$
$$\underline{a_5} = -(-a_{33}\, a_{41} + a_{31}\, a_{43});$$
$$\underline{a_4} = (-a_{34}\, a_{41} + a_{31}\, a_{44});$$
$$\underline{a_3} = (-a_{33}\, a_{42} + a_{32}\, a_{43});$$
$$\underline{a_2} = -(-a_{34}\, a_{42} + a_{32}\, a_{44});$$
$$\underline{a_1} = (-a_{34}\, a_{43} + a_{33}\, a_{44});$$

We can then calculate the determinant from:

$$D_2 = a_1\, \underline{a_1} + a_2\, \underline{a_2} + a_3\, \underline{a_3} + a_4\, \underline{a_4} + a_5\, \underline{a_5} + a_6\, \underline{a_6}$$

$$(-a_{14}\, a_{23} + a_{13}\, a_{24})\,(-a_{32}\, a_{41} + a_{31}\, a_{42}) + (-a_{14}\, a_{22} + a_{12}\, a_{24})\,(a_{33}\, a_{41} - a_{31}\, a_{43}) + (-a_{14}\, a_{21} + a_{11}\, a_{24})\,(-a_{33}\, a_{42} + a_{32}\, a_{43}) + (-a_{13}\, a_{22} + a_{12}\, a_{23})\,(-a_{34}\, a_{41} + a_{31}\, a_{44}) + (-a_{13}\, a_{21} + a_{11}\, a_{23})\,(a_{34}\, a_{42} - a_{32}\, a_{44}) + (-a_{12}\, a_{21} + a_{11}\, a_{22})\,(-a_{34}\, a_{43} + a_{33}\, a_{44})$$

Verification of the calculations

Finally, we verify that these two results are the same and correspond to that given by the inbuilt *Mathematica* function Det.

```
Expand[D1] == Expand[D2] == Det @@ D0
```

```
True
```

Transformations of cobases

In this section we show that if a basis of the underlying linear space is transformed by a transformation whose components are a_{ij} , then its cobasis is transformed by the induced transformation whose components are the cofactors of the a_{ij} . For simplicity in what follows, we use Einstein's summation convention in which a summation over repeated indices is understood.

Let a_{ij} be a transformation on the basis e_j to give the new basis ε_i . That is, $\varepsilon_i == a_{ij}\, e_j$.

Let $\underbrace{a_{ij}}$ be the corresponding transformation on the cobasis $\underline{e_j}$ to give the new cobasis $\underline{\varepsilon_i}$. That is, $\underline{\varepsilon_i} == \underbrace{a_{ij}}\ \underline{e_j}$.

Now take the exterior product of these two equations.

$$\varepsilon_i \wedge \underline{\varepsilon_k} == (a_{ip}\, e_p) \wedge \left(\underbrace{a_{kj}}\ \underline{e_j}\right) == a_{ip}\ \underbrace{a_{kj}}\ e_p \wedge \underline{e_j}$$

But the product of a basis element and its cobasis element is equal to the n-element of that basis. That is, $\varepsilon_i \wedge \underline{\varepsilon_k} == \delta_{ik}\, \underset{n}{\varepsilon}$ and $e_p \wedge \underline{e_j} == \delta_{pj}\, \underset{n}{e}$. Substituting in the previous equation gives:

$$\delta_{ik}\, \underset{n}{\varepsilon} == a_{ip}\ \underbrace{a_{kj}}\ \delta_{pj}\, \underset{n}{e}$$

Using the properties of the Kronecker delta we can simplify the right side to give:

$$\delta_{ik}\, \underset{n}{\varepsilon} == a_{ij}\ \underbrace{a_{kj}}\ \underset{n}{e}$$

We can now substitute $\underset{n}{\varepsilon} == D\, \underset{n}{e}$ where $D == \mathrm{Det}[a_{ij}]$ is the determinant of the transformation.

$$\delta_{ik}\, D == a_{ij}\ \underbrace{a_{kj}}$$

This is precisely the relationship derived in the previous section for the expansion of a determinant in terms of cofactors. Hence we have shown that $\underbrace{a_{kj}} == \underline{a_{kj}}$.

$$\boxed{\varepsilon_i == a_{ij}\, e_j \iff \underline{\varepsilon_i} == \underline{a_{ij}}\ \underline{e_j}} \qquad 2.30$$

In sum: If a basis of the underlying linear space is transformed by a transformation whose components are a_{ij} , then its cobasis is transformed by the induced transformation whose components are the cofactors $\underline{a_{ij}}$ of the a_{ij} .

2.9 Solution of Linear Equations

Grassmann's approach to solving linear equations

Because of its encapsulation of the properties of linear independence, Grassmann was able to use the exterior product to present a theory and formulae for the solution of linear equations well before anyone else.

Suppose *m* independent equations in *n* (*m* ≤ *n*) unknowns x_i :

$$a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1\,n}\,x_n == a_1$$

$$a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2\,n}\,x_n == a_2$$

$$\vdots \qquad \vdots \qquad \cdots \quad \vdots \qquad \vdots$$

$$a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n == a_m$$

Multiply these equations by e_1 , e_2 , ⋯, e_m respectively and define

$$C_i == a_{1\,i}\,e_1 + a_{2\,i}\,e_2 + \cdots + a_{mi}\,e_m$$

$$C_0 == a_1\,e_1 + a_2\,e_2 + \cdots + a_m\,e_m$$

The C_i and C_0 are therefore 1-elements in a linear space of dimension *m*.

Adding the resulting equations then gives the system in the form:

$$x_1\,C_1 + x_2\,C_2 + \cdots + x_n\,C_n == C_0 \qquad 2.31$$

To obtain an equation from which the unknowns x_i have been eliminated, it is only necessary to multiply the linear system through by the corresponding C_i .

If *m* = *n* and a solution for x_i exists, it is obtained by eliminating x_1 , ⋯, x_{i-1} , x_{i+1} , ⋯, x_n ; that is, by multiplying the linear system through by $C_1 \wedge \cdots \wedge C_{i-1} \wedge C_{i+1} \wedge \cdots \wedge C_n$:

$$x_i\,C_i \wedge C_1 \wedge \cdots \wedge C_{i-1} \wedge C_{i+1} \wedge \cdots \wedge C_n == C_0 \wedge C_1 \wedge \cdots \wedge C_{i-1} \wedge C_{i+1} \wedge \cdots \wedge C_n$$

Solving for x_i gives:

$$x_i == \frac{C_0 \wedge (C_1 \wedge \cdots \wedge C_{i-1} \wedge C_{i+1} \wedge \cdots \wedge C_n)}{C_i \wedge (C_1 \wedge \cdots \wedge C_{i-1} \wedge C_{i+1} \wedge \cdots \wedge C_n)} \qquad 2.32$$

In this form we only have to calculate the (*n*–1)-element $C_1 \wedge \cdots \wedge C_{i-1} \wedge C_{i+1} \wedge \cdots \wedge C_n$ once.

An alternative form more reminiscent of Cramer's Rule is:

$$x_i = \frac{C_1 \wedge \cdots \wedge C_{i-1} \wedge C_0 \wedge C_{i+1} \wedge \cdots \wedge C_n}{C_1 \wedge C_2 \wedge \cdots \wedge C_n} \qquad 2.33$$

All the well-known properties of solutions to systems of linear equations proceed directly from the properties of the exterior products of the C_i .

Example solution

Consider the following system of 3 equations in 4 unknowns.

$$w - 2x + 3y + 4z == 2$$

$$2w + 7y - 5z == 9$$

$$w + x + y + z == 8$$

To solve this system, first declare the unknowns **w**, **x**, **y** and **z** as scalars, and then declare a basis of dimension at least equal to the number of equations. In this example, we declare a 4-space so that we can add another equation later.

```
DeclareExtraScalars[{w, x, y, z}]
```

$\{a, b, c, d, e, f, g, h, w, x, y, z, \mathbb{k}, (\_\ominus\_)\ ?\text{InnerProductQ}, \underset{0}{\_}\}$

```
DeclareBasis[4]
```

$\{e_1, e_2, e_3, e_4\}$

Next define:

$C_w = e_1 + 2e_2 + e_3;$
$C_x = -2e_1 + e_3;$
$C_y = 3e_1 + 7e_2 + e_3;$
$C_z = 4e_1 - 5e_2 + e_3;$
$C_0 = 2e_1 + 9e_2 + 8e_3;$

The system equation then becomes:

$$w\,C_w + x\,C_x + y\,C_y + z\,C_z == C_0$$

Suppose we wish to eliminate **w** and **z** thus giving a relationship between **x** and **y**. To accomplish this we multiply the system equation through by $C_w \wedge C_z$.

$$(w\,C_w + x\,C_x + y\,C_y + z\,C_z) \wedge C_w \wedge C_z == C_0 \wedge C_w \wedge C_z$$

Or, since the terms involving **w** and **z** will obviously be eliminated by their product with $C_w \wedge C_z$, we have more simply:

$$(x\,C_x + y\,C_y) \wedge C_w \wedge C_z == C_0 \wedge C_w \wedge C_z$$

Evaluating this with the `GrassmannSimplify` ($\mathcal{G}$) function gives the required relationship between **x** and **y** from the coefficient of the product of the introduced basis elements.

$$\mathcal{G}[\,(x\, C_x + y\, C_y) \wedge C_w \wedge C_z\,] == \mathcal{G}[C_0 \wedge C_w \wedge C_z]\ \text{// Simplify}$$

$$(63 - 27\,x + 29\,y)\, e_1 \wedge e_2 \wedge e_3 == 0$$

Had we had a fourth equation, say $x - 3y + z == 7$, and wanted to solve for **x**, we can incorporate the new information into the above formulation and re-evaluate:

$$\mathcal{G}[\,x\,(C_x + e_4) \wedge C_w \wedge (C_y - 3\,e_4) \wedge (C_z + e_4)\,] == \mathcal{G}[(C_0 + 7\,e_4) \wedge C_w \wedge (C_y - 3\,e_4) \wedge (C_z + e_4)]\ \text{// Simplify}$$

$$0 == (30 + 41\,x)\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

Or we can redefine the C_i.

$$C_w = e_1 + 2\,e_2 + e_3;$$
$$C_x = -2\,e_1 + e_3 + e_4;$$
$$C_y = 3\,e_1 + 7\,e_2 + e_3 - 3\,e_4;$$
$$C_z = 4\,e_1 - 5\,e_2 + e_3 + e_4;$$
$$C_0 = 2\,e_1 + 9\,e_2 + 8\,e_3 + 7\,e_4;$$

In this case, since we will be using the product $C_w \wedge C_y \wedge C_z$ twice, it makes sense to calculate it just once:

$$C_{wyz} = \mathcal{G}[C_w \wedge C_y \wedge C_z]$$

$$-29\, e_1 \wedge e_2 \wedge e_3 - 38\, e_1 \wedge e_2 \wedge e_4 - 11\, e_1 \wedge e_3 \wedge e_4 + 16\, e_2 \wedge e_3 \wedge e_4$$

The final result is then obtained from the formula:

$$x == \frac{\mathcal{G}[C_0 \wedge C_{wyz}]}{\mathcal{G}[C_x \wedge C_{wyz}]}\ \text{// Simplify}$$

$$x == -\frac{30}{41}$$

2.10 Simplicity

The concept of simplicity

An important concept in the Grassmann algebra is that of *simplicity*. Earlier in the chapter we introduced the concept informally. Now we will discuss it in a little more detail.

An element is *simple* if it is the exterior product of 1-elements. We extend this definition to scalars by defining all scalars to be simple. Clearly also, since any *n*-element always reduces to a product of 1-elements, all *n*-elements are simple. Thus we see immediately that all 0-elements, 1-elements, and *n*-elements are simple. In the next section we show that all (*n*–1)-elements are also simple.

In 2-space therefore, all elements of $\underset{0}{\Lambda}$, $\underset{1}{\Lambda}$, and $\underset{2}{\Lambda}$ are simple. In 3-space all elements of $\underset{0}{\Lambda}$, $\underset{1}{\Lambda}$, $\underset{2}{\Lambda}$ and $\underset{3}{\Lambda}$ are simple. In higher dimensional spaces elements of grade $2 \le m \le n-2$ are therefore the only ones that may not be simple. In 4-space, the only elements that may not be simple are those of grade 2.

There is a straightforward way of testing the simplicity of 2-elements not shared by elements of higher grade. *A 2-element is simple if and only if its exterior square is zero.* (The exterior square of $\underset{m}{\alpha}$ is $\underset{m}{\alpha} \wedge \underset{m}{\alpha}$.) Since odd elements anti-commute, the exterior square of odd elements will be zero, even if they are not simple. An even element of grade 4 or higher may be of the form of the exterior product of a 1-element with a non-simple 3-element: whence its exterior square is zero without its being simple.

We will return to a further discussion of simplicity from the point of view of factorization in Chapter 3: The Regressive Product.

All (*n*–1)-elements are simple

In general, (*n*–1)-elements are simple. We can show this as follows.

Consider first two simple (*n*–1)-elements. Since they can differ by at most *one* 1-element factor (otherwise they would together contain more than *n* independent factors), we can express them as $\underset{n-2}{\alpha} \wedge \beta_1$ and $\underset{n-2}{\alpha} \wedge \beta_2$. Summing these, and factoring the common (*n*–2)-element gives a simple (*n*–1)-element.

$$\underset{n-2}{\alpha} \wedge \beta_1 + \underset{n-2}{\alpha} \wedge \beta_2 == \underset{n-2}{\alpha} \wedge (\beta_1 + \beta_2)$$

Any (*n*–1)-element can be expressed as the sum of simple (*n*–1)-elements. We can therefore prove the general case by supposing pairs of simple elements to be combined to form another simple element, until just one simple element remains.

Conditions for simplicity of a 2-element in a 4-space

Consider a simple 2-element in a 4-dimensional space. First we declare a 4-dimensional basis for the space, and then create a general 2-element $\underset{2}{\alpha}$. We can shortcut the entry of the 2-element by using the *GrassmannAlgebra* function `CreateBasisForm`. The first argument is the grade of the element required. The second is the symbol on which the scalar coefficients are to be based. `CreateBasisForm` automatically declares the coefficients to be scalars by including the new pattern `a_` in the list of declared scalars.

$$\mathbb{V}_4;\ \underset{2}{\alpha} = \mathbf{CreateBasisForm[2,\ a]}$$

$$a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4 + a_6\, e_3 \wedge e_4$$

Since $\underset{2}{\alpha}$ is simple, $\underset{2}{\alpha} \wedge \underset{2}{\alpha} == 0$. To see how this constrains the coefficients a_i of the terms of $\underset{2}{\alpha}$ we expand and simplify the product:

```
𝒢[α∧α] // Simplify
  2 2
```

$2\,(a_3\,a_4 - a_2\,a_5 + a_1\,a_6)\;e_1\wedge e_2\wedge e_3\wedge e_4$

Thus the condition that a 2-element in a 4-dimensional space be simple is that:

$$a_3\,a_4 - a_2\,a_5 + a_1\,a_6 == 0 \qquad 2.34$$

Conditions for simplicity of a 2-element in a 5-space

The situation in 5-space is a little more complex.

$\mathbb{V}_5;\; \underset{2}{\alpha} = \mathbf{CreateBasisForm[2,\ a]}$

$a_1\,e_1\wedge e_2 + a_2\,e_1\wedge e_3 + a_3\,e_1\wedge e_4 + a_4\,e_1\wedge e_5 + a_5\,e_2\wedge e_3 +$
$a_6\,e_2\wedge e_4 + a_7\,e_2\wedge e_5 + a_8\,e_3\wedge e_4 + a_9\,e_3\wedge e_5 + a_{10}\,e_4\wedge e_5$

$\mathcal{G}\left[\underset{2}{\alpha}\wedge\underset{2}{\alpha}\right]$ **// Simplify**

$$\begin{aligned}2\,(&(a_3\,a_5 - a_2\,a_6 + a_1\,a_8)\;e_1\wedge e_2\wedge e_3\wedge e_4 +\\ &(a_4\,a_5 - a_2\,a_7 + a_1\,a_9)\;e_1\wedge e_2\wedge e_3\wedge e_5 +\\ &(a_4\,a_6 - a_3\,a_7 + a_1\,a_{10})\;e_1\wedge e_2\wedge e_4\wedge e_5 +\\ &(a_4\,a_8 - a_3\,a_9 + a_2\,a_{10})\;e_1\wedge e_3\wedge e_4\wedge e_5 +\\ &(a_7\,a_8 - a_6\,a_9 + a_5\,a_{10})\;e_2\wedge e_3\wedge e_4\wedge e_5)\end{aligned}$$

For $\underset{2}{\alpha}$ to be simple we require this product to be zero, hence each of the five coefficients must be zero simultaneously. We can collect these five equations into a list **A** and obtain a solution to them by using *Mathematica*'s inbuilt `Solve` function:

$$\begin{aligned}\mathbf{A} = \{&a_3\,a_5 - a_2\,a_6 + a_1\,a_8 == 0,\\ &a_4\,a_5 - a_2\,a_7 + a_1\,a_9 == 0,\; a_4\,a_6 - a_3\,a_7 + a_1\,a_{10} == 0,\\ &a_4\,a_8 - a_3\,a_9 + a_2\,a_{10} == 0,\; a_7\,a_8 - a_6\,a_9 + a_5\,a_{10} == 0\};\end{aligned}$$

$\mathbf{S = Solve[A,\ \{a_8,\ a_9,\ a_{10}\}]}$

$$\left\{\left\{a_8 \to -\frac{a_3\,a_5 - a_2\,a_6}{a_1},\; a_9 \to -\frac{a_4\,a_5 - a_2\,a_7}{a_1},\; a_{10} \to -\frac{a_4\,a_6 - a_3\,a_7}{a_1}\right\}\right\}$$

By inspection we see that these three solutions correspond to the first three equations. Consequently, from the generality of the expression for the 2-element we expect *any* three equations to suffice.

In summary we can say that a 2-element $\underset{2}{\alpha}$ in a 5-space is simple if any three of the following equations is satisfied:

$$a_3\,a_5 - a_2\,a_6 + a_1\,a_8 == 0,\ a_4\,a_5 - a_2\,a_7 + a_1\,a_9 == 0,\ a_4\,a_6 - a_3\,a_7 + a_1\,a_{10} == 0,\ a_4\,a_8 - a_3\,a_9 + a_2\,a_{10} == 0,\ a_7\,a_8 - a_6\,a_9 + a_5\,a_{10} == 0 \qquad 2.35$$

2.11 Exterior Division

The definition of an exterior quotient

It will be seen in Chapter 4: Geometric Interpretations that one way of defining geometric entities like lines and planes is by the exterior quotient of two interpreted elements. Such a quotient does not yield a unique element. Indeed this is why it is useful for defining geometric entities, for they are composed of *sets* of elements.

The exterior quotient of a simple (*m*+*k*)-element $\underset{m+k}{\alpha}$ by another simple element $\underset{k}{\beta}$ which is contained in $\underset{m+k}{\alpha}$ is defined to be the most general *m*-element contained in $\underset{m+k}{\alpha}$ ($\underset{m}{\gamma}$, say) such that the exterior product of the quotient $\underset{m}{\gamma}$ with the denominator $\underset{k}{\beta}$ yields the numerator $\underset{m+k}{\alpha}$.

$$\underset{m}{\gamma} == \frac{\underset{m+k}{\alpha}}{\underset{k}{\beta}} \quad \Longrightarrow \quad \underset{m}{\gamma}\wedge\underset{k}{\beta} == \underset{m+k}{\alpha} \qquad 2.36$$

Note the convention adopted for the order of the factors.

In Chapter 9: Exploring Grassmann Algebras we shall generalize this definition of quotient to define the general division of Grassmann numbers.

Division by a 1-element

Consider the quotient of a simple (*m*+1)-element $\underset{m+1}{\alpha}$ by a 1-element β. Since $\underset{m+1}{\alpha}$ contains β we can write it as $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \wedge \beta$. However, we could also have written this numerator in the more general form:

$$(\alpha_1 + t_1\,\beta) \wedge (\alpha_2 + t_2\,\beta) \wedge \cdots \wedge (\alpha_m + t_m\,\beta) \wedge \beta$$

where the t_i are arbitrary scalars. It is in this more general form that the numerator must be written before β can be 'divided out'. Thus the quotient may be written:

$$\frac{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \wedge \beta}{\beta} == (\alpha_1 + t_1\,\beta) \wedge (\alpha_2 + t_2\,\beta) \wedge \cdots \wedge (\alpha_m + t_m\,\beta) \qquad 2.37$$

Division by a *k*-element

Suppose now we have a quotient with a simple *k*-element denominator. Since the denominator is contained in the numerator, we can write the quotient as:

$$\frac{(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k)}{(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k)}$$

To prepare for the 'dividing out' of the β factors we rewrite the numerator in the more general form:

$$(\alpha_1 + t_{11}\,\beta_1 + \cdots + t_{1\,k}\,\beta_k) \wedge (\alpha_2 + t_{21}\,\beta_1 + \cdots + t_{2\,k}\,\beta_k) \wedge \\ \cdots \wedge (\alpha_m + t_{m1}\,\beta_1 + \cdots + t_{mk}\,\beta_k) \wedge (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k)$$

where the t_{ij} are arbitrary scalars. Hence:

$$\frac{(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k)}{(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k)} == \\ (\alpha_1 + t_{11}\,\beta_1 + \cdots + t_{1\,k}\,\beta_k) \wedge \\ (\alpha_2 + t_{21}\,\beta_1 + \cdots + t_{2\,k}\,\beta_k) \wedge \cdots \wedge (\alpha_m + t_{m1}\,\beta_1 + \cdots + t_{mk}\,\beta_k) \qquad 2.38$$

In the special case of *m* equal to 1, this reduces to:

$$\frac{\alpha \wedge \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k}{\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k} == \alpha + t_1\,\beta_1 + \cdots + t_k\,\beta_k \qquad 2.39$$

We will later see that this formula neatly defines a hyperplane.

Special cases

In the special cases where $\underset{m}{\alpha}$ or $\underset{k}{\beta}$ is a scalar, the results are unique.

$$\frac{a\,\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k}{\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k} == a \qquad 2.40$$

$$\frac{b\,\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m}{b} == \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \qquad 2.41$$

3 The Regressive Product

3.1 Introduction

Since the linear spaces $\underset{m}{\Lambda}$ an $\underset{n-m}{\Lambda}$ are of the same dimension $\binom{n}{m} = \binom{n}{n-m}$ and hence are isomorphic, the opportunity exists to define a product operation (called the regressive product) dual to the exterior product such that theorems involving exterior products of *m*-elements have duals involving regressive products of (*n*–*m*)-elements.

Very roughly speaking, if the exterior product is associated with the notion of *union*, then the regressive product is associated with the notion of *intersection*.

The regressive product appears to be unused in the recent literature. Grassmann's original development did not distinguish *notationally* between the exterior and regressive product operations. Instead he capitalized on the inherent duality and used a notation which, depending on the grade of the elements in the product, could only sensibly be *interpreted* by one or the other operation. This was a very elegant idea, but its subtlety may have been one of the reasons the notion has become lost. (See the historical notes in Section 3.3.)

However, since the regressive product is a simple dual operation to the exterior product, an enticing and powerful symmetry is lost by ignoring it. We will find that its 'intersection' properties are a useful conceptual and algorithmic addition to non-metric geometry, and that its algebraic properties enable a firm foundation for the development of metric geometry. Some of the results which are proven in metric spaces via the inner and cross products can also be proven using the exterior and regressive products, thus showing that the result is independent of whether or not the space has a metric.

The approach adopted in this book is to distinguish between the two product operations by using different notations, just as Boolean algebra has its dual operations of union and intersection. We will find that this approach does not detract from the elegance of the results. We will also find that differentiating the two operations explicitly enhances the simplicity and power of the derivation of results.

Since the commonly accepted modern notation for the exterior product operation is the 'wedge' symbol $\wedge$, we will denote the regressive product operation by a 'vee' symbol $\vee$. Note however that this (unfortunately) does not correspond to the Boolean algebra usage of the 'vee' for union and the 'wedge' for intersection.

3.2 Duality

The notion of duality

In order to ensure that the regressive product is defined as an operation correctly dual to the exterior product, we give the defining axiom set for the regressive product the same formal symbolic structure as the axiom set for the exterior product. This may be accomplished by replacing $\wedge$ by $\vee$, and replacing the grades of elements and spaces by their complementary grades. The *complementary grade* of a grade *m* is defined in a linear space of *n* dimensions to be *n–m*.

$$\wedge \longrightarrow \vee \qquad \underset{m}{\alpha} \longrightarrow \underset{n-m}{\alpha} \qquad \underset{m}{\Lambda} \longrightarrow \underset{n-m}{\Lambda} \tag{3.1}$$

Note that here we are undertaking the construction of a mathematical structure and thus there is no specific mapping implied between individual elements at this stage. In Chapter 5: The Complement we will introduce a mapping between the elements of $\underset{m}{\Lambda}$ and $\underset{n-m}{\Lambda}$ which will lead to the definition of *complement* and *interior product*.

For concreteness, we take some examples.

Examples: Obtaining the dual of an axiom

The dual of axiom $\wedge$ 6

We begin with the exterior product axiom $\wedge$ 6:

The exterior product of an m-element and a k-element is an (m+k)-element.

$$\left\{\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{k}{\beta} \in \underset{k}{\Lambda}\right\} \Longrightarrow \underset{m}{\alpha} \wedge \underset{k}{\beta} \in \underset{m+k}{\Lambda}$$

To form the dual of this axiom, replace $\wedge$ with $\vee$, and the grades of elements and spaces by their complementary grades.

$$\left\{\underset{n-m}{\alpha} \in \underset{n-m}{\Lambda},\ \underset{n-k}{\beta} \in \underset{n-k}{\Lambda}\right\} \Longrightarrow \underset{n-m}{\alpha} \vee \underset{n-k}{\beta} \in \underset{n-(m+k)}{\Lambda}$$

Although this is indeed the dual of axiom $\wedge$ 6, it is not necessary to display the grades of what are arbitrary elements in the more complex form specifically involving the dimension *n* of the space. It will be more convenient to display them as grades denoted by simple symbols like *m* and *k* as were the grades of the elements of the original axiom. To effect this transformation most expeditiously we first let *m*' = *n–m*, *k*' = *n–k* to get

$$\left\{\underset{m'}{\alpha} \in \underset{m'}{\Lambda},\ \underset{k'}{\beta} \in \underset{k'}{\Lambda}\right\} \Longrightarrow \underset{m'}{\alpha} \vee \underset{k'}{\beta} \in \underset{(m'+k')-n}{\Lambda}$$

The grade of the space to which the regressive product belongs is $n-(m+k) = n-((n-m')+(n-k')) = (m'+k')-n$.

Finally, since the primes are no longer necessary we drop them. Then the final form of the axiom dual to $\wedge 6$, which we label $\vee 6$, becomes:

$$\left\{\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{k}{\beta} \in \underset{k}{\Lambda}\right\} \Longrightarrow \underset{m}{\alpha} \vee \underset{k}{\beta} \in \underset{m+k-n}{\Lambda}$$

In words, this says that the regressive product of an m-element and a k-element is an $(m+k-n)$-element.

The dual of axiom $\wedge 8$

Axiom $\wedge 8$ says:

There is a unit scalar which acts as an identity under the exterior product.

$$\mathcal{E}xists\left[1,\ 1 \in \underset{0}{\Lambda}\right] : \underset{m}{\alpha} == 1 \wedge \underset{m}{\alpha}$$

For simplicity we do not normally display designated scalars with an underscripted zero. However, the duality transformation will be clearer if we rewrite the axiom with $\underset{0}{1}$ in place of $\mathbf{1}$.

$$\mathcal{E}xists\left[\underset{0}{1},\ \underset{0}{1} \in \underset{0}{\Lambda}\right] : \underset{m}{\alpha} == \underset{0}{1} \wedge \underset{m}{\alpha}$$

Replace $\wedge$ with $\vee$, and the grades of elements and spaces by their complementary grades.

$$\mathcal{E}xists\left[\underset{n}{1},\ \underset{n}{1} \in \underset{n}{\Lambda}\right] : \underset{n-m}{\alpha} == \underset{n}{1} \vee \underset{n-m}{\alpha}$$

Replace arbitrary grades m with $n-m'$ (n is not an arbitrary grade).

$$\mathcal{E}xists\left[\underset{n}{1},\ \underset{n}{1} \in \underset{n}{\Lambda}\right] : \underset{m'}{\alpha} == \underset{n}{1} \vee \underset{m'}{\alpha}$$

Drop the primes.

$$\mathcal{E}xists\left[\underset{n}{1},\ \underset{n}{1} \in \underset{n}{\Lambda}\right] : \underset{m}{\alpha} == \underset{n}{1} \vee \underset{m}{\alpha}$$

In words, this says that there is a unit n-element which acts as an identity under the regressive product.

The dual of axiom $\wedge 10$

Axiom $\wedge 10$ says:

The exterior product of elements of odd grade is anti-commutative.

$$\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{m\,k}\, \underset{k}{\beta} \wedge \underset{m}{\alpha}$$

Replace $\wedge$ with $\vee$, and the grades of elements by their complementary grades.

$$\underset{n-m}{\alpha} \wedge \underset{n-k}{\beta} == (-1)^{m\,k} \underset{n-k}{\beta} \vee \underset{n-m}{\alpha}$$

Replace arbitrary grades *m* with *n–m*', *k* with *n–k*'.

$$\underset{m'}{\alpha} \vee \underset{k'}{\beta} == (-1)^{(n-m')\,(n-k')} \underset{k'}{\beta} \vee \underset{m'}{\alpha}$$

Drop the primes.

$$\underset{m}{\alpha} \vee \underset{k}{\beta} == (-1)^{(n-m)\,(n-k)} \underset{k}{\beta} \vee \underset{m}{\alpha}$$

In words this says that the regressive product of elements of odd complementary grade is anti-commutative.

Summary: The duality transformation algorithm

The algorithm for the duality transformation may be summarized as follows:

1. Replace $\wedge$ with $\vee$, and the grades of elements and spaces by their complementary grades.
2. Replace arbitrary grades *m* with *n–m*', *k* with *n–k*'. Drop the primes.

3.3 Properties of the Regressive Product

Axioms for the regressive product

In this section we collect the results of applying the duality algorithm above to the exterior product axioms $\wedge$ 6 to $\wedge$ 12. Axioms $\wedge$ 1 to $\wedge$ 5 transform unchanged since there are no products involved.

◆ **$\vee$ 6: The regressive product of an *m*-element and a *k*-element is an (*m*+*k*–*n*)-element.**

$$\left\{\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{k}{\beta} \in \underset{k}{\Lambda}\right\} \Longrightarrow \underset{m}{\alpha} \vee \underset{k}{\beta} \in \underset{m+k-n}{\Lambda} \qquad 3.2$$

◆ **$\vee$ 7: The regressive product is associative.**

$$\left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \vee \underset{r}{\gamma} == \underset{m}{\alpha} \vee \left(\underset{k}{\beta} \vee \underset{r}{\gamma}\right) \qquad 3.3$$

◆ ∨ 8: **There is a unit *n*-element which acts as an identity under the regressive product.**

$$\mathcal{E}xists\left[\underset{n}{1},\ \underset{n}{1}\in\underset{n}{\Lambda}\right]:\underset{m}{\alpha}=\underset{n}{1}\vee\underset{m}{\alpha} \qquad 3.4$$

◆ ∨ 9: **Non-zero *n*-elements have inverses with respect to the regressive product.**

$$\mathcal{E}xists\left[\frac{1}{\underset{n}{\alpha}},\ \frac{1}{\underset{n}{\alpha}}\in\underset{n}{\Lambda}\right]:\underset{n}{1}=\underset{n}{\alpha}\vee\frac{1}{\underset{n}{\alpha}},\ \mathcal{F}or\mathcal{A}ll\left[\underset{n}{\alpha},\ \underset{n}{\alpha}\neq\underset{n}{0}\in\underset{n}{\Lambda}\right] \qquad 3.5$$

◆ ∨ 10: **The regressive product of elements of odd complementary grade is anti-commutative.**

$$\underset{m}{\alpha}\vee\underset{k}{\beta}=(-1)^{(n-m)\,(n-k)}\,\underset{k}{\beta}\vee\underset{m}{\alpha} \qquad 3.6$$

◆ ∨ 11: **Additive identities act as multiplicative zero elements under the regressive product.**

$$\mathcal{E}xists\left[\underset{k}{0},\ \underset{k}{0}\in\underset{k}{\Lambda}\right]:\underset{k}{0}\vee\underset{m}{\alpha}=\underset{k+m-n}{0} \qquad 3.7$$

◆ ∨ 12: **The regressive product is both left and right distributive under addition.**

$$\left(\underset{m}{\alpha}+\underset{m}{\beta}\right)\vee\underset{r}{\gamma}=\underset{m}{\alpha}\vee\underset{r}{\gamma}+\underset{m}{\beta}\vee\underset{r}{\gamma}$$
$$\underset{m}{\alpha}\vee\left(\underset{r}{\beta}+\underset{r}{\gamma}\right)=\underset{m}{\alpha}\vee\underset{r}{\beta}+\underset{m}{\alpha}\vee\underset{r}{\gamma} \qquad 3.8$$

◆ ∨ 13: **Scalar multiplication commutes with the regressive product.**

$$\underset{m}{\alpha}\vee\left(a\,\underset{k}{\beta}\right)=\left(a\,\underset{m}{\alpha}\right)\vee\underset{k}{\beta}=a\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right) \qquad 3.9$$

The unit n-element

◆ **The unit n-element is congruent to any basis n-element.**

The duality algorithm has generated a unit n-element $\underset{n}{1}$ which acts as the multiplicative identity for the regressive product, just as the unit scalar 1 (or $\underset{0}{1}$) acts as the multiplicative identity for the exterior product (axioms $\wedge$ 8 and $\vee$ 8).

We have already seen that any basis of $\underset{n}{\Lambda}$ contains only one element. If the basis of $\underset{1}{\Lambda}$ is $\{e_1, e_2, \cdots, e_n\}$, then the single basis element of $\underset{n}{\Lambda}$ is congruent to $e_1 \wedge e_2 \wedge \cdots \wedge e_n$. If the basis of $\underset{1}{\Lambda}$ is changed by an arbitrary (non-singular) linear transformation, then the basis of $\underset{n}{\Lambda}$ changes by a scalar factor which is the determinant of the transformation. Any basis of $\underset{n}{\Lambda}$ may therefore be expressed as a scalar multiple of some given basis, say $e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Hence we can therefore also express $\underset{n}{1}$ as some scalar multiple $\mathbb{k}$ of $e_1 \wedge e_2 \wedge \cdots \wedge e_n$.

$$\underset{n}{1} == \mathbb{k}\; e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 3.10$$

Defining $\underset{n}{1}$ any more specifically than this is normally done by imposing a metric onto the space. This we do in Chapter 5: The Complement, and Chapter 6: The Interior Product. It turns out then that $\underset{n}{1}$ is the n-element whose *measure* (magnitude, volume) *is unity*.

On the other hand, for geometric application in spaces without a metric, for example the calculation of intersections of lines, planes, and hyperplanes, it is inconsequential that we only know $\underset{n}{1}$ up to congruence, because we will see that *if an element defines a geometric entity then any element congruent to it will define the same geometric entity*.

◆ **The unit n-element is idempotent under the regressive product.**

By putting $\underset{m}{\alpha}$ equal to $\underset{n}{1}$ in the regressive product axiom $\vee$ 8 we have immediately that:

$$\underset{n}{1} == \underset{n}{1} \vee \underset{n}{1} \qquad 3.11$$

◆ ***n*-elements allow a sort of associativity.**

By letting $\underset{n}{\alpha} == a\,\underset{n}{1}$ we can show that $\underset{n}{\alpha}$ allows a sort of associativity with the exterior product.

$$\left(\underset{n}{\alpha} \vee \underset{k}{\beta}\right) \wedge \underset{p}{\gamma} == \underset{n}{\alpha} \vee \left(\underset{k}{\beta} \wedge \underset{p}{\gamma}\right) \qquad 3.12$$

The inverse of an *n*-element

Axiom $\vee 9$ says that every (non-zero) *n*-element has an inverse with respect to the regressive product. Suppose we have an *n*-element $\underset{n}{\alpha}$ expressed in terms of some basis. Then, according to 1.10 we can express it as a scalar multiple (**a**, say) of the unit n-element $\underset{n}{1}$.

We may then write the inverse $\underset{n}{\alpha}^{-1}$ of $\underset{n}{\alpha}$ with respect to the regressive product as the scalar multiple $\frac{1}{a}$ of $\underset{n}{1}$.

$$\underset{n}{\alpha} == a\,\underset{n}{1} \quad \Leftrightarrow \quad \underset{n}{\alpha}^{-1} == \frac{1}{a}\,\underset{n}{1}$$

We can see this by taking the regressive product of $a\,\underset{n}{1}$ with $\frac{1}{a}\,\underset{n}{1}$:

$$\underset{n}{\alpha} \vee \underset{n}{\alpha}^{-1} == \left(a\,\underset{n}{1}\right) \vee \left(\frac{1}{a}\,\underset{n}{1}\right) == \underset{n}{1} \vee \underset{n}{1} == \underset{n}{1}$$

If $\underset{n}{\alpha}$ is now expressed in terms of a basis we have:

$$\underset{n}{\alpha} == b\, e_1 \wedge e_2 \wedge \cdots \wedge e_n == \frac{b}{\mathbb{k}}\,\underset{n}{1}$$

Hence $\underset{n}{\alpha}^{-1}$ can be written as:

$$\underset{n}{\alpha}^{-1} == \frac{\mathbb{k}}{b}\,\underset{n}{1} == \frac{\mathbb{k}^2}{b}\, e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

Summarizing these results:

$$\boxed{\underset{n}{\alpha} == a\,\underset{n}{1} \quad \Leftrightarrow \quad \underset{n}{\alpha}^{-1} == \frac{1}{a}\,\underset{n}{1}} \qquad 3.13$$

$$\boxed{\underset{n}{\alpha} == b\, e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad \Leftrightarrow \quad \underset{n}{\alpha}^{-1} == \frac{\mathbb{k}^2}{b}\, e_1 \wedge e_2 \wedge \cdots \wedge e_n} \qquad 3.14$$

Historical Note

In Grassmann's *Ausdehnungslehre* of 1862 Section 95 (translated by Lloyd C. Kannenberg) he states:

> If *q* and *r* are the orders of two magnitudes A and B, and *n* that of the principal domain, then the order of the product [A B] is *first* equal to *q*+*r* if *q*+*r* is smaller than *n*, and *second* equal to *q*+*r*–*n* if *q*+*r* is greater than or equal to *n*.

Translating this into the terminology used in this book we have:

> If *q* and *r* are the *grades* of two *elements* A and B, and *n* that of the *underlying linear space*, then the *grade* of the product [A B] is *first* equal to *q+r* if *q+r* is smaller than *n*, and *second* equal to *q+r–n* if *q+r* is greater than or equal to *n*.

Translating this further into the notation used in this book we have:

$$[\mathbf{A}\,\mathbf{B}] \Longleftrightarrow \underset{p}{\mathbf{A}} \wedge \underset{q}{\mathbf{B}} \qquad p+q<n$$

$$[\mathbf{A}\,\mathbf{B}] \Longleftrightarrow \underset{p}{\mathbf{A}} \vee \underset{q}{\mathbf{B}} \qquad p+q\geq n$$

Grassmann called the product [A B] in the first case a *progressive* product, and in the second case a *regressive* product.

In the equivalence above, Grassmann has opted to define the product of two elements whose grades sum to the dimension *n* of the space as *regressive*, and thus a *scalar*. However, the more explicit notation that we have adopted identifies that some definition is still required for the *progressive* (exterior) product of two such elements.

The advantage of denoting the two products differently enables us to correctly define the exterior product of two elements whose grades sum to *n*, *as an n-element*. Grassmann by his choice of notation has had to define it as a scalar. In modern terminology this is equivalent to confusing scalars with pseudo-scalars. A separate notation for the two products thus avoids this tensorially invalid confusion.

We can see how then, in not being explicitly denoted, the regressive product may have become lost.

3.4 The Duality Principle

The dual of a dual

The duality of the axiom sets for the exterior and regressive products is completed by requiring that the dual of a dual of an axiom is the axiom itself. The dual of a regressive product axiom may be obtained by applying the following algorithm:

1. Replace $\vee$ with $\wedge$, and the grades of elements and spaces by their complementary grades.
2. Replace arbitrary grades *m* with *n–m*', *k* with *n–k*'. Drop the primes.

This differs from the algorithm for obtaining the dual of an exterior product axiom only in the replacement of $\vee$ with $\wedge$ instead of *vice versa*.

It is easy to see that applying this algorithm to the regressive product axioms generates the original exterior product axiom set.

We can combine both transformation algorithms by restating them as:

1. Replace each product operation by its dual operation, and the grades of elements and spaces by their complementary grades.
2. Replace arbitrary grades *m* with *n–m*', *k* with *n–k*'. Drop the primes.

Since this algorithm applies to both sets of axioms, it also applies to any theorem. Thus to each theorem involving exterior or regressive products corresponds a dual theorem obtained by applying the algorithm. We call this the Duality Principle.

The Duality Principle

To every theorem involving exterior and regressive products, a dual theorem may be obtained by:
1. Replacing each product operation by its dual operation, and the grades of elements and spaces by their complementary grades.
2. Replacing arbitrary grades *m* with *n–m*', *k* with *n–k*', then dropping the primes.

Example 1

The following theorem says that the exterior product of two elements is zero if the sum of their grades is greater than the dimension of the linear space.

$$\left\{\underset{m}{\alpha} \wedge \underset{k}{\beta} == 0,\ m + k - n > 0\right\}$$

The dual theorem states that the regressive product of two elements is zero if the sum of their grades is less than the dimension of the linear space.

$$\left\{\underset{m}{\alpha} \vee \underset{k}{\beta} == 0,\ n - (k + m) > 0\right\}$$

We can recover the original theorem as the dual of this one.

Example 2

This theorem below says that the exterior product of an element with itself is zero if it is of odd grade.

$$\left\{\underset{m}{\alpha} \wedge \underset{m}{\alpha} == 0,\ m \in \{\text{OddIntegers}\}\right\}$$

The dual theorem states that the regressive product of an element with itself is zero if its complementary grade is odd.

$$\left\{\underset{m}{\alpha} \vee \underset{m}{\alpha} == 0,\ (n - m) \in \{\text{OddIntegers}\}\right\}$$

Again, we can recover the original theorem by calculating the dual of this one.

Example 3

The following theorem that states that the exterior product of unity with itself any number of times remains unity.

$$\mathbf{1} \wedge \mathbf{1} \wedge \cdots \wedge \mathbf{1} \wedge \underset{m}{\alpha} == \underset{m}{\alpha}$$

The dual theorem states the corresponding fact for the regressive product of unit n-elements $\underset{n}{\mathbf{1}}$.

$$\underset{n}{\mathbf{1}} \vee \underset{n}{\mathbf{1}} \vee \cdots \vee \underset{n}{\mathbf{1}} \vee \underset{m}{\alpha} == \underset{m}{\alpha}$$

Using the *GrassmannAlgebra* function Dual

The algorithm of the Duality Principle has been encapsulated in the function `Dual` in *GrassmannAlgebra*. `Dual` takes an expression or list of expressions which comprise an axiom or theorem and generates the list of dual expressions by transforming them according to the replacement rules of the Duality Principle.

Example 1

Our first example is to show how `Dual` may be used to develop dual axioms. For example, to take the dual of axiom $\wedge$ 10 simply enter:

$$\mathbf{Dual}\left[\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{m\,k}\, \underset{k}{\beta} \wedge \underset{m}{\alpha}\right]$$

$$\underset{m}{\alpha} \vee \underset{k}{\beta} == (-1)^{(-k+n)\,(-m+n)}\, \underset{k}{\beta} \vee \underset{m}{\alpha}$$

Example 2

Before applying `Dual` to axiom $\wedge$ 8 however, it is necessary to designate $\mathbf{1}$ specifically as the unit of $\underset{0}{\Lambda}$ (that is, $\underset{0}{\mathbf{1}}$), otherwise it will be treated as any other scalar.

$$\mathbf{Dual}\left[\mathcal{E}xists\left[\underset{0}{1},\ \underset{0}{1} \in \underset{0}{\Lambda}\right] : \underset{m}{\alpha} == \underset{0}{1} \wedge \underset{m}{\alpha}\right]$$

$$\mathcal{E}xists\left[\underset{n}{1},\ \underset{n}{1} \in \underset{n}{\Lambda}\right] : \underset{m}{\alpha} == \underset{n}{1} \vee \underset{m}{\alpha}$$

Example 3

To apply `Dual` to an axiom involving more than one statement, collect the statements in a list. For example, the dual of axiom $\wedge$ 9 is obtained as:

$$\mathbf{Dual}\left[\left\{\mathcal{E}xists\left[\underset{0}{\alpha}^{-1},\ \underset{0}{\alpha}^{-1} \in \underset{0}{\Lambda}\right] : 1 == \underset{0}{\alpha} \wedge \underset{0}{\alpha}^{-1},\ \mathcal{F}or\mathcal{A}ll\left[\underset{0}{\alpha},\ \underset{0}{\alpha} \neq \underset{0}{0} \in \underset{0}{\Lambda}\right]\right\}\right]$$

$$\left\{\mathcal{E}xists\left[\frac{1}{\underset{n}{\alpha}},\ \frac{1}{\underset{n}{\alpha}} \in \underset{n}{\Lambda}\right] : 1 == \underset{n}{\alpha} \vee \frac{1}{\underset{n}{\alpha}},\ \mathcal{F}or\mathcal{A}ll\left[\underset{n}{\alpha},\ \underset{n}{\alpha} \neq \underset{n}{0} \in \underset{n}{\Lambda}\right]\right\}$$

Note that in order to simplify the algorithm for `Dual`, we are purposely frustrating *Mathematica*'s inbuilt definition of `Exists` and `ForAll` by writing them in script.

Example 4

Our fourth example is to generate the dual of the Common Factor Axiom. This axiom will be introduced in the next section.

CommonFactorAxiom =

$$\left\{\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma},\ m+k+p-n == 0\right\};$$

Dual[CommonFactorAxiom]

$$\left\{\left(\underset{m}{\alpha}\vee\underset{p}{\gamma}\right)\wedge\left(\underset{k}{\beta}\vee\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha}\vee\underset{k}{\beta}\vee\underset{p}{\gamma}\right)\wedge\underset{p}{\gamma},\ -k-m+2n-p == 0\right\}$$

We can verify that the dual of the dual of the axiom is the axiom itself.

CommonFactorAxiom == Dual[Dual[CommonFactorAxiom]]

`True`

Example 5

Our fifth example is to obtain the dual of one of the product formulae derived in a later section.

$$\mathbf{F} = \left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\vee\underset{n-1}{\mathbf{x}} == \left(\underset{m}{\alpha}\vee\underset{n-1}{\mathbf{x}}\right)\wedge\underset{k}{\beta} + (-1)^{m}\,\underset{m}{\alpha}\wedge\left(\underset{k}{\beta}\vee\underset{n-1}{\mathbf{x}}\right);$$

Dual[F]

$$\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge\mathbf{x} == (-1)^{-m+n}\,\underset{m}{\alpha}\vee\underset{k}{\beta}\wedge\mathbf{x} + \underset{m}{\alpha}\wedge\mathbf{x}\vee\underset{k}{\beta}$$

F == Dual[Dual[F]]

`True`

Setting up your own input to the Dual function

The *GrassmannAlgebra* function `Dual` depends on the syntax of the statements given it as arguments. The following are some suggestions for setting up your own input to `Dual`.

- Scalars under the usual field multiplication should be declared as scalars.
- Other symbols which you want to protect from change can be temporarily declared as scalars.
- A scalar denoted as a 0-element should be expressly multiplied using the exterior product.
- Separate statements should be combined into a list of statements.
- The statements should not include anything which will evaluate when entered.
- For example, summation signs should not use *Mathematica*'s `Sum` function.
- If you use simple concatenated symbols, *Mathematica* will assume multiplication and arrange the order as it pleases. Be prepared for this.

3.5 The Common Factor Axiom

Motivation

Although the axiom sets for the progressive and regressive products have been developed, it still remains to propose an axiom explicitly relating the two types of products. This axiom will have wide-ranging applications in the Grassmann algebra. We will find it particularly useful in the derivation of theoretical results involving the interior product, the factorization of simple elements, and in the calculation of geometric intersections.

The axiom effectively endows the dual exterior and regressive product structures with the properties we would like associated with unions and intersections of elements and spaces.

The regressive product of two intersecting elements is equal to the regressive product of their union and their intersection.

The formula we need is called the Common Factor Axiom. In this section we motivate its basic form up to the determination of a (possible) scalar factor with a *plausible* argument. (If we had been able to present a rigorous argument, then the axiom would instead have attained the status of a theorem!)

Specifically, since axiom $\vee 6$ states that the regressive product of an *m*-element and a *k*-element is an (*m*+*k*–*n*)-element, the formula should enable determination of that element as an exterior product.

For concreteness, suppose we are in a 4-dimensional space with basis $\{e_1, e_2, e_3, e_4\}$. Consider the regressive product $(e_1 \wedge e_2 \wedge e_3) \vee (e_2 \wedge e_3 \wedge e_4)$ with the common factor $e_2 \wedge e_3$. By axiom $\vee 6$, $(e_1 \wedge e_2 \wedge e_3) \vee (e_2 \wedge e_3 \wedge e_4)$ is a 2-element. We will show that this 2-element is a scalar multiple of the common factor $e_2 \wedge e_3$.

The basis of $\underset{2}{\Lambda}$ is $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$. The most general 2-element expressed in terms of this basis and which maintains the homogeneity of the original expression (that is, displays the elements in products the same number of times) is:

$$\begin{aligned}
&a_1\ (e_3 \wedge e_2 \wedge e_3 \wedge e_4) \vee (e_1 \wedge e_2)\\
+&a_2\ (e_2 \wedge e_2 \wedge e_3 \wedge e_4) \vee (e_1 \wedge e_3)\\
+&a_3\ (e_2 \wedge e_3 \wedge e_2 \wedge e_3) \vee (e_1 \wedge e_4)\\
+&a_4\ (e_1 \wedge e_2 \wedge e_3 \wedge e_4) \vee (e_2 \wedge e_3)\\
+&a_5\ (e_1 \wedge e_2 \wedge e_3 \wedge e_3) \vee (e_2 \wedge e_4)\\
+&a_6\ (e_1 \wedge e_2 \wedge e_3 \wedge e_2) \vee (e_3 \wedge e_4)
\end{aligned}$$

Notice however that all but one of the terms in this sum are zero due to repeated factors in the 4-element. Thus:

$$(e_1 \wedge e_2 \wedge e_3) \vee (e_2 \wedge e_3 \wedge e_4) == a_4\ (e_1 \wedge e_2 \wedge e_3 \wedge e_4) \vee (e_2 \wedge e_3)$$

Here we see that the regressive product of two intersecting elements is (apart from a possible scalar factor) equal to the regressive product of their union $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and their intersection $e_2 \wedge e_3$.

It will turn out (as might be expected), that a value of a_4 equal to 1 gives us the neatest results in the subsequent development of the Grassmann algebra.

Furthermore since $\underset{4}{\Lambda}$ has only the one basis element, the unit *n*-element $\underset{n}{1}$ can be written $\underset{n}{1} == \Bbbk\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$, where $\Bbbk$ is some scalar yet to be determined. Applying axiom $\vee 8$ then gives us:

$$(e_1 \wedge e_2 \wedge e_3) \vee (e_2 \wedge e_3 \wedge e_4) == \frac{1}{\Bbbk}\, e_2 \wedge e_3$$

This is the formula for $(e_1 \wedge e_2 \wedge e_3) \vee (e_2 \wedge e_3 \wedge e_4)$ that we were looking for. But it only determines the 2-element to within an undetermined scalar multiple.

The Common Factor Axiom

Let $\underset{m}{\alpha}$, $\underset{k}{\beta}$, and $\underset{p}{\gamma}$ be *simple* elements with *m*+*k*+*p* = *n*, where *n* is the dimension of the space. Then the Common Factor Axiom states that:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \vee \left(\underset{k}{\beta} \wedge \underset{p}{\gamma}\right) == \left(\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma}\right) \vee \underset{p}{\gamma} \qquad m + k + p == n \qquad \mathbf{3.15}$$

Thus, the regressive product of two elements $\underset{m}{\alpha} \wedge \underset{p}{\gamma}$ and $\underset{k}{\beta} \wedge \underset{p}{\gamma}$ with a common factor $\underset{p}{\gamma}$ is equal to the regressive product of the 'union' of the elements $\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma}$ with the common factor $\underset{p}{\gamma}$ (their 'intersection').

If $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ still contain some simple elements in common, then the product $\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma}$ is zero, hence, by the definition above $\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \vee \left(\underset{k}{\beta} \wedge \underset{p}{\gamma}\right)$ is also zero. In what follows, we suppose that $\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma}$ is not zero.

Since the union $\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma}$ is an *n*-element, we can write it as some scalar factor κ, say, of the unit *n*-element: $\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma} == \kappa\, \underset{n}{1}$. Hence by axiom $\vee 8$ we derive immediately that the regressive product of two elements $\underset{m}{\alpha} \wedge \underset{p}{\gamma}$ and $\underset{k}{\beta} \wedge \underset{p}{\gamma}$ with a common factor $\underset{p}{\gamma}$ is congruent to that factor.

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \vee \left(\underset{k}{\beta} \wedge \underset{p}{\gamma}\right) \equiv \underset{p}{\gamma} \qquad m + k + p == n \qquad \mathbf{3.16}$$

It is easy to see that by using the anti-commutativity axiom $\wedge$ 10, that the axiom may be arranged in any of a number of alternative forms, the most useful of which are:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\vee\underset{p}{\gamma} \qquad m+k+p == n \qquad 3.17$$

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\vee\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\vee\underset{p}{\gamma} \qquad m+k+p == n \qquad 3.18$$

Historical Note
The approach we have adopted in this chapter of treating the common factor relation as an axiom is effectively the same as Grassmann used in his first *Ausdehnungslehre* (1844) but differs from the approach that he used in his second *Ausdehnungslehre* (1862). (See Chapter 3, Section 5 in Kannenberg.) In the 1862 version Grassmann *proves* this relation from another which is (almost) the same as the Complement Axiom that we introduce in Chapter 5: The Complement. Whitehead [ ], and other writers in the Grassmannian tradition follow his 1862 approach.
The relation which Grassmann used in the 1862 *Ausdehnungslehre* is in effect equivalent to assuming the space has a Euclidean metric (his *Ergänzung* or *supplement*). However the Common Factor Axiom does not depend on the space having a metric; that is, it is completely independent of any correspondence we set up between $\underset{m}{\Lambda}$ and $\underset{n-m}{\Lambda}$. Hence we would rather not adopt an approach which introduces an unnecessary constraint, especially since we want to show later that his work is easily extended to more general metrics than the Euclidean.

Extension of the Common Factor Axiom to general elements

The axiom has been stated for simple elements. In this section we show that it remains valid for general (possibly non-simple) elements, *provided that the common factor remains simple*.

Consider two simple elements $\underset{m}{\alpha_1}$ and $\underset{m}{\alpha_2}$. Then the Common Factor Axiom can be written:

$$\left(\underset{m}{\alpha_1}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha_1}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma}$$

$$\left(\underset{m}{\alpha_2}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha_2}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma}$$

Adding these two equations and using the distributivity of $\wedge$ and $\vee$ gives:

$$\left(\left(\underset{m}{\alpha_1}+\underset{m}{\alpha_2}\right)\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\left(\underset{m}{\alpha_1}+\underset{m}{\alpha_2}\right)\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma}$$

Extending this process, we see that the formula remains true for arbitrary $\underset{m}{\alpha}$ and $\underset{k}{\beta}$, *providing* $\underset{p}{\gamma}$ *is simple*.

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma} \equiv \underset{p}{\gamma}$$
$$m+k+p-n == 0 \qquad 3.19$$

This is an extended version of the Common Factor Axiom. It states that: t*he regressive product of two arbitrary elements containing a simple common factor is congruent to that factor*.

For applications involving computations in a non-metric space, particularly those with a geometric interpretation, we will see that the congruence form is not restrictive. Indeed, it will be quite elucidating. For more general applications in metric spaces we will see that the associated scalar factor is no longer arbitrary but is determined by the metric imposed.

Special cases of the Common Factor Axiom

In this section we list some special cases of the Common Factor Axiom. We assume, without explicitly stating, that the common factor is simple.

When the grades do not conform to the requirement shown, the product is zero.

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) == 0 \qquad m+k+p-n \neq 0 \qquad 3.20$$

If there is no common factor (other than the scalar **1**), then the axiom reduces to:

$$\underset{m}{\alpha}\vee\underset{k}{\beta} == \left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\vee 1 \quad \in \underset{0}{\Lambda} \qquad m+k-n == 0 \qquad 3.21$$

The following version yields a sort of associativity for products which are scalar.

$$\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\vee\underset{p}{\gamma} == \underset{m}{\alpha}\vee\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right) \quad \in \underset{0}{\Lambda} \qquad m+k+p-n == 0 \qquad 3.22$$

This may be proven from the previous case by noting that each side is equal to $\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\vee 1$.

Dual versions of the Common Factor Axiom

The dual Common Factor Axiom is:

$$\left(\underset{m}{\alpha}\vee\underset{p}{\gamma}\right)\wedge\left(\underset{k}{\beta}\vee\underset{p}{\gamma}\right) == \left(\underset{m}{\alpha}\vee\underset{k}{\beta}\vee\underset{p}{\gamma}\right)\wedge\underset{p}{\gamma} \quad \in \underset{p}{\Lambda} \qquad m+k+p-2n == 0 \qquad 3.23$$

The duals of the above three cases are:

$$\left(\underset{m}{\alpha} \vee \underset{p}{\gamma}\right) \wedge \left(\underset{k}{\beta} \vee \underset{p}{\gamma}\right) == 0 \qquad m + k + p - 2n \neq 0 \qquad 3.24$$

$$\underset{m}{\alpha} \wedge \underset{k}{\beta} == \left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \underset{n}{1} \quad \in \underset{n}{\Lambda} \qquad m + k - n == 0 \qquad 3.25$$

$$\left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \wedge \underset{p}{\gamma} == \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \vee \underset{p}{\gamma}\right) \quad \in \underset{n}{\Lambda} \qquad m + k + p - 2n == 0 \qquad 3.26$$

Application of the Common Factor Axiom

Suppose we have two general 2-elements **X** and **Y** in a 3-space and we wish to find a formula for their 1-element intersection **Z**. Because **X** and **Y** are in a 3-space, we are assured that they are simple.

$$X = x_1\, e_1 \wedge e_2 + x_2\, e_1 \wedge e_3 + x_3\, e_2 \wedge e_3;$$

$$Y = y_1\, e_1 \wedge e_2 + y_2\, e_1 \wedge e_3 + y_3\, e_2 \wedge e_3;$$

We calculate **Z** as the regressive product of **X** and **Y**:

$$\begin{aligned} &Z == X \vee Y \\ &\;== (x_1\, e_1 \wedge e_2 + x_2\, e_1 \wedge e_3 + x_3\, e_2 \wedge e_3) \\ &\vee (y_1\, e_1 \wedge e_2 + y_2\, e_1 \wedge e_3 + y_3\, e_2 \wedge e_3) \end{aligned}$$

Expanding this product, and remembering that the regressive product of identical basis elements is zero, we obtain:

$$\begin{aligned} &Z == (x_1\, e_1 \wedge e_2) \vee (y_2\, e_1 \wedge e_3) + (x_1\, e_1 \wedge e_2) \vee (y_3\, e_2 \wedge e_3) \\ &+ (x_2\, e_1 \wedge e_3) \vee (y_1\, e_1 \wedge e_2) + (x_2\, e_1 \wedge e_3) \vee (y_3\, e_2 \wedge e_3) \\ &+ (x_3\, e_2 \wedge e_3) \vee (y_1\, e_1 \wedge e_2) + (x_3\, e_2 \wedge e_3) \vee (y_2\, e_1 \wedge e_3) \end{aligned}$$

In a 3 space, regressive products of 2-elements are anti-commutative since $(-1)^{(n-m)\,(n-k)} = (-1)^{(3-2)\,(3-2)} = -1$. Hence we can collect pairs of terms with the same factors:

$$\begin{aligned} &Z == (x_1\, y_2 - x_2\, y_1)\, (e_1 \wedge e_2) \vee (e_1 \wedge e_3) \\ &+ (x_1\, y_3 - x_3\, y_1)\, (e_1 \wedge e_2) \vee (e_2 \wedge e_3) \\ &+ (x_2\, y_3 - x_3\, y_2)\, (e_1 \wedge e_3) \vee (e_2 \wedge e_3) \end{aligned}$$

We can now apply the Common Factor Axiom to each of these regressive products:

$$\begin{aligned} &Z == (x_1\, y_2 - x_2\, y_1)\, (e_1 \wedge e_2 \wedge e_3) \vee e_1 \\ &+ (x_1\, y_3 - x_3\, y_1)\, (e_1 \wedge e_2 \wedge e_3) \vee e_2 \\ &+ (x_2\, y_3 - x_3\, y_2)\, (e_1 \wedge e_2 \wedge e_3) \vee e_3 \end{aligned}$$

Finally, by putting $\Bbbk\, e_1 \wedge e_2 \wedge e_3 == \underset{n}{1}$, we have Z expressed as a 1-element:

$$Z = \frac{1}{\Bbbk}\,((x_1\, y_2 - x_2\, y_1)\, e_1 + (x_1\, y_3 - x_3\, y_1)\, e_2 + (x_2\, y_3 - x_3\, y_2)\, e_3);$$

Thus, in sum, we have the general congruence relation for the intersection of two 2-elements in a 3-space.

$$\boxed{\begin{gathered}(x_1\, e_1 \wedge e_2 + x_2\, e_1 \wedge e_3 + x_3\, e_2 \wedge e_3)\\ \vee\, (y_1\, e_1 \wedge e_2 + y_2\, e_1 \wedge e_3 + y_3\, e_2 \wedge e_3)\\ \equiv (x_1\, y_2 - x_2\, y_1)\, e_1 + (x_1\, y_3 - x_3\, y_1)\, e_2 + (x_2\, y_3 - x_3\, y_2)\, e_3\end{gathered}} \qquad 3.27$$

Check by calculating the exterior products

We can check that Z is indeed a common element to X and Y by determining if the exterior product of Z with each of X and Y is zero. We use `GrassmannSimplify` to do the check.

```
DeclareExtraScalars[{x1, x2, x3, y1, y2, y3}];
```

```
GrassmannSimplify[{Z ∧ X, Z ∧ Y}]
```

```
{0, 0}
```

3.6 The Common Factor Theorem

Development of the Common Factor Theorem

The example in the previous section applied the Common Factor Axiom to two general elements by expanding all the terms in their regressive product, applying the Common Factor Axiom to each of the terms and then factoring the result. In the case one of the factors in the regressive product is simple, as is often the case, and expressed as a product of 1-element factors, there is usually a more effective way to determine a simple common factor. The formula for doing this we call the Common Factor Theorem.

This theorem is one of the most important in the Grassmann algebra. We will see later that it has a counterpart expressed in terms of exterior and interior products which forms the principal expansion theorem for interior products.

Consider a regressive product $\underset{m}{\alpha} \vee \underset{k}{\beta}$ where $\underset{m}{\alpha}$ is given as a simple product of 1-element factors and $m+k = n+p$, $p > 0$. Since the common factor ($\underset{p}{\gamma}$, say) is a p-element it can be expressed as a linear combination of all the essentially different p-elements formed from the 1-element factors of $\underset{m}{\alpha}$: $\underset{p}{\gamma} == \sum a_i\, \underset{p}{\alpha_i}$, where the $\underset{p}{\alpha_i}$ are defined by:

$$\underset{m}{\alpha} == \underset{m-p}{\alpha_1} \wedge \underset{p}{\alpha_1} == \underset{m-p}{\alpha_2} \wedge \underset{p}{\alpha_2} == \cdots == \underset{m-p}{\alpha_\nu} \wedge \underset{p}{\alpha_\nu} \qquad \nu == \binom{m}{p}$$

We can also express $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ in terms of $\underset{p}{\gamma}$ by:

$$\underset{m}{\alpha} == \underset{m-p}{\alpha} \wedge \underset{p}{\gamma} \qquad \underset{k}{\beta} == \underset{p}{\gamma} \wedge \underset{k-p}{\beta}$$

By starting with the regressive product $\underset{m}{\alpha} \vee \underset{k}{\beta}$ and performing a series of straightforward algebraic manipulations using these definitions, we obtain the result required. Summation is always over the index *i*.

$$\underset{m}{\alpha} \vee \underset{k}{\beta} == \left(\underset{m-p}{\alpha} \wedge \underset{p}{\gamma}\right) \vee \left(\underset{p}{\gamma} \wedge \underset{k-p}{\beta}\right) == \left(\underset{m-p}{\alpha} \wedge \underset{p}{\gamma} \wedge \underset{k-p}{\beta}\right) \vee \underset{p}{\gamma}$$

$$== \left(\underset{m}{\alpha} \wedge \underset{k-p}{\beta}\right) \vee \underset{p}{\gamma} == \left(\underset{m}{\alpha} \wedge \underset{k-p}{\beta}\right) \vee \left(\Sigma\, a_i \underset{p}{\alpha_i}\right)$$

$$== \Sigma \left(a_i \underset{m}{\alpha} \wedge \underset{k-p}{\beta}\right) \vee \underset{p}{\alpha_i} == \Sigma \left(a_i \underset{m-p}{\alpha_i} \wedge \underset{p}{\alpha_i} \wedge \underset{k-p}{\beta}\right) \vee \underset{p}{\alpha_i}$$

$$== \Sigma \left(\underset{m-p}{\alpha_i} \wedge \left(\Sigma\, a_i \underset{p}{\alpha_i}\right) \wedge \underset{k-p}{\beta}\right) \vee \underset{p}{\alpha_i} == \Sigma \left(\underset{m-p}{\alpha_i} \wedge \underset{p}{\gamma} \wedge \underset{k-p}{\beta}\right) \vee \underset{p}{\alpha_i}$$

$$== \Sigma \left(\underset{m-p}{\alpha_i} \wedge \underset{k}{\beta}\right) \vee \underset{p}{\alpha_i}$$

In sum, we can write:

$$\boxed{\begin{array}{ll} \underset{m}{\alpha} \vee \underset{k}{\beta} == \displaystyle\sum_{i=1}^{\nu} \left(\underset{m-p}{\alpha_i} \wedge \underset{k}{\beta}\right) \vee \underset{p}{\alpha_i} & p == m + k - n \\ \underset{m}{\alpha} == \underset{m-p}{\alpha_1} \wedge \underset{p}{\alpha_1} == \underset{m-p}{\alpha_2} \wedge \underset{p}{\alpha_2} == \cdots == \underset{m-p}{\alpha_\nu} \wedge \underset{p}{\alpha_\nu} & \nu == \binom{m}{p} \end{array}} \qquad 3.28$$

An analogous formula may be obtained *mutatis mutandis* by factoring $\underset{k}{\beta}$ rather than $\underset{m}{\alpha}$. Hence $\underset{k}{\beta}$ must now be simple.

$$\boxed{\begin{array}{ll} \underset{m}{\alpha} \vee \underset{k}{\beta} == \displaystyle\sum_{i=1}^{\nu} \left(\underset{m}{\alpha} \wedge \underset{k-p}{\beta_i}\right) \vee \underset{p}{\beta_i} & p == m + k - n \\ \underset{k}{\beta} == \underset{p}{\beta_1} \wedge \underset{k-p}{\beta_1} == \underset{p}{\beta_2} \wedge \underset{k-p}{\beta_2} == \cdots == \underset{p}{\beta_\nu} \wedge \underset{k-p}{\beta_\nu} & \nu == \binom{k}{p} \end{array}} \qquad 3.29$$

It is evident that if both elements are simple, expansion in factors of the element of lower grade will be the more efficient.

When neither $\underset{m}{\alpha}$ nor $\underset{k}{\beta}$ is simple, the Common Factor Rule can still be applied to the simple component terms of the product.

Multiple regressive products may be treated by successive applications of the formulae above.

Example: The decomposition of a 1-element

The special case of the regressive product of a 1-element β with an n-element $\underset{n}{\alpha}$ enables us to decompose β directly in terms of the factors of $\underset{n}{\alpha}$. The Common Factor Theorem gives:

$$\underset{n}{\alpha} \vee \beta == \sum_{i=1}^{n} \left(\underset{n-1}{\alpha_i} \wedge \beta\right) \vee \alpha_i$$

where:

$$\underset{n}{\alpha} == \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n == (-1)^{n-i} \,(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_n) \wedge \alpha_i$$

The symbol $\square_i$ means that the ith factor is missing from the product. Substituting in the Common Factor Theorem gives:

$$\underset{n}{\alpha} \vee \beta == \sum_{i=1}^{n} (-1)^{n-i} \,((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_n) \wedge \beta) \vee \alpha_i$$

$$== \sum_{i=1}^{n} (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{i-1} \wedge \beta \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_n) \vee \alpha_i$$

Hence the decomposition formula becomes:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \vee \beta == \sum_{i=1}^{n} (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{i-1} \wedge \beta \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_n) \vee \alpha_i \qquad 3.30$$

Writing this out in full shows that we can expand the expression simply by interchanging β successively with each of the factors of $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$, and summing the results.

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \vee \beta == (\beta \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \vee \alpha_1 + (\alpha_1 \wedge \beta \wedge \cdots \wedge \alpha_n) \vee \alpha_2 + \cdots + (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \beta) \vee \alpha_n \qquad 3.31$$

We can make the result express more explicitly the decomposition of β in terms of the α_i by 'dividing through' by $\underset{n}{\alpha}$.

$$\beta == \sum_{i=1}^{n} \left(\frac{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{i-1} \wedge \beta \wedge \alpha_{i+1} \wedge \cdots \wedge \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n}\right) \alpha_i \qquad 3.32$$

Here the quotient of two n-elements has been defined previously as the quotient of their scalar coefficients.

Example: Applying the Common Factor Theorem

In this section we show how the Common Factor Theorem generally leads to a more efficient computation of results than repeated application of the Common Factor Axiom, particularly when done manually, since there are fewer terms in the Common Factor Theorem expansion, and many are evidently zero by inspection.

Again, we take the problem of finding the 1-element common to two 2-elements in a 3-space. This time, however, we take a numerical example and suppose that we know the factors of at least one of the 2-elements. Let:

$$\xi_1 == 3\, e_1 \wedge e_2 + 2\, e_1 \wedge e_3 + 3\, e_2 \wedge e_3$$
$$\xi_2 == (5\, e_2 + 7\, e_3) \wedge e_1$$

We keep ξ_1 fixed initially and apply the Common Factor Theorem to rewrite the regressive product as a sum of the two products, each due to one of the essentially different rearrangements of ξ_2 :

$$\xi_1 \vee \xi_2 == (\xi_1 \wedge e_1) \vee (5\, e_2 + 7\, e_3) - (\xi_1 \wedge (5\, e_2 + 7\, e_3)) \vee e_1$$

The next step is to expand out the exterior products with ξ_1 . Often this step can be done by inspection, since all products with a repeated basis element factor will be zero.

$$\xi_1 \vee \xi_2 == (3\, e_2 \wedge e_3 \wedge e_1) \vee (5\, e_2 + 7\, e_3) -$$
$$(10\, e_1 \wedge e_3 \wedge e_2 + 21\, e_1 \wedge e_2 \wedge e_3) \vee e_1$$

Factorizing out the 3-element $e_1 \wedge e_2 \wedge e_3$ then gives:

$$\xi_1 \vee \xi_2 == (e_1 \wedge e_2 \wedge e_3) \vee (-11\, e_1 + 15\, e_2 + 21\, e_3)$$
$$== \frac{1}{\mathbb{k}} (-11\, e_1 + 15\, e_2 + 21\, e_3) \equiv -11\, e_1 + 15\, e_2 + 21\, e_3$$

Thus all factors common to both ξ_1 and ξ_2 are congruent to $-11\, e_1 + 15\, e_2 + 21\, e_3$.

◆ Check using GrassmannSimplify

We can check that this result is correct by taking its exterior product with each of ξ_1 and ξ_2 . We should get zero in both cases, indicating that the factor determined is indeed common to both original 2-elements. Here we use `GrassmannSimplify` in its alias form $\mathcal{G}$.

$$\mathcal{G}[(3\, e_1 \wedge e_2 + 2\, e_1 \wedge e_3 + 3\, e_2 \wedge e_3) \wedge (-11\, e_1 + 15\, e_2 + 21\, e_3)]$$

0

$$\mathcal{G}[(5\, e_2 + 7\, e_3) \wedge e_1 \wedge (-11\, e_1 + 15\, e_2 + 21\, e_3)]$$

0

Automating the application of the Common Factor Theorem

The Common Factor Theorem is automatically applied to regressive products by `GrassmannSimplify`. In the example of the previous section we could therefore have used `GrassmannSimplify` directly to determine the common factor.

$$\mathbf{X = \mathcal{G}[(3\, e_1 \wedge e_2 + 2\, e_1 \wedge e_3 + 3\, e_2 \wedge e_3) \vee (5\, e_2 + 7\, e_3) \wedge e_1]}$$

$$e_1 \wedge e_2 \wedge e_3 \vee (-11\, e_1 + 15\, e_2 + 21\, e_3)$$

If the resulting expression explicitly involves the basis *n*-element of the currently declared basis we can use formula 3.10: $\underset{n}{1} == \mathbb{k}\;\; e_1 \wedge e_2 \wedge \cdots \wedge e_n$ to eliminate any regressive products of the basis *n*-element and replace them with scalar multiples of the congruence factor $\mathbb{k}$.

$$(e_1 \wedge e_2 \wedge \cdots \wedge e_n) \vee \underset{m}{\alpha} == \frac{1}{\mathbb{k}}\, \underset{n}{1} \vee \underset{m}{\alpha} == \frac{1}{\mathbb{k}}\, \underset{m}{\alpha}$$

We can automate this step by using the *GrassmannAlgebra* function `ToCongruenceForm`:

```
ToCongruenceForm[X]
```

$$\frac{-11\, e_1 + 15\, e_2 + 21\, e_3}{\mathbb{k}}$$

Example: Multiple regressive products

Some expressions may simplify to a form in which there are regressive products of multiple copies of the basis *n*-element. For example, the regressive product of three bivectors in a 3-space is a scalar. To compute this scalar we first create three general bivectors **X**, **Y**, and **Z**, by using `CreateBasisForm`:

```
{X, Y, Z} = CreateBasisForm[2, {x, y, z}]
```

$$\{x_1\, e_1 \wedge e_2 + x_2\, e_1 \wedge e_3 + x_3\, e_2 \wedge e_3,\; y_1\, e_1 \wedge e_2 + y_2\, e_1 \wedge e_3 + y_3\, e_2 \wedge e_3,\; z_1\, e_1 \wedge e_2 + z_2\, e_1 \wedge e_3 + z_3\, e_2 \wedge e_3\}$$

Next, we compute the regressive product **R**, say, of these three bivectors by using `GrassmannSimplify`.

$$\mathbf{R = \mathcal{G}[X \vee Y \vee Z]}$$

$$(x_3\, (-y_2\, z_1 + y_1\, z_2) + x_2\, (y_3\, z_1 - y_1\, z_3) + x_1\, (-y_3\, z_2 + y_2\, z_3))\; e_1 \wedge e_2 \wedge e_3 \vee e_1 \wedge e_2 \wedge e_3 \vee 1$$

We can convert this expression into one which more clearly portrays its scalar nature by using `ToCongruenceForm`.

```
ToCongruenceForm[R]
```

$$\frac{x_3\, (-y_2\, z_1 + y_1\, z_2) + x_2\, (y_3\, z_1 - y_1\, z_3) + x_1\, (-y_3\, z_2 + y_2\, z_3)}{\mathbb{k}^2}$$

Or, we could have used `ToCongruenceForm` on the original regressive product to give the same result.

```
ToCongruenceForm[X ∨ Y ∨ Z]
```

$$\frac{x_3\,(-y_2\,z_1+y_1\,z_2)+x_2\,(y_3\,z_1-y_1\,z_3)+x_1\,(-y_3\,z_2+y_2\,z_3)}{\mathbb{k}^2}$$

We see that this scalar is in fact proportional to the determinant of the coefficients of the bivectors. Thus, if the bivectors are not linearly independent, their regressive product will be zero.

3.7 The Regressive Product of Simple Elements

The regressive product of simple elements

Consider the regressive product $\underset{m}{\alpha}\vee\underset{k}{\beta}$, where $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are simple, $m+k \geq n$. The 1-element factors of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ must then have a common subspace of dimension $m+k-n = p$. Let $\underset{p}{\gamma}$ be a simple p-element which spans this common subspace. We can then write:

$$\underset{m}{\alpha}\vee\underset{k}{\beta} == \left(\underset{m-p}{\alpha}\wedge\underset{p}{\gamma}\right)\vee\left(\underset{k-p}{\beta}\wedge\underset{p}{\gamma}\right) == \left(\underset{m-p}{\alpha}\wedge\underset{k-p}{\beta}\wedge\underset{p}{\gamma}\right)\vee\underset{p}{\gamma} \equiv \underset{p}{\gamma}$$

The Common Factor Axiom then assures us that since $\underset{p}{\gamma}$ is simple, then so is the original product of simple elements $\underset{m}{\alpha}\vee\underset{k}{\beta}$.

In sum: The regressive product of simple elements is simple.

The regressive product of (n–1)-elements

Since we have shown in Chapter 2: The Exterior Product that all (n–1)-elements are simple, and in the previous subsection that the regressive product of simple elements is simple, it follows immediately that the regressive product of any number of (n–1)-elements is simple.

Example: The regressive product of two 3-elements in a 4-space is simple

As an example of the foregoing result we calculate the regressive product of two 3-elements in a 4-space. We begin by declaring a 4-space and creating two general 3-elements.

```
DeclareBasis[4]
```

$\{e_1, e_2, e_3, e_4\}$

```
X = CreateBasisForm[3, x]
```

$x_1\,e_1\wedge e_2\wedge e_3 + x_2\,e_1\wedge e_2\wedge e_4 + x_3\,e_1\wedge e_3\wedge e_4 + x_4\,e_2\wedge e_3\wedge e_4$

```
Y = CreateBasisForm[3, y]
```

$y_1\, e_1 \wedge e_2 \wedge e_3 + y_2\, e_1 \wedge e_2 \wedge e_4 + y_3\, e_1 \wedge e_3 \wedge e_4 + y_4\, e_2 \wedge e_3 \wedge e_4$

Note that the *GrassmannAlgebra* function `CreateBasisForm` automatically declares the coefficients of the form x_i and y_i to be scalars by declaring the patterns `x_` and `y_` to be scalars. We can check this by entering **`Scalars`** and observing that the patterns have been added to the default list of scalars.

```
Scalars
```

$\{a, b, c, d, e, f, g, h, \mathbb{k}, (\_ \ominus \_)\,?\text{InnerProductQ}, x\_, y\_, \underset{0}{\_}\}$

Next, we calculate the regressive product of these two 3-elements. We can use `GrassmannSimplify` since it will convert regressive products of basis elements.

```
Z = 𝒢[X ∨ Y]
```

$$-\frac{(x_2 y_1 - x_1 y_2)\, e_1 \wedge e_2}{\mathbb{k}} - \frac{(x_3 y_1 - x_1 y_3)\, e_1 \wedge e_3}{\mathbb{k}} - \frac{(x_3 y_2 - x_2 y_3)\, e_1 \wedge e_4}{\mathbb{k}} - \frac{(x_4 y_1 - x_1 y_4)\, e_2 \wedge e_3}{\mathbb{k}} - \frac{(x_4 y_2 - x_2 y_4)\, e_2 \wedge e_4}{\mathbb{k}} - \frac{(x_4 y_3 - x_3 y_4)\, e_3 \wedge e_4}{\mathbb{k}}$$

We want to extract the coefficients of these terms. Here we use the *GrassmannAlgebra* function `ExtractFlesh`.

```
{A1, A2, A3, A4, A5, A6} = Z // ExtractFlesh
```

$$\left\{\frac{-x_2 y_1 + x_1 y_2}{\mathbb{k}}, \frac{-x_3 y_1 + x_1 y_3}{\mathbb{k}}, \frac{-x_3 y_2 + x_2 y_3}{\mathbb{k}}, \frac{-x_4 y_1 + x_1 y_4}{\mathbb{k}}, \frac{-x_4 y_2 + x_2 y_4}{\mathbb{k}}, \frac{-x_4 y_3 + x_3 y_4}{\mathbb{k}}\right\}$$

Finally we substitute these coefficients into the constraint equation for simplicity of 2-elements in a 4-space discussed in Section 2.10, Chapter 2, to show that the constraint is satisfied.

```
Simplify[A3 A4 - A2 A5 + A1 A6 == 0]
```

```
True
```

The cobasis form of the Common Factor Axiom

The Common Factor Axiom has a significantly suggestive form when written in terms of cobasis elements. This form will later help us extend the definition of interior product to arbitrary elements.

We start with three basis elements whose exterior product is equal to the basis n-element.

$$\underset{m}{e_i} \wedge \underset{k}{e_j} \wedge \underset{p}{e_s} == e_1 \wedge e_2 \wedge \cdots \wedge e_n == \underline{1} == \frac{1}{\mathbb{k}} \underset{n}{1}$$

The Common Factor Axiom can be written for these basis elements as:

$$\left(\underset{m}{e_i}\wedge\underset{p}{e_s}\right)\vee\left(\underset{k}{e_j}\wedge\underset{p}{e_s}\right) == \left(\underset{m}{e_i}\wedge\underset{k}{e_j}\wedge\underset{p}{e_s}\right)\vee\underset{p}{e_s}, \qquad m+k+p==n \tag{3.33}$$

From the definition of cobasis elements we have that:

$$\underset{m}{e_i}\wedge\underset{p}{e_s} == (-1)^{m\,k}\,\underline{\underset{k}{e_j}}$$

$$\underset{k}{e_j}\wedge\underset{p}{e_s} == \underline{\underset{m}{e_i}}$$

$$\underset{p}{e_s} == \underline{\underset{m}{e_i}\wedge\underset{k}{e_j}}$$

$$\underset{m}{e_i}\wedge\underset{k}{e_j}\wedge\underset{p}{e_s} == \underline{1}$$

Substituting these four elements into the Common Factor Axiom above gives:

$$(-1)^{m\,k}\,\underline{\underset{k}{e_j}}\vee\underline{\underset{m}{e_i}} == \underline{1}\vee\left(\underline{\underset{m}{e_i}\wedge\underset{k}{e_j}}\right)$$

Or, more symmetrically, by interchanging the first two factors:

$$\underline{\underset{m}{e_i}}\vee\underline{\underset{k}{e_j}} == \underline{1}\vee\left(\underline{\underset{m}{e_i}\wedge\underset{k}{e_j}}\right) == \frac{1}{\Bbbk}\,\underline{\underset{m}{e_i}\wedge\underset{k}{e_j}} \tag{3.34}$$

It can be seen that in this form the Common Factor Axiom does not specifically display the common factor, and indeed remains valid for all basis elements, independent of their grades.

In sum: Given any two basis elements, the cobasis element of their exterior product is congruent to the regressive product of their cobasis elements.

The regressive product of cobasis elements

In Chapter 5: The Complement, we will have cause to calculate the regressive product of cobasis elements. From the formula derived below we will have an instance of the fact that the regressive product of $(n-1)$-elements is simple, and we will determine that simple element.

First, consider basis elements e_1 and e_2 of an n-space and their cobasis elements $\underline{e_1}$ and $\underline{e_2}$. The regressive product of $\underline{e_1}$ and $\underline{e_2}$ is given by:

$$\underline{e_1}\vee\underline{e_2} == (e_2\wedge e_3\wedge\cdots\wedge e_n)\vee(-\,e_1\wedge e_3\wedge\cdots\wedge e_n)$$

Applying the Common Factor Axiom enables us to write:

$$\underline{e_1}\vee\underline{e_2} == (e_1\wedge e_2\wedge e_3\wedge\cdots\wedge e_n)\vee(\,e_3\wedge\cdots\wedge e_n)$$

We can write this either in the form already derived in the section above:

$$\underline{e_1} \vee \underline{e_2} == \underline{1} \vee (\underline{e_1 \wedge e_2})$$

or, by writing $\mathbb{k}\,\underline{1} == \underset{n}{1}$ as:

$$\underline{e_1} \vee \underline{e_2} == \frac{1}{\mathbb{k}} (\underline{e_1 \wedge e_2})$$

Taking the regressive product of this equation with $\underline{e_3}$ gives:

$$\underline{e_1} \vee \underline{e_2} \vee \underline{e_3} == \frac{1}{\mathbb{k}} (\underline{e_1 \wedge e_2}) \vee \underline{e_3} ==$$
$$\frac{1}{\mathbb{k}}\,\underline{1} \vee (\underline{e_1 \wedge e_2 \wedge e_3}) == \frac{1}{\mathbb{k}^2} (\underline{e_1 \wedge e_2 \wedge e_3})$$

Continuing this process, we arrive finally at the result:

$$\boxed{\underline{e_1} \vee \underline{e_2} \vee \cdots \vee \underline{e_m} == \frac{1}{\mathbb{k}^{m-1}} (\underline{e_1 \wedge e_2 \wedge \cdots \wedge e_m})} \qquad 3.35$$

A special case which we will have occasion to use in Chapter 5 is where the result reduces to a 1-element.

$$\boxed{(-1)^{j-1}\,\underline{e_1} \vee \underline{e_2} \vee \cdots \vee \square_j \vee \cdots \vee \underline{e_n} == \frac{1}{\mathbb{k}^{n-2}} (-1)^{n-1}\,e_j} \qquad 3.36$$

In sum: The regressive product of cobasis elements of basis 1-elements is congruent to the cobasis element of their exterior product.

In fact, this formula is just an instance of a more general result which says that: *The regressive product of cobasis elements of any grade is congruent to the cobasis element of their exterior product*. We will discuss a result very similar to this in more detail after we have defined the complement of an element in Chapter 5.

3.8 Factorization of Simple Elements

Factorization using the regressive product

The Common Factor Axiom asserts that in an *n*-space, the regressive product of a simple *m*-element with a simple (*n*–*m*+1)-element will give either zero, or a 1-element belonging to them both, and hence a factor of the *m*-element. If *m* such factors can be obtained which are independent, their product will therefore constitute (apart from a scalar factor easily determined) a factorization of the *m*-element.

Let $\underset{m}{\alpha}$ be the simple element to be factorized. Choose first an (*n*–*m*)-element $\underset{n-m}{\beta}$ whose product with $\underset{m}{\alpha}$ is non-zero. Next, choose a set of *m* independent 1-elements β_j whose products with $\underset{n-m}{\beta}$ are also non-zero.

A factorization α_j of $\underset{m}{\alpha}$ is then obtained from:

$$\underset{m}{\alpha} == \mathbf{a}\, \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \qquad\qquad \alpha_j == \underset{m}{\alpha} \vee \left(\beta_j \wedge \underset{n-m}{\beta}\right) \tag{3.37}$$

The scalar factor $\mathbf{a}$ may be determined simply by equating any two corresponding terms of the original element and the factorized version.

Note that no factorization is unique. Had different β_j been chosen, a different factorization would have been obtained. Nevertheless, any one factorization may be obtained from any other by adding multiples of the factors to each factor.

If an element is simple, then the exterior product of the element with itself is zero. The converse, however, is not true in general for elements of grade higher than 2, for it only requires the element to have just one simple factor to make the product with itself zero.

If the method is applied to the factorization of a non-simple element, the result will still be a simple element. Thus an element may be tested to see if it is simple by applying the method of this section: if the factorization is not equivalent to the original element, the hypothesis of simplicity has been violated.

■ Example: Factorization of a simple 2-element

Suppose we have a 2-element $\underset{2}{\alpha}$ which we wish to show is simple and which we wish to factorize.

$$\underset{2}{\alpha} == \mathbf{v} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{x} + \mathbf{v} \wedge \mathbf{y} + \mathbf{v} \wedge \mathbf{z} + \mathbf{w} \wedge \mathbf{z} + \mathbf{x} \wedge \mathbf{z} + \mathbf{y} \wedge \mathbf{z}$$

There are 5 independent 1-elements in the expression for $\underset{2}{\alpha}$: $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, hence we can choose *n* to be 5. Next, we choose $\underset{n-m}{\beta}$ (= $\underset{3}{\beta}$) arbitrarily as $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$, β_1 as $\mathbf{v}$, and β_2 as $\mathbf{w}$. Our two factors then become:

$$\alpha_1 \equiv \underset{2}{\alpha} \vee \left(\beta_1 \wedge \underset{3}{\beta}\right) \equiv \underset{2}{\alpha} \vee (\mathbf{v} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z})$$

$$\alpha_2 \equiv \underset{2}{\alpha} \vee \left(\beta_2 \wedge \underset{3}{\beta}\right) \equiv \underset{2}{\alpha} \vee (\mathbf{w} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z})$$

In determining α_1 the Common Factor Theorem permits us to write for arbitrary 1-elements ξ and ψ:

$$(\xi \wedge \psi) \vee (\mathbf{v} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) == (\xi \wedge \mathbf{v} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) \vee \psi - (\psi \wedge \mathbf{v} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) \vee \xi$$

Applying this to each of the terms of $\underset{2}{\alpha}$ gives:

$$\alpha_1 \equiv -(\mathbf{w} \wedge \mathbf{v} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) \vee \mathbf{v} + (\mathbf{w} \wedge \mathbf{v} \wedge \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) \vee \mathbf{z} \equiv \mathbf{v} - \mathbf{z}$$

Similarly for α_2 we have the same formula, except that $\mathbf{w}$ replaces $\mathbf{v}$.

$$(\xi \wedge \psi) \vee (w \wedge x \wedge y \wedge z) == (\xi \wedge w \wedge x \wedge y \wedge z) \vee \psi - (\psi \wedge w \wedge x \wedge y \wedge z) \vee \xi$$

Again applying this to each of the terms of $\underset{2}{\alpha}$ gives:

$$\alpha_2 \equiv (v \wedge w \wedge x \wedge y \wedge z) \vee w + (v \wedge w \wedge x \wedge y \wedge z) \vee x + \\ (v \wedge w \wedge x \wedge y \wedge z) \vee y + (v \wedge w \wedge x \wedge y \wedge z) \vee z \equiv w + x + y + z$$

Hence the factorization is congruent to:

$$\alpha_1 \wedge \alpha_2 \equiv (v - z) \wedge (w + x + y + z)$$

Verification by expansion of this product shows that this is indeed the factorization required.

Factorizing elements expressed in terms of basis elements

We now take the special case where the element to be factorized is expressed in terms of basis elements. This will enable us to develop formulae from which we can write down the factorization of an element (almost) by inspection.

The development is most clearly apparent from a specific example, but one that is general enough to cover the general concept. Consider a 3-element in a 5-space, where we suppose the coefficients to be such as to ensure the simplicity of the element.

$$\underset{3}{\alpha} == a_1\, e_1 \wedge e_2 \wedge e_3 + a_2\, e_1 \wedge e_2 \wedge e_4 + a_3\, e_1 \wedge e_2 \wedge e_5 \\ + a_4\, e_1 \wedge e_3 \wedge e_4 + a_5\, e_1 \wedge e_3 \wedge e_5 + a_6\, e_1 \wedge e_4 \wedge e_5 \\ + a_7\, e_2 \wedge e_3 \wedge e_4 + a_8\, e_2 \wedge e_3 \wedge e_5 + a_9\, e_2 \wedge e_4 \wedge e_5 \\ + a_{10}\, e_3 \wedge e_4 \wedge e_5$$

Choose $\beta_1 == e_1$, $\beta_2 == e_2$, $\beta_3 == e_3$, $\underset{2}{\beta} == e_4 \wedge e_5$, then we can write each of the factors $\beta_i \wedge \underset{2}{\beta}$ as a cobasis element.

$$\beta_1 \wedge \underset{2}{\beta} == e_1 \wedge e_4 \wedge e_5 == \underline{e_2 \wedge e_3}$$

$$\beta_2 \wedge \underset{2}{\beta} == e_2 \wedge e_4 \wedge e_5 == -\underline{e_1 \wedge e_3}$$

$$\beta_3 \wedge \underset{2}{\beta} == e_3 \wedge e_4 \wedge e_5 == \underline{e_1 \wedge e_2}$$

Consider the cobasis element $\underline{e_2 \wedge e_3}$, and a typical term of $\underset{3}{\alpha} \vee \underline{e_2 \wedge e_3}$, which we write as $(a\, e_i \wedge e_j \wedge e_k) \vee \underline{e_2 \wedge e_3}$. The Common Factor Theorem tells us that this product is zero if $e_i \wedge e_j \wedge e_k$ does not contain $e_2 \wedge e_3$. We can thus simplify the product $\underset{3}{\alpha} \vee \underline{e_2 \wedge e_3}$ by dropping out the terms of $\underset{3}{\alpha}$ which do not contain $e_2 \wedge e_3$. Thus:

$$\underset{3}{\alpha} \vee \underline{e_2 \wedge e_3} == (a_1\, e_1 \wedge e_2 \wedge e_3 + a_7\, e_2 \wedge e_3 \wedge e_4 + a_8\, e_2 \wedge e_3 \wedge e_5) \vee \underline{e_2 \wedge e_3}$$

Furthermore, the Common Factor Theorem applied to a typical term $(e_2 \wedge e_3 \wedge e_i) \vee \underline{e_2 \wedge e_3}$ of the expansion in which *both* $e_2 \wedge e_3$ and $\underline{e_2 \wedge e_3}$ occur yields a 1-element congruent to the remaining basis 1-element e_i in the product. This effectively cancells out (up to congruence) the product $e_2 \wedge e_3$ from the original term.

$$(e_2 \wedge e_3 \wedge e_i) \vee \underline{e_2 \wedge e_3} == ((e_2 \wedge e_3) \wedge (\underline{e_2 \wedge e_3})) \vee e_i \equiv e_i$$

Thus we can further reduce $\underset{3}{\alpha} \vee \underline{e_2 \wedge e_3}$ to give:

$$\alpha_1 \equiv \underset{3}{\alpha} \vee \underline{e_2 \wedge e_3} \equiv a_1\, e_1 + a_7\, e_4 + a_8\, e_5$$

Similarly we can determine other factors:

$$\alpha_2 \equiv \underset{3}{\alpha} \vee -\underline{e_1 \wedge e_3} \equiv a_1\, e_2 - a_4\, e_4 - a_5\, e_5$$

$$\alpha_3 \equiv \underset{3}{\alpha} \vee \underline{e_1 \wedge e_2} \equiv a_1\, e_3 + a_2\, e_4 + a_3\, e_5$$

It is clear from inspecting the product of the first terms in each 1-element that the product requires a scalar divisor of a_1^2. The final result is then:

$$\underset{3}{\alpha} == \frac{1}{a_1{}^2}\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3$$

$$== \frac{1}{a_1{}^2}\, (a_1\, e_1 + a_7\, e_4 + a_8\, e_5) \wedge$$
$$(a_1\, e_2 - a_4\, e_4 - a_5\, e_5) \wedge (a_1\, e_3 + a_2\, e_4 + a_3\, e_5)$$

Verification and derivation of conditions for simplicity

We verify the factorization by multiplying out the factors and comparing the result with the original expression. When we do this we obtain a result which still requires some conditions to be met: those ensuring the original element is simple.

First we declare a 5-dimensional basis and create a basis form for $\underset{3}{\alpha}$. Using `CreateBasisForm` automatically declares all the coefficients to be scalar.

$$\mathbb{V}_5;\ \underset{3}{\alpha} = \mathbf{CreateBasisForm[3, a]}$$

$$a_1\, e_1 \wedge e_2 \wedge e_3 + a_2\, e_1 \wedge e_2 \wedge e_4 + a_3\, e_1 \wedge e_2 \wedge e_5 +$$
$$a_4\, e_1 \wedge e_3 \wedge e_4 + a_5\, e_1 \wedge e_3 \wedge e_5 + a_6\, e_1 \wedge e_4 \wedge e_5 +$$
$$a_7\, e_2 \wedge e_3 \wedge e_4 + a_8\, e_2 \wedge e_3 \wedge e_5 + a_9\, e_2 \wedge e_4 \wedge e_5 + a_{10}\, e_3 \wedge e_4 \wedge e_5$$

To effect the multiplication of the factored form we use `GrassmannSimplify` (in its shorthand form).

$$\underset{3}{A} = \mathcal{G}\left[\frac{1}{a_1{}^2}\ (a_1\ e_1 + a_7\ e_4 + a_8\ e_5) \wedge (a_1\ e_2 - a_4\ e_4 - a_5\ e_5) \wedge (a_1\ e_3 + a_2\ e_4 + a_3\ e_5)\right]$$

$$a_1\ e_1 \wedge e_2 \wedge e_3 + a_2\ e_1 \wedge e_2 \wedge e_4 + a_3\ e_1 \wedge e_2 \wedge e_5 + a_4\ e_1 \wedge e_3 \wedge e_4 + a_5\ e_1 \wedge e_3 \wedge e_5 + \frac{(-a_3\ a_4 + a_2\ a_5)\ e_1 \wedge e_4 \wedge e_5}{a_1} + a_7\ e_2 \wedge e_3 \wedge e_4 + a_8\ e_2 \wedge e_3 \wedge e_5 + \frac{(-a_3\ a_7 + a_2\ a_8)\ e_2 \wedge e_4 \wedge e_5}{a_1} + \frac{(-a_5\ a_7 + a_4\ a_8)\ e_3 \wedge e_4 \wedge e_5}{a_1}$$

For this expression to be congruent to the original expression we clearly need to apply the simplicity conditions for a general 3-element in a 5-space. Once this is done we retrieve the original 3-element $\underset{3}{\alpha}$ with which we began. Simplicity conditions will be discussed in the next section.

$$\underset{3}{A}\ /.\ \{a_2\ a_5 - a_3\ a_4 \to a_1\ a_6,\ a_2\ a_8 - a_3\ a_7 \to a_1\ a_9,\ a_4\ a_8 - a_5\ a_7 \to a_1\ a_{10}\}$$

$$a_1\ e_1 \wedge e_2 \wedge e_3 + a_2\ e_1 \wedge e_2 \wedge e_4 + a_3\ e_1 \wedge e_2 \wedge e_5 + a_4\ e_1 \wedge e_3 \wedge e_4 + a_5\ e_1 \wedge e_3 \wedge e_5 + a_6\ e_1 \wedge e_4 \wedge e_5 + a_7\ e_2 \wedge e_3 \wedge e_4 + a_8\ e_2 \wedge e_3 \wedge e_5 + a_9\ e_2 \wedge e_4 \wedge e_5 + a_{10}\ e_3 \wedge e_4 \wedge e_5$$

In sum: A 3-element in a 5-space of the form:

$$a_1\ e_1 \wedge e_2 \wedge e_3 + a_2\ e_1 \wedge e_2 \wedge e_4 + a_3\ e_1 \wedge e_2 \wedge e_5 + a_4\ e_1 \wedge e_3 \wedge e_4 + a_5\ e_1 \wedge e_3 \wedge e_5 + a_6\ e_1 \wedge e_4 \wedge e_5 + a_7\ e_2 \wedge e_3 \wedge e_4 + a_8\ e_2 \wedge e_3 \wedge e_5 + a_9\ e_2 \wedge e_4 \wedge e_5 + a_{10}\ e_3 \wedge e_4 \wedge e_5$$

whose coefficients are constrained by the relations:

$$a_2\ a_5 - a_3\ a_4 - a_1\ a_6 == 0$$
$$a_2\ a_8 - a_3\ a_7 - a_1\ a_9 == 0$$
$$a_4\ a_8 - a_5\ a_7 - a_1\ a_{10} == 0$$

is simple, and has a factorization of:

$$\frac{1}{a_1{}^2}\ (a_1\ e_1 + a_7\ e_4 + a_8\ e_5) \wedge (a_1\ e_2 - a_4\ e_4 - a_5\ e_5) \wedge (a_1\ e_3 + a_2\ e_4 + a_3\ e_5)$$

The factorization algorithm

In this section we take the development of the previous sections and extract an algorithm for writing down a factorization by inspection.

Suppose we have a *simple* *m*-element expressed as a sum of terms, each of which is of the form **a x∧y∧⋯∧z**, where **a** denotes a scalar and the **x**, **y**, ⋯,**z** are symbols denoting 1-element factors.

To obtain a 1-element factor of the *m*-element

- Select an (*m*–1)-element belonging to at least one of the terms, for example $y \wedge \cdots \wedge z$.
- Drop any terms not containing the selected (*m*–1)-element.
- Factor this (*m*–1)-element from the resulting expression and eliminate it.
- The 1-element remaining is a 1-element factor of the *m*-element.

To factorize the *m*-element

- Select an *m*-element belonging to at least one of the terms, for example $x \wedge y \wedge \cdots \wedge z$.
- Create *m* different (*m*–1)-elements by dropping a different 1-element factor each time. The sign of the result is not important, since a scalar factor will be determined in the last step.
- Obtain *m* independent 1-element factors corresponding to each of these (*m*–1)-elements.
- The original *m*-element is *congruent* to the exterior product of these 1-element factors.
- Compare this product to the original m-element to obtain the correct scalar factor and hence the final factorization.

Example 1: Factorizing a 2-element in a 4-space

Suppose we have a 2-element in a 4-space, and we wish to apply this algorithm to obtain a factorization. We have already seen that such an element is in general *not* simple. We may however use the preceding algorithm to obtain the simplicity conditions on the coefficients.

$$\underset{2}{\alpha} == a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4 + a_6\, e_3 \wedge e_4$$

- Select a 2-element belonging to at least one of the terms, say $e_1 \wedge e_2$.
- Drop e_2 to create e_1. Then drop e_1 to create e_2.

- Select e_1.
- Drop the terms $a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4 + a_6\, e_3 \wedge e_4$ since they do not contain e_1.
- Factor e_1 from $a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4$ and eliminate it to give factor α_1.

$$\alpha_1 == a_1\, e_2 + a_2\, e_3 + a_3\, e_4$$

- Select e_2.
- Drop the terms $a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_6\, e_3 \wedge e_4$ since they do not contain e_2.
- Factor e_2 from $a_1\, e_1 \wedge e_2 + a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4$ and eliminate it to give factor α_2.

$$\alpha_2 == -a_1\, e_1 + a_4\, e_3 + a_5\, e_4$$

- The exterior product of these 1-element factors is

$$\begin{aligned}\alpha_1 \wedge \alpha_2 &== (a_1\, e_2 + a_2\, e_3 + a_3\, e_4) \wedge (-a_1\, e_1 + a_4\, e_3 + a_5\, e_4)\\ &== a_1 \Big(a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 +\\ &\quad a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4 + \left(\frac{-a_3\, a_4 + a_2\, a_5}{a_1}\right) e_3 \wedge e_4\Big)\end{aligned}$$

- Comparing this product to the original 2-element $\underset{2}{\alpha}$ gives the final factorization as

$$\underset{2}{\alpha} == \frac{1}{a_1}\,(a_1\, e_2 + a_2\, e_3 + a_3\, e_4) \wedge (-a_1\, e_1 + a_4\, e_3 + a_5\, e_4)$$

Provided that the simplicity conditions on the coefficients are satisfied:

$$a_1\, a_6 + a_3\, a_4 - a_2\, a_5 == 0$$

In sum: A 2-element in a 4-space may be factorized if and only if a condition on the coefficients is satisfied.

$$\begin{gathered} a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + \\ a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4 + a_6\, e_3 \wedge e_4 \\ == \frac{1}{a_1} (a_1\, e_1 - a_4\, e_3 - a_5\, e_4) \wedge (a_1\, e_2 + a_2\, e_3 + a_3\, e_4), \\ \Longleftrightarrow \quad a_3\, a_4 - a_2\, a_5 + a_1\, a_6 == 0 \end{gathered} \tag{3.38}$$

Example 2: Factorizing a 3-element in a 5-space

This time we have a numerical example of a 3-element in 5-space. We wish to determine if the element is simple, and if so, to obtain a factorization of it. To achieve this, we first *assume* that it is simple, and apply the factorization algorithm. We will verify its simplicity (or its non-simplicity) by comparing the results of this process to the original element.

$$\begin{aligned} \underset{3}{\alpha} == & -3\, e_1 \wedge e_2 \wedge e_3 - 4\, e_1 \wedge e_2 \wedge e_4 + \\ & 12\, e_1 \wedge e_2 \wedge e_5 + 3\, e_1 \wedge e_3 \wedge e_4 - 3\, e_1 \wedge e_3 \wedge e_5 + \\ & 8\, e_1 \wedge e_4 \wedge e_5 + 6\, e_2 \wedge e_3 \wedge e_4 - 18\, e_2 \wedge e_3 \wedge e_5 + 12\, e_3 \wedge e_4 \wedge e_5 \end{aligned}$$

- Select a 3-element belonging to at least one of the terms, say $e_1 \wedge e_2 \wedge e_3$.
- Drop e_1 to create $e_2 \wedge e_3$. Drop e_2 to create $e_1 \wedge e_3$. Drop e_3 to create $e_1 \wedge e_2$.

- Select $e_2 \wedge e_3$.
- Drop the terms not containing it, factor it from the remainder, and eliminate it to give α_1.

$$\alpha_1 \equiv -3\, e_1 + 6\, e_4 - 18\, e_5 \equiv -e_1 + 2\, e_4 - 6\, e_5$$

- Select $e_1 \wedge e_3$.
- Drop the terms not containing it, factor it from the remainder, and eliminate it to give α_2.

$$\alpha_2 \equiv 3\, e_2 + 3\, e_4 - 3\, e_5 \equiv e_2 + e_4 - e_5$$

- Select $e_1 \wedge e_2$.
- Drop the terms not containing it, factor it from the remainder, and eliminate it to give α_3.

$$\alpha_3 \equiv -3\, e_3 - 4\, e_4 + 12\, e_5$$

- The exterior product of these 1-element factors is:

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 == (-e_1 + 2\, e_4 - 6\, e_5) \wedge (e_2 + e_4 - e_5) \wedge (-3\, e_3 - 4\, e_4 + 12\, e_5)$$

Multiplying this out (here using `GrassmannSimplify`) gives:

$$\mathcal{G}[(-e_1 + 2 e_4 - 6 e_5) \wedge (e_2 + e_4 - e_5) \wedge (-3 e_3 - 4 e_4 + 12 e_5)]$$

$$3 e_1 \wedge e_2 \wedge e_3 + 4 e_1 \wedge e_2 \wedge e_4 - 12 e_1 \wedge e_2 \wedge e_5 - 3 e_1 \wedge e_3 \wedge e_4 + 3 e_1 \wedge e_3 \wedge e_5 - 8 e_1 \wedge e_4 \wedge e_5 - 6 e_2 \wedge e_3 \wedge e_4 + 18 e_2 \wedge e_3 \wedge e_5 - 12 e_3 \wedge e_4 \wedge e_5$$

- Comparing this product to the original 3-element $\underset{3}{\alpha}$ verifies a final factorization as:

$$\underset{3}{\alpha} == (e_1 - 2 e_4 + 6 e_5) \wedge (e_2 + e_4 - e_5) \wedge (-3 e_3 - 4 e_4 + 12 e_5)$$

This factorization is, of course, not unique. For example, a slightly simpler factorization could be obtained by subtracting twice the first factor from the third factor to obtain:

$$\underset{3}{\alpha} == (e_1 - 2 e_4 + 6 e_5) \wedge (e_2 + e_4 - e_5) \wedge (-3 e_3 - 2 e_1)$$

Factorization of (*n*–1)-elements

The foregoing method may be used to prove constructively that any (*n*–1)-element is simple by obtaining a factorization and subsequently verifying its validity. Let the general (*n*–1)-element be:

$$\underset{n-1}{\alpha} == \sum_{i=1}^{n} a_i \underline{e_i}, \qquad a_1 \neq 0$$

where the $\underline{e_i}$ are the cobasis elements of the e_i.

Choose $\underset{1}{\beta} == e_1$ and $\beta_j == e_j$ and apply the Common Factor Theorem to obtain (for $j \neq 1$):

$$\underset{n-1}{\alpha} \vee (e_1 \wedge e_j) == \left(\underset{n-1}{\alpha} \wedge e_j\right) \vee e_1 - \left(\underset{n-1}{\alpha} \wedge e_1\right) \vee e_j$$

$$== \left(\sum_{i=1}^{n} a_i \underline{e_i} \wedge e_j\right) \vee e_1 - \left(\sum_{i=1}^{n} a_i \underline{e_i} \wedge e_1\right) \vee e_j$$

$$== (-1)^{n-1} \left(a_j \underset{n}{e} \vee e_1 - a_1 \underset{n}{e} \vee e_j\right)$$

$$== (-1)^{n-1} \underset{n}{e} \vee (a_j e_1 - a_1 e_j)$$

Factors of $\underset{n-1}{\alpha}$ are therefore of the form $a_j e_1 - a_1 e_j$, $j \neq 1$, so that $\underset{n-1}{\alpha}$ is congruent to:

$$\underset{n-1}{\alpha} \equiv (a_2 e_1 - a_1 e_2) \wedge (a_3 e_1 - a_1 e_3) \wedge \cdots \wedge (a_n e_1 - a_1 e_n)$$

The result may be summarized as follows:

$$\boxed{\begin{gathered} a_1 \underline{e_1} + a_2 \underline{e_2} + \cdots + a_n \underline{e_n} \\ \equiv (a_2 e_1 - a_1 e_2) \wedge (a_3 e_1 - a_1 e_3) \wedge \cdots \wedge (a_n e_1 - a_1 e_n), \\ a_1 \neq 0 \end{gathered}} \qquad 3.39$$

$$\sum_{i=1}^{n} a_i \, \underline{e_i} \equiv \bigwedge_{i=2}^{n} (a_i \, e_1 - a_1 \, e_i), \qquad a_1 \neq 0 \qquad 3.40$$

Example: Application to 4-space

We check the validity of the formula above in a 4-space. Without loss of generality, let $\mathbf{a_1}$ be unity.

$\mathcal{G}[e_2 \wedge e_3 \wedge e_4 - a_2 \, e_1 \wedge e_3 \wedge e_4 + a_3 \, e_1 \wedge e_2 \wedge e_4 - a_4 \, e_1 \wedge e_2 \wedge e_3 == -(a_2 \, e_1 - e_2) \wedge (a_3 \, e_1 - e_3) \wedge (a_4 \, e_1 - e_4)]$

```
True
```

Obtaining a factorization of a simple *m*-element

For simple *m*-elements

For *m*-elements known to be simple we can use the *GrassmannAlgebra* function `Factorize` to obtain a factorization. Note that the result obtained from `Factorize` will be only one factorization out of the generally infinitely many possibilities. However, every factorization may be obtained from any other by adding scalar multiples of each factor to the other factors.

We have shown above that every (*n*–1)-element in an *n*-space is simple. Applying `Factorize` to a series of general (*n*–1)-elements in spaces of 4, 5, and 6 dimensions corroborates the formula derived in the previous section.

In each case we first declare a basis of the requisite dimension by entering $\mathbb{V}_n$, and then create a general (*n*–1)-element **X** with scalars $\mathbf{a_i}$ using the *GrassmannAlgebra* function `CreateBasisForm`.

◆ A 3-element in a 4-space:

$\mathbb{V}_4$**; X = CreateBasisForm[3, a]**

$a_1 \, e_1 \wedge e_2 \wedge e_3 + a_2 \, e_1 \wedge e_2 \wedge e_4 + a_3 \, e_1 \wedge e_3 \wedge e_4 + a_4 \, e_2 \wedge e_3 \wedge e_4$

Factorize[X]

$$\frac{(a_1 \, e_3 + a_2 \, e_4) \wedge (-a_1 \, e_2 + a_3 \, e_4) \wedge (a_1 \, e_1 + a_4 \, e_4)}{a_1^2}$$

◆ A 4-element in a 5-space:

$\mathbb{V}_5$**; X = CreateBasisForm[4, a]**

$a_1 \, e_1 \wedge e_2 \wedge e_3 \wedge e_4 + a_2 \, e_1 \wedge e_2 \wedge e_3 \wedge e_5 + a_3 \, e_1 \wedge e_2 \wedge e_4 \wedge e_5 + a_4 \, e_1 \wedge e_3 \wedge e_4 \wedge e_5 + a_5 \, e_2 \wedge e_3 \wedge e_4 \wedge e_5$

```
Factorize[X]
```

$$\frac{(a_1\,e_4 + a_2\,e_5)\wedge(-a_1\,e_3 + a_3\,e_5)\wedge(a_1\,e_2 + a_4\,e_5)\wedge(-a_1\,e_1 + a_5\,e_5)}{a_1^3}$$

For non-simple *m*-elements

If `Factorize` is applied to an element which is not simple, the element will simply be returned.

```
Factorize[x ∧ y + u ∧ v]
```

$u\wedge v + x\wedge y$

For contingently simple *m*-elements

If `Factorize` is applied to an element which may be simple *conditional* on the values taken by some of its scalar coefficients, the conditional result will be returned.

◆ A 2-element in a 4-space:

```
𝕍4; X = CreateBasisForm[2, a]
```

$a_1\,e_1\wedge e_2 + a_2\,e_1\wedge e_3 + a_3\,e_1\wedge e_4 + a_4\,e_2\wedge e_3 + a_5\,e_2\wedge e_4 + a_6\,e_3\wedge e_4$

```
X1 = Factorize[X]
```

$$\mathrm{If}\Big[\frac{a_3\,a_4 - a_2\,a_5}{a_1} + a_6 == 0,\ \frac{(a_1\,e_2 + a_2\,e_3 + a_3\,e_4)\wedge(-a_1\,e_1 + a_4\,e_3 + a_5\,e_4)}{a_1},\ a_1\,e_1\wedge e_2 + a_2\,e_1\wedge e_3 + a_3\,e_1\wedge e_4 + a_4\,e_2\wedge e_3 + a_5\,e_2\wedge e_4 + a_6\,e_3\wedge e_4\Big]$$

The *Mathematica* `If` function has syntax `If[Q,T,F]`, where **Q** is a predicate, **T** is the result if **Q** is true, **F** is the result if **Q** is false. Hence the above may be read: *if* the simplicity condition is satisfied, *then* the factorization is as given, *else* the element is not factorizable.

If we are able to assert the condition required, that a_6 is equal to $-\frac{a_3\,a_4 - a_2\,a_5}{a_1}$, then the 2-element is indeed simple and the factorization is valid. Substituting this condition into the `If` statement, yields true for the predicate, hence the factorization is returned.

$$X_1\ /.\ a_6 \to -\frac{a_3\,a_4 - a_2\,a_5}{a_1}$$

$$\frac{(a_1\,e_2 + a_2\,e_3 + a_3\,e_4)\wedge(-a_1\,e_1 + a_4\,e_3 + a_5\,e_4)}{a_1}$$

◆ **A 2-element in a 5-space:**

```
𝕍5; X = CreateBasisForm[2, a]
```

$a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_4\, e_1 \wedge e_5 + a_5\, e_2 \wedge e_3 + a_6\, e_2 \wedge e_4 + a_7\, e_2 \wedge e_5 + a_8\, e_3 \wedge e_4 + a_9\, e_3 \wedge e_5 + a_{10}\, e_4 \wedge e_5$

```
X1 = Factorize[X]
```

$$\mathrm{If}\Big[\frac{a_3\, a_5 - a_2\, a_6}{a_1} + a_8 == 0\ \&\&\ \frac{a_4\, a_5 - a_2\, a_7}{a_1} + a_9 == 0\ \&\&\ \frac{a_4\, a_6 - a_3\, a_7}{a_1} + a_{10} == 0,\ \frac{(a_1\, e_2 + a_2\, e_3 + a_3\, e_4 + a_4\, e_5) \wedge (-a_1\, e_1 + a_5\, e_3 + a_6\, e_4 + a_7\, e_5)}{a_1},\ a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_4\, e_1 \wedge e_5 + a_5\, e_2 \wedge e_3 + a_6\, e_2 \wedge e_4 + a_7\, e_2 \wedge e_5 + a_8\, e_3 \wedge e_4 + a_9\, e_3 \wedge e_5 + a_{10}\, e_4 \wedge e_5\Big]$$

In this case of a 2-element in a 5-space, we have a predicate involving three conditions on the coefficients to satisfy before obtaining a simple factorizable element.

$$\mathbf{X_1\ /.\ \Big\{a_8 \to -\frac{a_3\, a_5 - a_2\, a_6}{a_1},\ a_9 \to -\frac{a_4\, a_5 - a_2\, a_7}{a_1},\ a_{10} \to -\frac{a_4\, a_6 - a_3\, a_7}{a_1}\Big\}}$$

$$\frac{(a_1\, e_2 + a_2\, e_3 + a_3\, e_4 + a_4\, e_5) \wedge (-a_1\, e_1 + a_5\, e_3 + a_6\, e_4 + a_7\, e_5)}{a_1}$$

Determining if an element is simple

To determine if an element is simple, we can use the *GrassmannAlgebra* function `SimpleQ`. If an element is simple `SimpleQ` returns `True`. If it is not simple, it returns `False`. If it may be simple conditional on the values taken by some of its scalar coefficients, `SimpleQ` returns the conditions required.

We take the same examples as we discussed in the section on factorization.

For simple *m*-elements

◆ **A 3-element in a 4-space:**

```
𝕍4; X = CreateBasisForm[3, a]
```

$a_1\, e_1 \wedge e_2 \wedge e_3 + a_2\, e_1 \wedge e_2 \wedge e_4 + a_3\, e_1 \wedge e_3 \wedge e_4 + a_4\, e_2 \wedge e_3 \wedge e_4$

```
SimpleQ[X]
```

```
True
```

◆ **A 4-element in a 5-space:**

```
𝕍5; X = CreateBasisForm[4, a]
```

$a_1\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 + a_2\, e_1 \wedge e_2 \wedge e_3 \wedge e_5 + a_3\, e_1 \wedge e_2 \wedge e_4 \wedge e_5 + a_4\, e_1 \wedge e_3 \wedge e_4 \wedge e_5 + a_5\, e_2 \wedge e_3 \wedge e_4 \wedge e_5$

```
SimpleQ[X]
```

True

For non-simple *m*-elements

If `SimpleQ` is applied to an element which is not simple, the element will simply be returned.

```
SimpleQ[x ∧ y + u ∧ v]
```

False

For contingently simple *m*-elements

If `SimpleQ` is applied to an element which may be simple *conditional* on the values taken by some of its scalar coefficients, the conditional result will be returned.

◆ **A 2-element in a 4-space:**

```
𝕍4; X = CreateBasisForm[2, a]
```

$a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_4\, e_2 \wedge e_3 + a_5\, e_2 \wedge e_4 + a_6\, e_3 \wedge e_4$

```
SimpleQ[X]
```

$\frac{a_3\, a_4 - a_2\, a_5}{a_1} + a_6 == 0$

If we are able to assert the condition required, that $\frac{a_3\, a_4 - a_2\, a_5}{a_1} + a_6 == 0$, then the 2-element is indeed simple.

$\texttt{SimpleQ}\left[X\ /.\ a_6 \to -\frac{a_3\, a_4 - a_2\, a_5}{a_1}\right]$

True

◆ **A 2-element in a 5-space:**

```
𝕍5; X = CreateBasisForm[2, a]
```

$a_1\, e_1 \wedge e_2 + a_2\, e_1 \wedge e_3 + a_3\, e_1 \wedge e_4 + a_4\, e_1 \wedge e_5 + a_5\, e_2 \wedge e_3 + a_6\, e_2 \wedge e_4 + a_7\, e_2 \wedge e_5 + a_8\, e_3 \wedge e_4 + a_9\, e_3 \wedge e_5 + a_{10}\, e_4 \wedge e_5$

```
SimpleQ[X]
```

$$\frac{a_3\,a_5 - a_2\,a_6}{a_1} + a_8 == 0\ \&\&\ \frac{a_4\,a_5 - a_2\,a_7}{a_1} + a_9 == 0\ \&\&\ \frac{a_4\,a_6 - a_3\,a_7}{a_1} + a_{10} == 0$$

In this case of a 2-element in a 5-space, we have three conditions on the coefficients to satisfy before being able to assert that the element is simple.

3.9 Product Formulae for Regressive Products

The Product Formula

The Common Factor Theorem forms the basis for many of the important formulae relating the various products of the Grassmann algebra. In this section, some of the more basic formulae which involve just the exterior and regressive products will be developed. These in turn will be shown in Chapter 6: The Interior Product to have their counterparts in terms of exterior and interior products. The formulae are usually directed at obtaining alternative expressions (or expansions) for an element, or for products of elements.

The first formula to be developed (which forms a basis for much of the rest) is an expansion for the regressive product of an (*n*–1)-element with the exterior product of two arbitrary elements. We call this (and its dual) the Product Formula.

Let $\underset{n-1}{x}$ be an (*n*–1)-element, then:

$$\boxed{\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \vee \underset{n-1}{x} == \left(\underset{m}{\alpha} \vee \underset{n-1}{x}\right) \wedge \underset{k}{\beta} + (-1)^m\, \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \vee \underset{n-1}{x}\right)} \qquad 3.41$$

To prove this, suppose initially that $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are simple and can be expressed as $\underset{m}{\alpha} == \alpha_1 \wedge \cdots \wedge \alpha_m$ and $\underset{k}{\beta} == \beta_1 \wedge \cdots \wedge \beta_k$. Applying the Common Factor Theorem gives

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \vee \underset{n-1}{x} == ((\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \cdots \wedge \beta_k)) \vee \underset{n-1}{x}$$

$$== \sum_{i=1}^{m} (-1)^{i-1} \left(\alpha_i \wedge \underset{n-1}{x}\right) \vee ((\alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \cdots \wedge \beta_k))$$

$$+ \sum_{j=1}^{k} (-1)^{m+j-1} \left(\beta_j \wedge \underset{n-1}{x}\right) \vee ((\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \cdots \wedge \square_j \wedge \cdots \wedge \beta_k))$$

$$== \left(\underset{m}{\alpha} \vee \underset{n-1}{x}\right) \wedge (\beta_1 \wedge \cdots \wedge \beta_k) + (-1)^m (\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge \left(\underset{k}{\beta} \vee \underset{n-1}{x}\right)$$

Here we have used formula 3.13.

This result may be extended in a straightforward manner to the case where $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are not simple: since a non-simple element may be expressed as the sum of simple terms, and the formula is valid for each term, then by addition it can be shown to be valid for the sum.

We can calculate the dual of this formula by applying the *GrassmannAlgebra* function `Dual`:

$$\mathbf{Dual}\left[\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\vee\underset{n-1}{x} == \left(\underset{m}{\alpha}\vee\underset{n-1}{x}\right)\wedge\underset{k}{\beta}+(-1)^{m}\,\underset{m}{\alpha}\wedge\left(\underset{k}{\beta}\vee\underset{n-1}{x}\right)\right]$$

$$\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge x == (-1)^{-m+n}\,\underset{m}{\alpha}\vee\underset{k}{\beta}\wedge x+\underset{m}{\alpha}\wedge x\vee\underset{k}{\beta}$$

which we rearrange slightly to make it more readable as:

$$\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge x == \left(\underset{m}{\alpha}\wedge x\right)\vee\underset{k}{\beta}+(-1)^{n-m}\,\underset{m}{\alpha}\vee\left(\underset{k}{\beta}\wedge x\right) \qquad 3.42$$

Note that here x is a 1-element.

The General Product Formula

If x is of a grade higher than 1, then similar relations hold, but with extra terms on the right-hand side. For example, if we replace x by $x_1\wedge x_2$ and note that:

$$\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge(x_1\wedge x_2) == \left(\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge x_1\right)\wedge x_2$$

then the right-hand side may be expanded by applying the Product Formula [3.42] successively to obtain:

$$\begin{aligned}\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge(x_1\wedge x_2) == {}&\left(\underset{m}{\alpha}\wedge(x_1\wedge x_2)\right)\vee\underset{k}{\beta}+\underset{m}{\alpha}\vee\left(\underset{k}{\beta}\wedge(x_1\wedge x_2)\right)\\&+(-1)^{n-m}\left(\left(\underset{m}{\alpha}\wedge x_2\right)\vee\left(\underset{k}{\beta}\wedge x_1\right)-\left(\underset{m}{\alpha}\wedge x_1\right)\vee\left(\underset{k}{\beta}\wedge x_2\right)\right)\end{aligned} \qquad 3.43$$

Continued reapplication of this process expresses for simple $\underset{p}{x}$ the expression $\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge\underset{p}{x}$ as the sum of 2^{p} terms.

$$\begin{gathered}\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge\underset{p}{x} == \sum_{r==0}^{p}(-1)^{r\,(n-m)}\sum_{i==1}^{\nu}\left(\underset{m}{\alpha}\wedge\underset{p-r}{x_i}\right)\vee\left(\underset{k}{\beta}\wedge\underset{r}{x_i}\right)\\ \underset{p}{x} == \underset{r}{x_1}\wedge\underset{p-r}{x_1} == \underset{r}{x_2}\wedge\underset{p-r}{x_2} == \cdots == \underset{r}{x_\nu}\wedge\underset{p-r}{x_\nu},\quad \nu == \binom{p}{r}\end{gathered} \qquad 3.44$$

We call formula 3.44 the General Product Formula.

The dual of the General Product Formula may be obtained following the duality algorithm, or by using the `Dual` function. Input to the `Dual` function requires the summations to be expressed in such a way as to frustrate *Mathematica* interpreting them as the `Sum` function. To make the formula more readable we have modified the actual dual by replacing *r* with *n–r*.

$$\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\vee\underset{p}{x} == \sum_{r==0}^{n-p}(-1)^{m\,r}\sum_{i==1}^{\nu}\left(\underset{m}{\alpha}\vee\underset{p+r}{x_i}\right)\wedge\left(\underset{k}{\beta}\vee\underset{n-r}{x_i}\right)$$

$$\underset{p}{x} == \underset{n-r}{x_1}\vee\underset{p+r}{x_1} == \underset{n-r}{x_2}\vee\underset{p+r}{x_2} == \cdots == \underset{n-r}{x_\nu}\vee\underset{p+r}{x_\nu}, \quad \nu == \binom{n-p}{r} \qquad 3.45$$

These formulae apply only for simple $\underset{p}{x}$, but may be extended to the non-simple case by application to each component of a non-simple $\underset{p}{x}$.

Exploring the General Product Formula

The development of the General Product Formula [3.44] from the Product Formula [3.42] is a good example of how *Mathematica* and *GrassmannAlgebra* may be used to explore and create new results. The code below is typical of how you can combine functions from *Mathematica* (for example Fold) with functions from *GrassmannAlgebra*.

```
ToProductFormula[(α_∨β_)∧x_,n_]:=
Module[{S,F},S=Scalars;DeclareExtraScalars[{Grade[α],n}];
  ExpandProductFormula[X_,y_]:=
  SimplifySigns[Expand[FactorScalars[ExpandProducts[X∧y]]
    /.((u_∨v_)∧z_:→(u∧z)∨v+(-1)^(n-Grade[α]) u∨(v∧z))]];
F=Fold[ExpandProductFormula,α∨β,List@@CreateVariableForm[x]];
DeclareScalars[S];F]
```

To get some immediate information from *Mathematica* on the syntax of functions to use, you can type in a question mark **?** followed by the name of the function.

```
? ToProductFormula
```

```
ToProductFormula[(α∨β)∧x,n] expands the product (α∨β)∧x in terms
  of the simple factors of x. The elements α and β may be
  expressed as graded symbols with symbolic or numeric grade.
  The element x must be either a simple element or a graded
  symbol with a numeric grade. The expansion is independent
  of the dimension of the currently declared space, but
  requires the specification of a symbol n for this dimension.
```

Example: The General Product Formula for a 3-element

We take the example of the General Product Formula for a 3-element.

```
ToProductFormula[(α ∨ β) ∧ x, n]
                  m   k     3
```

$$(-1)^{m+n}\,\underset{m}{\alpha}\vee\underset{k}{\beta}\wedge x_1\wedge x_2\wedge x_3+\underset{m}{\alpha}\wedge x_1\vee\underset{k}{\beta}\wedge x_2\wedge x_3+$$
$$\underset{m}{\alpha}\wedge x_2\vee\underset{k}{\beta}\wedge x_1\wedge x_3+\underset{m}{\alpha}\wedge x_3\vee\underset{k}{\beta}\wedge x_1\wedge x_2+$$
$$(-1)^{m+n}\,\underset{m}{\alpha}\wedge x_1\wedge x_2\vee\underset{k}{\beta}\wedge x_3+(-1)^{m+n}\,\underset{m}{\alpha}\wedge x_1\wedge x_3\vee\underset{k}{\beta}\wedge x_2+$$
$$(-1)^{m+n}\,\underset{m}{\alpha}\wedge x_2\wedge x_3\vee\underset{k}{\beta}\wedge x_1+\underset{m}{\alpha}\wedge x_1\wedge x_2\wedge x_3\vee\underset{k}{\beta}$$

This output is not particularly easy to read, as *Mathematica* adopts the precedence that the exterior product has over the regressive product, and hence does not show the parentheses. Replacing the parentheses and grouping terms manually gives:

$$\begin{aligned}&\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\wedge(x_1\wedge x_2\wedge x_3)\\
&= (-1)^{m+n}\,\underset{m}{\alpha}\vee\left(\underset{k}{\beta}\wedge x_1\wedge x_2\wedge x_3\right)\\
&+\left(\underset{m}{\alpha}\wedge x_1\right)\vee\left(\underset{k}{\beta}\wedge x_2\wedge x_3\right)+\\
&\left(\underset{m}{\alpha}\wedge x_2\right)\vee\left(\underset{k}{\beta}\wedge x_1\wedge x_3\right)+\left(\underset{m}{\alpha}\wedge x_3\right)\vee\left(\underset{k}{\beta}\wedge x_1\wedge x_2\right)\\
&+(-1)^{m+n}\left(\left(\underset{m}{\alpha}\wedge x_1\wedge x_2\right)\vee\left(\underset{k}{\beta}\wedge x_3\right)+\right.\\
&\left.\left(\underset{m}{\alpha}\wedge x_1\wedge x_3\right)\vee\left(\underset{k}{\beta}\wedge x_2\right)+\left(\underset{m}{\alpha}\wedge x_2\wedge x_3\right)\vee\left(\underset{k}{\beta}\wedge x_1\right)\right)\\
&+\left(\underset{m}{\alpha}\wedge x_1\wedge x_2\wedge x_3\right)\vee\\
&\underset{k}{\beta}\end{aligned}\qquad 3.46$$

The decomposition formula

In the General Product Formula [3.43], putting k equal to $n-m$ permits the left-hand side to be expressed as a scalar multiple of $\underset{p}{x}$ and hence expresses a type of 'decomposition' of $\underset{p}{x}$ into components.

$$\underset{p}{x}=\sum_{r=0}^{p}(-1)^{r\,(n-m)}\sum_{i=1}^{\nu}\frac{\left(\underset{m}{\alpha}\wedge\underset{p-r}{x_i}\right)\vee\left(\underset{n-m}{\beta}\wedge\underset{r}{x_i}\right)}{\left(\underset{m}{\alpha}\vee\underset{n-m}{\beta}\right)}\qquad 3.47$$

$$\underset{p}{x}=\underset{r}{x_1}\wedge\underset{p-r}{x_1}=\underset{r}{x_2}\wedge\underset{p-r}{x_2}=\cdots=\underset{r}{x_\nu}\wedge\underset{p-r}{x_\nu},\quad \nu=\binom{p}{r}$$

The first and last terms of this sum, those for which $r=0$ and $r=p$ are:

$$\underset{p}{x} == \frac{\left(\underset{m}{\alpha} \wedge \underset{p}{x}\right) \vee \underset{n-m}{\beta}}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} + \cdots + \frac{\underset{m}{\alpha} \vee \left(\underset{p}{x} \wedge \underset{n-m}{\beta}\right)}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} \qquad 3.48$$

If $p = 1$, there are just two terms, reducing the decomposition formula to:

$$x == \frac{\left(\underset{m}{\alpha} \wedge x\right) \vee \underset{n-m}{\beta}}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} + \frac{\underset{m}{\alpha} \vee \left(x \wedge \underset{n-m}{\beta}\right)}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} \qquad 3.49$$

In Chapter 4: Geometric Interpretations we will explore some of the geometric significance of these formulae, especially the decomposition formula. The decomposition formula will also find application in Chapter 6: The Interior Product.

Expressing an element in terms of another basis

The Common Factor Theorem may also be used to express an *m*-element in terms of another basis with basis *n*-element $\underset{n}{\beta}$ by expanding the product $\underset{n}{\beta} \vee \underset{m}{\alpha}$.

Let the new basis be $\{\beta_1, \cdots, \beta_n\}$ and let $\underset{n}{\beta} == \beta_1 \wedge \cdots \wedge \beta_n$, then the Common Factor Theorem permits us to write:

$$\underset{n}{\beta} \vee \underset{m}{\alpha} == \sum_{i=1}^{\nu} \left(\underset{n-m}{\beta_i} \wedge \underset{m}{\alpha}\right) \vee \underset{m}{\beta_i}$$

where $\nu = \binom{n}{m}$ and $\underset{n}{\beta} == \underset{n-m}{\beta_1} \wedge \underset{m}{\beta_1} == \underset{n-m}{\beta_2} \wedge \underset{m}{\beta_2} == \cdots == \underset{n-m}{\beta_\nu} \wedge \underset{m}{\beta_\nu}$

We can visualize how the formula operates by writing $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ as simple products and then exchanging the β_i with the α_i in all the essentially different ways possible whilst always retaining the original ordering. To make this more concrete, suppose *n* is 5 and *m* is 2:

$$\begin{aligned}
&(\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 \wedge \beta_5) \vee \alpha_1 \wedge \alpha_2 == \\
&\quad (\alpha_1 \wedge \alpha_2 \wedge \beta_3 \wedge \beta_4 \wedge \beta_5) \vee (\beta_1 \wedge \beta_2) \\
&\quad + (\alpha_1 \wedge \beta_2 \wedge \alpha_2 \wedge \beta_4 \wedge \beta_5) \vee (\beta_1 \wedge \beta_3) \\
&\quad + (\alpha_1 \wedge \beta_2 \wedge \beta_3 \wedge \alpha_2 \wedge \beta_5) \vee (\beta_1 \wedge \beta_4) \\
&\quad + (\alpha_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4 \wedge \alpha_2) \vee (\beta_1 \wedge \beta_5) \\
&\quad + (\beta_1 \wedge \alpha_1 \wedge \alpha_2 \wedge \beta_4 \wedge \beta_5) \vee (\beta_2 \wedge \beta_3) \\
&\quad + (\beta_1 \wedge \alpha_1 \wedge \beta_3 \wedge \alpha_2 \wedge \beta_5) \vee (\beta_2 \wedge \beta_4) \\
&\quad + (\beta_1 \wedge \alpha_1 \wedge \beta_3 \wedge \beta_4 \wedge \alpha_2) \vee (\beta_2 \wedge \beta_5) \\
&\quad + (\beta_1 \wedge \beta_2 \wedge \alpha_1 \wedge \alpha_2 \wedge \beta_5) \vee (\beta_3 \wedge \beta_4) \\
&\quad + (\beta_1 \wedge \beta_2 \wedge \alpha_1 \wedge \beta_4 \wedge \alpha_2) \vee (\beta_3 \wedge \beta_5) \\
&\quad + (\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \alpha_1 \wedge \alpha_2) \vee (\beta_4 \wedge \beta_5)
\end{aligned}$$

Now let $\left(\underset{n-m}{\beta_i} \wedge \underset{m}{\alpha}\right) == b_i \underset{n}{\beta}$, where the b_i are scalars, and substitute into the above equation to give the required expression for $\underset{m}{\alpha}$.

$$\underset{m}{\alpha} == \sum_{i=1}^{\nu} b_i \underset{m}{\beta_i} \qquad 3.50$$

Example: Expressing a 3-element in terms of another basis

Consider a 4-space with basis $\{e_1, e_2, e_3, e_4\}$ and let:

$$X = 2\, e_1 \wedge e_2 \wedge e_3 - 3\, e_1 \wedge e_3 \wedge e_4 + -2\, e_2 \wedge e_3 \wedge e_4$$

be a 3-element expressed in terms of this basis.

Suppose now that we have another basis related to the e basis by:

$$\epsilon_1 == 2\, e_1 + 3\, e_3$$
$$\epsilon_2 == 5\, e_3 - e_4$$
$$\epsilon_3 == e_1 - e_3$$
$$\epsilon_4 == e_1 + e_2 + e_3 + e_4$$

We wish to express X in terms of the new basis. Instead of inverting the transformation to find the e_i in terms of the ϵ_i, substituting for the e_i in X, and simplifying, we can almost write the result required by inspection.

$$X == (\epsilon_1 \wedge X) \vee (\epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4) - (\epsilon_2 \wedge X) \vee (\epsilon_1 \wedge \epsilon_3 \wedge \epsilon_4) + (\epsilon_3 \wedge X) \vee (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_4) - (\epsilon_4 \wedge X) \vee (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3)$$

$$== (5\, e_1 \wedge e_2 \wedge e_3 \wedge e_4) \vee (\epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4) - (2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4) \vee (\epsilon_1 \wedge \epsilon_3 \wedge \epsilon_4) + (-2\, e_1 \wedge e_2 \wedge e_3 \wedge e_4) \vee (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_4) - (-e_1 \wedge e_2 \wedge e_3 \wedge e_4) \vee (\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3)$$

But from the transformation we find that $\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 == 19\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$ whence X is:

$$X == \frac{5}{19}\, \epsilon_2 \wedge \epsilon_3 \wedge \epsilon_4 - \frac{2}{19}\, \epsilon_1 \wedge \epsilon_3 \wedge \epsilon_4 - \frac{2}{19}\, \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_4 + \frac{1}{19}\, \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$$

Expressing a 1-element in terms of another basis

For the special case of a 1-element, the scalar coefficients b_i in formula 3.49 can be written more explicitly, and without undue complexity, resulting in a decomposition of α in terms of n independent elements β_i equivalent to expressing α in the basis β_i.

$$\alpha == \sum_{i=1}^{n} \left(\frac{\beta_1 \wedge \beta_2 \wedge \cdots \wedge \alpha \wedge \cdots \wedge \beta_n}{\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_i \wedge \cdots \wedge \beta_n} \right) \beta_i \qquad 3.51$$

Here the denominators are the same in each term but are expressed this way to show the positioning of the factor α in the numerator.

The symmetric expansion of a 1-element in terms of another basis

For the case $m = 1$, $\underset{n}{\beta} \vee \alpha$ may be expanded by the Common Factor Theorem to give:

$$(\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n) \vee \alpha == \sum_{i=1}^{\vee} (-1)^{n-i} (\beta_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \beta_n \wedge \alpha) \vee \beta_i$$

or, in terms of the mnemonic expansion in the previous section:

$$\begin{aligned}
&(\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \cdots \wedge \beta_n) \vee \alpha \\
&== (\alpha \wedge \beta_2 \wedge \beta_3 \wedge \cdots \wedge \beta_n) \vee \beta_1 \\
&\quad + (\beta_1 \wedge \alpha \wedge \beta_3 \wedge \cdots \wedge \beta_n) \vee \beta_2 \\
&\quad + (\beta_1 \wedge \beta_2 \wedge \alpha \wedge \cdots \wedge \beta_n) \vee \beta_3 \\
&\quad + \cdots \\
&\quad + (\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \cdots \wedge \alpha) \vee \beta_n
\end{aligned}$$

Putting α equal to β_0, this may be written more symmetrically as:

$$\sum_{i=0}^{n} (-1)^i (\beta_0 \wedge \beta_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \beta_n) \vee \beta_i == 0 \qquad 3.52$$

Example: A relationship between four 1-elements which span a 3-space

Suppose β_0, β_1, β_2, β_3 are four dependent 1-elements which span a 3-space, then formula 3.50 reduces to the identity:

$$\begin{aligned}
&(\beta_1 \wedge \beta_2 \wedge \beta_3) \vee \beta_0 - (\beta_0 \wedge \beta_2 \wedge \beta_3) \vee \beta_1 \\
&\quad + (\beta_0 \wedge \beta_1 \wedge \beta_3) \vee \beta_2 - (\beta_0 \wedge \beta_1 \wedge \beta_2) \vee \beta_3 == 0
\end{aligned} \qquad 3.53$$

Product formulae leading to scalar results

A formula particularly useful in its interior product form to be derived later is obtained by application of the Common Factor Theorem to the product $(\alpha_1 \wedge \cdots \wedge \alpha_m) \vee \left(\underset{n-1}{\beta_1} \vee \cdots \vee \underset{n-1}{\beta_m}\right)$ where the α_i are 1-elements and the $\underset{n-1}{\beta_i}$ are $(n-1)$-elements.

$$(\alpha_1 \wedge \cdots \wedge \alpha_m) \vee \left(\underset{n-1}{\beta_1} \vee \cdots \vee \underset{n-1}{\beta_m}\right)$$
$$== \left((\alpha_1 \wedge \cdots \wedge \alpha_m) \vee \underset{n-1}{\beta_j}\right) \vee \left((-1)^{j-1} \underset{n-1}{\beta_1} \vee \cdots \vee \square_j \vee \cdots \vee \underset{n-1}{\beta_m}\right)$$
$$== \left(\sum_i (-1)^{i-1} \left(\alpha_i \vee \underset{n-1}{\beta_j}\right) (\alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m)\right) \vee$$
$$\left((-1)^{j-1} \underset{n-1}{\beta_1} \vee \cdots \vee \square_j \vee \cdots \vee \underset{n-1}{\beta_m}\right)$$
$$== \sum_i (-1)^{i+j} \left(\alpha_i \vee \underset{n-1}{\beta_j}\right) \left((\alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m) \vee \left(\underset{n-1}{\beta_1} \vee \cdots \vee \square_j \vee \cdots \vee \underset{n-1}{\beta_m}\right)\right)$$

By repeating this process one obtains finally that:

$$\boxed{(\alpha_1 \wedge \cdots \wedge \alpha_m) \vee \left(\underset{n-1}{\beta_1} \vee \cdots \vee \underset{n-1}{\beta_m}\right) == \mathbf{Det}\left[\alpha_i \vee \underset{n-1}{\beta_j}\right]} \qquad 3.54$$

where $\mathbf{Det}\left[\alpha_i \vee \underset{n-1}{\beta_j}\right]$ is the determinant of the matrix whose elements are the scalars $\alpha_i \vee \underset{n-1}{\beta_j}$.

For the case $m = 2$ we have:

$$(\alpha_1 \wedge \alpha_2) \vee \left(\underset{n-1}{\beta_1} \vee \underset{n-1}{\beta_2}\right) == \left(\alpha_1 \vee \underset{n-1}{\beta_1}\right) \left(\alpha_2 \vee \underset{n-1}{\beta_2}\right) - \left(\alpha_1 \vee \underset{n-1}{\beta_2}\right) \left(\alpha_2 \vee \underset{n-1}{\beta_1}\right)$$

Although the right-hand side is composed of products of scalars, if we want to obtain the correct dual we need to use the exterior product instead of the usual field multiplication for these scalars.

$$\mathbf{Dual}\Big[$$
$$(\alpha_1 \wedge \alpha_2) \vee \left(\underset{n-1}{\beta_1} \vee \underset{n-1}{\beta_2}\right) == \left(\alpha_1 \vee \underset{n-1}{\beta_1}\right) \wedge \left(\alpha_2 \vee \underset{n-1}{\beta_2}\right) - \left(\alpha_1 \vee \underset{n-1}{\beta_2}\right) \wedge \left(\alpha_2 \vee \underset{n-1}{\beta_1}\right)\Big]$$

$$\left(\underset{-1+n}{\alpha_1} \vee \underset{-1+n}{\alpha_2}\right) \wedge \beta_1 \wedge \beta_2 == \underset{-1+n}{\alpha_1} \wedge \beta_1 \vee \underset{-1+n}{\alpha_2} \wedge \beta_2 - \underset{-1+n}{\alpha_1} \wedge \beta_2 \vee \underset{-1+n}{\alpha_2} \wedge \beta_1$$

Formula 3.54 is of central importance in the computation of inner products to be discussed in Chapter 6.

4 Geometric Interpretations

4.1 Introduction

In Chapter 2, the exterior product operation was introduced onto the elements of a linear space $\underset{1}{\Lambda}$, enabling the construction of a series of new linear spaces $\underset{m}{\Lambda}$ possessing new types of elements. In this chapter, the elements of $\underset{1}{\Lambda}$ will be *interpreted*, some as *vectors*, some as *points*. This will be done by singling out one particular element of $\underset{1}{\Lambda}$ and conferring upon it the *interpretation* of *origin point*. All the other elements of the linear space then divide into two categories. Those that involve the origin point will be called (weighted) points, and those that do not will be called vectors. As this distinction is developed in succeeding sections it will be seen to be both illuminating and consistent with accepted notions. Vectors and points will be called *geometric interpretations* of the elements of $\underset{1}{\Lambda}$.

Some of the more important consequences however, of the distinction between vectors and points arise from the distinctions thereby generated in the higher grade spaces $\underset{m}{\Lambda}$. It will be shown that a simple element of $\underset{m}{\Lambda}$ takes on two interpretations. The first, that of a *multivector* (or *m-vector*) is when the *m*-element can be expressed in terms of vectors alone. The second, that of a *bound multivector* is when the *m*-element requires both points and vectors to express it. These simple interpreted elements will be found useful for defining geometric entities such as lines and planes and their higher dimensional analogues known as *multiplanes* (or *m-planes*). Unions and intersections of multiplanes may then be calculated straightforwardly by using the bound multivectors which define them. A multivector may be visualized as a 'free' entity with no location. A bound multivector may be visualized as 'bound' through a location in space.

It is not only simple interpreted elements which will be found useful in applications however. In Chapter 8: Exploring Screw Algebra and Chapter 9: Exploring Mechanics, a basis for a theory of mechanics is developed whose principal quantities (for example, systems of forces, momentum of a system of particles, velocity of a rigid body etc.) may be represented by a general interpreted 2-element, that is, by the *sum* of a bound vector and a bivector.

In the literature of the nineteenth century, wherever vectors and points were considered together, vectors were introduced as point differences. When it is required to designate physical quantities it is not satisfactory that all vectors should arise as the differences of points. In later literature, this problem appeared to be overcome by designating points by their position vectors alone, making vectors the fundamental entities [Gibbs 1886]. This approach is not satisfactory either, since by excluding points much of the power of the calculus for dealing with free and located entities *together* is excluded. In this book we do not require that vectors be defined in terms of points, but rather, postulate a difference of interpretation between the origin element and those elements not involving the origin. This approach permits the existence of points and vectors together without the vectors necessarily arising as point differences.

In this chapter, as in the preceding chapters, $\underset{1}{\Lambda}$ and the spaces $\underset{m}{\Lambda}$ do not yet have a metric. That is, there is no way of calculating a measure or magnitude associated with an element. The interpretation discussed therefore may also be supposed non-metric. In the next chapter a metric will be introduced onto the uninterpreted spaces and the consequences of this for the interpreted elements developed.

In summary then, it is the aim of this chapter to set out the distinct non-metric geometric interpretations of *m*-elements brought about by the interpretation of one specific element of $\underset{1}{\Lambda}$ as an origin point.

4.2 Geometrically Interpreted 1-elements

Vectors

The most common current geometric interpretation for an element of a linear space is that of a *vector*. We suppose in this chapter that the linear space does not have a metric (that is, we cannot calculate magnitudes). Such a vector has the geometric properties of *direction* and *sense* (but no *location*), and will be graphically *depicted* by a directed and sensed line segment thus:

Graphic showing an arrow.

This depiction is unsatisfactory in that the line segment has a definite location and length whilst the vector it is depicting is supposed to possess neither of these properties. Nevertheless we are not aware of any more faithful alternative.

A linear space, all of whose elements are interpreted as vectors, will be called a *vector space*. The geometric interpretation of the addition operation of a vector space is the parallelogram rule.

Since an element of $\underset{1}{\Lambda}$ may be written $a \wedge x$ where a is a scalar, then the interpretation of $a \wedge x$ as a vector suggests that it may be viewed as the product of two quantities.

1. A direction factor x.
2. An intensity factor a.

The term 'intensity' was coined by Alfed North Whitehead [ ]. The sign of a may be associated with the sense of the vector. Of course this division of vector into direction and intensity factors is arbitrarily definable, but is conceptually useful in non-metric spaces. In a non-metric space, two vectors may be compared if and only if they have the same direction. Then it is their scalar intensities that are compared. Intensity should not be confused with the metric quantity of length or magnitude.

Points

In order to describe *position* in space it is necessary to have a *reference point*. This point is usually called the *origin*.

Rather than the standard technique of implicitly assuming the origin and working only with vectors to describe position, we find it important for later applications to *augment the vector space with the origin as a new element* to create a new linear space with one more element in its basis. For reasons which will appear later, such a linear space will be called a *bound vector space*.

The only difference between the origin element and the vector elements of the linear space is their interpretation. The origin element is interpreted as a point. We will denote it in *GrassmannAlgebra* by $\mathbb{O}$, which we can access from the palette or type as ESCdsOESC (double-struck capital O). The bound vector space in addition to its vectors and its origin now possesses a new set of elements requiring interpretation: those formed from the sum of the origin and a vector.

$$\mathbf{P} == \mathbb{O} + \mathbf{x}$$

It is these elements that will be used to describe position and that we will call *points*. The vector $\mathbf{x}$ is called the *position vector* of the point $\mathbf{P}$.

It follows immediately from this definition of a point that *the difference of two points is a vector*:

$$\mathbf{P}_1 - \mathbf{P}_2 == (\mathbb{O} + \mathbf{x}_1) - (\mathbb{O} + \mathbf{x}_2) == \mathbf{x}_1 - \mathbf{x}_2$$

Remember that the bound vector space does not yet have a metric. That is, the distance between two points (the measure of the vector equal to the point difference) is not meaningful.

The *sum of a point and a vector* is another point.

$$\mathbf{P} + \mathbf{y} == \mathbb{O} + \mathbf{x} + \mathbf{y} == \mathbb{O} + (\mathbf{x} + \mathbf{y})$$

and so a vector may be viewed as a *carrier of points*. That is, the addition of a vector to a point carries or transforms it to another point.

A scalar multiple of a point (of the form $\mathbf{a}\ \mathbf{p}$ or $\mathbf{a} \wedge \mathbf{p}$) will be called a *weighted point*. Weighted points are summed just like a set of point masses is deemed equivalent to their total mass situated at their centre of mass. For example:

$$\sum \mathbf{m}_i\ \mathbf{P}_i == \sum \mathbf{m}_i\ (\mathbb{O} + \mathbf{x}_i) == \left(\sum \mathbf{m}_i\right) \left(\mathbb{O} + \frac{\Sigma\, \mathbf{m}_i\ \mathbf{x}_i}{\Sigma\, \mathbf{m}_i}\right)$$

This equation may be interpreted as saying that the sum of a number of mass-weighted points $\sum \mathbf{m}_i\ \mathbf{p}_i$ is equivalent to the centre of gravity $\mathbb{O} + \frac{\Sigma\, \mathbf{m}_i\ \mathbf{x}_i}{\Sigma\, \mathbf{m}_i}$ weighted by the total mass $\sum \mathbf{m}_i$.

As will be seen in Section 4.4 below, a weighted point may also be viewed as a bound scalar.

Historical Note

Sir William Rowan Hamilton in his *Lectures on Quaternions* [Hamilton 1853] was the first to introduce the notion of vector as 'carrier' of points.

> ... I regard the symbol **B-A** as denoting "the *step* from **B** to **A**": namely, that step by making which, from the given point **A**, we should reach or arrive at the sought point **B**; and so determine, generate, mark or *construct* that point. This step, (which we always suppose to be a straight line) may also in my opinion be properly called a *vector*; or more fully, it may be called "the vector of the point **B** from the point **A**": because it may be considered as having for its office, function, work, task or business, to transport or *carry* (in Latin *vehere*) a moveable point, from the given or initial position **A**, to the sought or final position **B**.

A shorthand for declaring standard bases

Any geometry that we do with points will require us to declare the origin $\mathbb{O}$ as one of the elements of the basis. We have already seen that a shorthand way of declaring a basis $\{e_1, e_2, \cdots, e_n\}$ is by entering $\mathbb{V}_n$. Declaring the augmented basis $\{\mathbb{O}, e_1, e_2, \cdots, e_n\}$ can be accomplished by entering $\mathbb{P}_n$. These are double-struck capital letters subscripted with the integer *n* denoting the desired 'vectorial' dimension of the space. For example, entering $\mathbb{V}_3$ or $\mathbb{P}_3$:

$\mathbb{V}_3$

$\{e_1, e_2, e_3\}$

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

We may often precede a calculation with one of these followed by a semi-colon. This accomplishes the declaration of the basis but for brevity suppresses the confirming output. For example:

$\mathbb{P}_2$**; Basis∧[]**

$\{1, \mathbb{O}, e_1, e_2, \mathbb{O}\wedge e_1, \mathbb{O}\wedge e_2, e_1\wedge e_2, \mathbb{O}\wedge e_1\wedge e_2\}$

Example: Calculation of the centre of mass

Suppose a space $\mathbb{P}_3$ with basis $\{\mathbb{O}, e_1, e_2, e_3\}$ and a set of masses M_i situated at points P_i. It is required to find their centre of mass. First declare the basis, then enter the mass points.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

$$\mathbf{M_1 = 2\,P_1;\quad P_1 = \mathbb{O} + e_1 + 3\,e_2 - 4\,e_3;}$$
$$\mathbf{M_2 = 4\,P_2;\quad P_2 = \mathbb{O} + 2\,e_1 - e_2 - 2\,e_3;}$$
$$\mathbf{M_3 = 7\,P_3;\quad P_3 = \mathbb{O} - 5\,e_1 + 3\,e_2 - 6\,e_3;}$$
$$\mathbf{M_4 = 5\,P_4;\quad P_4 = \mathbb{O} + 4\,e_1 + 2\,e_2 - 9\,e_3;}$$

We simply add the mass-weighted points.

$$\mathbf{M = \sum_{i=1}^{4} M_i}$$

$$5\,(\mathbb{O} + 4\,e_1 + 2\,e_2 - 9\,e_3) + 7\,(\mathbb{O} - 5\,e_1 + 3\,e_2 - 6\,e_3) + 2\,(\mathbb{O} + e_1 + 3\,e_2 - 4\,e_3) + 4\,(\mathbb{O} + 2\,e_1 - e_2 - 2\,e_3)$$

Simplifying this gives a weighted point with weight 18, the scalar attached to the origin.

```
M = Simplify[M]
```

$$18\,\mathbb{O} - 5\,e_1 + 33\,e_2 - 103\,e_3$$

Taking the weight out as a factor, that is, expressing the result in the form *mass* × *point*

```
M = WeightedPointForm[M]
```

$$18\left(\mathbb{O} + \frac{1}{18}\,(-5\,e_1 + 33\,e_2 - 103\,e_3)\right)$$

Thus the total mass is 18 situated at the point $\mathbb{O} + \frac{1}{18}\,(-5\,e_1 + 33\,e_2 - 103\,e_3)$.

4.3 Geometrically Interpreted 2-Elements

Simple geometrically interpreted 2-elements

It has been seen in Chapter 2 that the linear space $\underset{2}{\Lambda}$ may be generated from $\underset{1}{\Lambda}$ by the exterior product operation. In the preceding section the elements of $\underset{1}{\Lambda}$ have been given two geometric interpretations: that of a *vector* and that of a *point*. These interpretations in turn generate various other interpretations for the elements of $\underset{2}{\Lambda}$.

In $\underset{2}{\Lambda}$ there are at first sight three possibilities for simple elements:

1. $\mathbf{x \wedge y}$ (vector by vector).
2. $\mathbf{P \wedge x}$ (point by vector).
3. $\mathbf{P_1 \wedge P_2}$ (point by point).

However, $\mathbf{P_1 \wedge P_2}$ may be expressed as $\mathbf{P_1 \wedge (P_2 - P_1)}$ which is a point by a vector and thus reduces to the second case.

There are thus two *simple* interpreted elements in $\underset{2}{\Lambda}$:

1. $\mathbf{x \wedge y}$ (*the simple bivector*).
2. $\mathbf{P \wedge x}$ (*the bound vector*).

A note on terminology

The term *bound* as in the *bound vector* **P∧x** indicates that the vector **x** is conceived of as bound *through* the point **P**, rather than *to* the point **P**, since the latter conception would give the incorrect impression that the vector was located *at* the point **P**. By adhering to the terminology *bound through*, we get a slightly more correct impression of the 'freedom' that the vector enjoys.

The term 'bound vector' is often used in engineering to specify the type of quantity a force is.

The bivector

Earlier in this chapter, a vector **x** was depicted graphically by a directed and sensed line segment supposed to be located nowhere in particular.

Graphic showing an arrow with symbol **x** attached.

In like manner a simple bivector may be depicted graphically by an *oriented plane segment* also supposed located nowhere in particular.

Graphic showing a parallelogram constructed from two vectors **x** and **y** with common tails. The symbol **x∧y** is attached to the parallelogram.

Orientation is a relative concept. The plane segment depicting the bivector **y∧x** is of opposite orientation to that depicting **x∧y**.

Graphic showing a parallelogram **y∧x** .

The oriented plane segment or parallelogram depiction of the simple bivector is misleading in two main respects. It incorrectly suggests a specific *location* in the plane and *shape* of the parallelogram. Indeed, since **x∧y** == **x∧(x+y)**, another valid depiction of this simple bivector would be a parallelogram with sides constructed from vectors **x** and **x+y**.

Graphic showing parallelograms **x∧y** and **x∧(x+y)** superimposed.

In the following chapter a metric will be introduced onto $\underset{1}{\Lambda}$ from which a metric is induced onto $\underset{2}{\Lambda}$. This will permit the definition of the measure of a vector (its length) and the measure of the simple bivector (its area).

The measure of a simple bivector is geometrically interpreted as the *area of the parallelogram* formed by any two vectors in terms of which the simple bivector can be expressed. For example, the area of the parallelograms in the previous two figures are the same. From this point of view the parallelogram depiction is correctly *suggestive*, although the parallelogram is not of fixed shape. However, a bivector is as independent of the vector factors used to express it as any area is of its shape. Strictly speaking therefore, a bivector may be interpreted as a portion of an (unlocated) plane of any shape. In a metric space, this portion will have an area.

Earlier in the chapter it was remarked that a vector may be viewed as a 'carrier' of points. Analogously, a simple bivector may be viewed as a *carrier of bound vectors*. This view will be more fully explored in the next section.

A *sum of simple bivectors* is called a *bivector*. In two and three dimensions all bivectors are simple. This will have important consequences for our exploration of screw algebra in Chapter 7 and its application to mechanics in Chapter 8.

The bound vector

In mechanics the concept of force is paramount. In Chapter 9: Exploring Mechanics we will show that a force may be represented by a bound vector, and that a system of forces may be represented by a sum of bound vectors.

It has already been shown that a bound vector may be expressed either as the product of a point with a vector or as the product of two points.

$$\mathbf{P_1} \wedge \mathbf{x} == \mathbf{P_1} \wedge (\mathbf{P_2} - \mathbf{P_1}) == \mathbf{P_1} \wedge \mathbf{P_2} \qquad \mathbf{P_2} == \mathbf{P_1} + \mathbf{x}$$

The bound vector in the form $\mathbf{P_1} \wedge \mathbf{x}$ defines a line through the point $\mathbf{P_1}$ in the direction of $\mathbf{x}$. Similarly, in the form $\mathbf{P_1} \wedge \mathbf{P_2}$ it defines the line through $\mathbf{P_1}$ and $\mathbf{P_2}$.

Graphic showing 1. A line through a point parallel to a vector. 2. A line through two points. Both lines are parallel.

It is an important property of the Grassmann algebra that *any* point on this line may be used to represent the bound vector. Since if $\mathbf{P}$ and $\mathbf{P^*}$ are any two points in the line, $\mathbf{P-P^*}$ is a vector of the same direction as $\mathbf{x}$ so that:

$$(\mathbf{P} - \mathbf{P}^*) \wedge \mathbf{x} == 0 \quad \Longrightarrow \quad \mathbf{P} \wedge \mathbf{x} == \mathbf{P}^* \wedge \mathbf{x}$$

A bound vector may thus be depicted by a line, a point on it, and a vector parallel to it. For compactness, we will usually depict the vector of the bound vector as lying in the line.

Graphic showing a line through a point with an arrow in the line (not attached to the point).

This graphical depiction is misleading in that it suggests that the vector $\mathbf{x}$ has a specific location, and that the point $\mathbf{P}$ is of specific importance over other points in the line. Nevertheless we are not aware of any more faithful alternative.

It has been mentioned in the previous section that a simple bivector may be viewed as a 'carrier' of bound vectors. To see this, take any bound vector $\mathbf{P} \wedge \mathbf{x}$ and a bivector whose space contains $\mathbf{x}$. The bivector may be expressed in terms of $\mathbf{x}$ and some other vector, $\mathbf{y}$ say, yielding $\mathbf{y} \wedge \mathbf{x}$. Thus:

$$\mathbf{P} \wedge \mathbf{x} + \mathbf{y} \wedge \mathbf{x} == (\mathbf{P} + \mathbf{y}) \wedge \mathbf{x} == \mathbf{P}^* \wedge \mathbf{x}$$

Graphic showing a simple bivector added to a bound vector to give another bivector.

The geometric interpretation of the addition of such a simple bivector to a bound vector is then similar to that for the addition of a vector to a point, that is, a shift in position.

Sums of bound vectors

A *sum of bound vectors* $\Sigma P_i \wedge x_i$ (except in the case $\Sigma x_i == 0$) may always be reduced to the *sum of a bound vector and a bivector*, since, by choosing an *arbitrary* point P, $\Sigma P_i \wedge x_i$ may always be written in the form:

$$\Sigma P_i \wedge x_i == P \wedge (\Sigma x_i) + \Sigma (P_i - P) \wedge x_i \qquad 4.1$$

If $\Sigma x_i == 0$ then the sum is a bivector. This transformation is of fundamental importance in our exploration of mechanics in Chapter 8.

Example: Reducing a sum of bound vectors

In this example we verify that a sum of any number of bound vectors can always be reduced to the sum of a bound vector and a bivector. Note however that it is only in two or three vector dimensions that the bivector is necessarily simple. We begin by declaring the bound vector space of three vector dimensions, $\mathbb{P}_3$.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

Next, we define, enter, then sum four bound vectors.

$\beta_1 = P_1 \wedge x_1;\ \ P_1 = \mathbb{O} + e_1 + 3\, e_2 - 4\, e_3;\ \ x_1 = e_1 - e_3;$
$\beta_2 = P_2 \wedge x_2;\ \ P_2 = \mathbb{O} + 2\, e_1 - e_2 - 2\, e_3;\ \ x_2 = e_1 - e_2 + e_3;$
$\beta_3 = P_3 \wedge x_3;\ \ P_3 = \mathbb{O} - 5\, e_1 + 3\, e_2 - 6\, e_3;\ \ x_3 = 2\, e_1 + 3\, e_2;$
$\beta_4 = P_4 \wedge x_4;\ \ P_4 = \mathbb{O} + 4\, e_1 + 2\, e_2 - 9\, e_3;\ \ x_4 = 5\, e_3;$

$$B = \sum_{i=1}^{4} \beta_i$$

$(\mathbb{O} + 4\, e_1 + 2\, e_2 - 9\, e_3) \wedge (5\, e_3) + (\mathbb{O} - 5\, e_1 + 3\, e_2 - 6\, e_3) \wedge (2\, e_1 + 3\, e_2) +$
$(\mathbb{O} + e_1 + 3\, e_2 - 4\, e_3) \wedge (e_1 - e_3) + (\mathbb{O} + 2\, e_1 - e_2 - 2\, e_3) \wedge (e_1 - e_2 + e_3)$

By expanding these products, simplifying and collecting terms, we obtain the sum of a bound vector (through the origin) $\mathbb{O} \wedge (4\, e_1 + 2\, e_2 + 5\, e_3)$ and a bivector $-25\, e_1 \wedge e_2 + 39\, e_1 \wedge e_3 + 22\, e_2 \wedge e_3$. We can use `GrassmannSimplify` to do the computations for us.

$\mathcal{G}[B]$

$\mathbb{O} \wedge (4\, e_1 + 2\, e_2 + 5\, e_3) - 25\, e_1 \wedge e_2 + 39\, e_1 \wedge e_3 + 22\, e_2 \wedge e_3$

We could just as well have expressed this 2-element as bound through (for example) the point $\mathbb{O} + e_1$. To do this, we simply add $e_1 \wedge (4\, e_1 + 2\, e_2 + 5\, e_3)$ to the bound vector and subtract it from the bivector to get:

$$(\mathbb{O} + \mathbf{e}_1) \wedge (4\,\mathbf{e}_1 + 2\,\mathbf{e}_2 + 5\,\mathbf{e}_3) - 27\,\mathbf{e}_1 \wedge \mathbf{e}_2 + 34\,\mathbf{e}_1 \wedge \mathbf{e}_3 + 22\,\mathbf{e}_2 \wedge \mathbf{e}_3$$

4.4 Geometrically Interpreted *m*-elements

Types of geometrically interpreted *m*-elements

In $\underset{m}{\Lambda}$ the situation is analogous to that in $\underset{2}{\Lambda}$. A simple product of *m* points and vectors may always be reduced to the product of a point and a simple (*m*–1)-vector by subtracting one of the points from all of the others. For example, take the exterior product of three points and two vectors. By subtracting the first point, say, from the other two, we can cast the product into the form of a bound simple 4-vector.

$$\mathbf{P} \wedge \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{x} \wedge \mathbf{y} == \mathbf{P} \wedge (\mathbf{P}_1 - \mathbf{P}) \wedge (\mathbf{P}_2 - \mathbf{P}) \wedge \mathbf{x} \wedge \mathbf{y}$$

There are thus only two *simple* interpreted elements in $\underset{m}{\Lambda}$:

1. $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \mathbf{x}_m$ (the simple *m*-vector).
2. $\mathbf{P} \wedge \mathbf{x}_2 \wedge \cdots \mathbf{x}_m$ (the bound simple (*m*–1)-vector).

A sum of simple *m*-vectors is called an *m*-vector and the non-simple interpreted elements of $\underset{m}{\Lambda}$ are sums of *m*-vectors.

If $\underset{m}{\alpha}$ is a (not necessarily simple) *m*-vector, then $\mathbf{P} \wedge \underset{m}{\alpha}$ is called a *bound m-vector*.

A sum of bound *m*-vectors may always be reduced to the sum of a bound *m*-vector and an (*m*+1)-vector.

These interpreted elements and their relationships will be discussed further in the following sections.

The *m*-vector

The simple *m*-vector, or *multivector*, is the multidimensional equivalent of the vector. As with a vector, it does not have the property of location. The *m*-dimensional vector space of a simple *m*-vector may be used to define the multidimensional *direction* of the *m*-vector.

A simple *m*-vector may be depicted by an *oriented m-space segment*. A 3-vector may be depicted by a parallelepiped.

Graphic showing a parallelepiped formed from three vectors.

The orientation is given by the order of the factors in the simple *m*-vector. An interchange of any two factors produces an *m*-vector of opposite orientation. By the anti-symmetry of the exterior product, there are just two distinct orientations.

In Chapter 6: The Interior Product, it will be shown that the measure of a 3-vector may be geometrically interpreted as the volume of this parallelepiped. However, the depiction of the simple 3-vector in the manner above suffers from similar defects to those already described for the bivector: namely, it incorrectly suggests a specific location and shape of the parallelepiped.

A simple *m*-vector may also be viewed as a carrier of bound simple (*m*–1)-vectors in a manner analogous to that already described for the bivector.

A sum of simple *m*-vectors (that is, an *m*-vector) is not necessarily reducible to a simple *m*-vector, except in $\underset{0}{\Lambda}, \underset{1}{\Lambda}, \underset{n-1}{\Lambda}, \underset{n}{\Lambda}$.

The bound *m*-vector

The exterior product of a point and an *m*-vector is called a *bound m-vector*. Note that it belongs to $\underset{m+1}{\Lambda}$. A sum of bound *m*-vectors is not necessarily a bound *m*-vector. However, it may in general be reduced to the sum of a bound *m*-vector and an (*m*+1)-vector as follows:

$$\boxed{\sum \mathbf{P_i} \wedge \underset{m}{\alpha_i} == \sum (\mathbf{P} + \beta_i) \wedge \underset{m}{\alpha_i} == \mathbf{P} \wedge \sum \underset{m}{\alpha_i} + \sum \beta_i \wedge \underset{m}{\alpha_i}} \qquad \mathbf{4.2}$$

The first term $\mathbf{P} \wedge \sum \underset{m}{\alpha_i}$ is a bound *m*-vector providing $\sum \underset{m}{\alpha_i} \neq \mathbf{0}$ and the second term is an (*m*+1)-vector.

When *m* = 0, a bound 0-vector or *bound scalar* $\mathbf{p \wedge a}$ (= $\mathbf{ap}$) is seen to be equivalent to a weighted point.

When *m* = *n*, any bound *n*-vector is but a scalar multiple of $\mathbb{O} \wedge \mathbf{e_1} \wedge \mathbf{e_2} \wedge \cdots \wedge \mathbf{e_n}$ for any basis.

The graphical depiction of bound simple *m*-vectors presents even greater difficulties than those already discussed for bound vectors. As in the case of the bound vector, the point used to express the bound simple *m*-vector is not unique.

In Section 4.6 we will see how bound simple *m*-vectors may be used to define multiplanes.

Bound simple *m*-vectors expressed by points

A bound simple *m*-vector may always be expressed as a product of *m*+1 points.
Let $\mathbf{P_i} = \mathbf{P_0} + \mathbf{x_i}$, then:

$$\begin{aligned}
&\mathbf{P_0} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m} \\
&\quad == \mathbf{P_0} \wedge (\mathbf{P_1} - \mathbf{P_0}) \wedge (\mathbf{P_2} - \mathbf{P_0}) \wedge \cdots \wedge (\mathbf{P_m} - \mathbf{P_0}) \\
&\quad == \mathbf{P_0} \wedge \mathbf{P_1} \wedge \mathbf{P_2} \wedge \cdots \wedge \mathbf{P_m}
\end{aligned}$$

Conversely, as we have already seen, a product of *m*+1 points may always be expressed as the product of a point and a simple *m*-vector by subtracting one of the points from all of the others.

The *m-vector of a bound simple m-vector* $\mathbf{P_0} \wedge \mathbf{P_1} \wedge \mathbf{P_2} \wedge \cdots \wedge \mathbf{P_m}$ may thus be expressed in terms of these points as:

$$(\mathbf{P_1} - \mathbf{P_0}) \wedge (\mathbf{P_2} - \mathbf{P_0}) \wedge \cdots \wedge (\mathbf{P_m} - \mathbf{P_0})$$

A particularly symmetrical formula results from the expansion of this product reducing it to a form no longer showing preference for P_0.

$$(P_1 - P_0) \wedge (P_2 - P_0) \wedge \cdots \wedge (P_m - P_0) == \sum_{i=0}^{m} (-1)^i P_0 \wedge P_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge P_m \qquad 4.3$$

Here $\square_i$ denotes deletion of the factor P_i from the product.

4.5 Decomposition into Components

The shadow

In Chapter 3 we developed a formula which expressed a *p*-element as a sum of components, the element being decomposed with respect to a pair of elements $\underset{m}{\alpha}$ and $\underset{n-m}{\beta}$ which together span the whole space. The first and last terms of this decomposition were given as:

$$\underset{p}{x} == \frac{\left(\underset{m}{\alpha} \wedge \underset{p}{x}\right) \vee \underset{n-m}{\beta}}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} + \cdots + \frac{\underset{m}{\alpha} \vee \left(\underset{p}{x} \wedge \underset{n-m}{\beta}\right)}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}}$$

Grassmann called the last term the *shadow of* $\underset{p}{x}$ *on* $\underset{m}{\alpha}$ *excluding* $\underset{n-m}{\beta}$.

It can be seen that the first term can be rearranged as the *shadow of* $\underset{p}{x}$ *on* $\underset{n-m}{\beta}$ *excluding* $\underset{m}{\alpha}$.

$$\frac{\left(\underset{m}{\alpha} \wedge \underset{p}{x}\right) \vee \underset{n-m}{\beta}}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} == \frac{\underset{n-m}{\beta} \vee \left(\underset{p}{x} \wedge \underset{m}{\alpha}\right)}{\underset{n-m}{\beta} \vee \underset{m}{\alpha}}$$

If $p = 1$, the decomposition formula reduces to the sum of just two components, x_α and x_β, where x_α lies in $\underset{m}{\alpha}$ and x_β lies in $\underset{n-m}{\beta}$.

$$x == x_\alpha + x_\beta == \frac{\underset{m}{\alpha} \vee \left(x \wedge \underset{n-m}{\beta}\right)}{\underset{m}{\alpha} \vee \underset{n-m}{\beta}} + \frac{\underset{n-m}{\beta} \vee \left(x \wedge \underset{m}{\alpha}\right)}{\underset{n-m}{\beta} \vee \underset{m}{\alpha}} \qquad 4.4$$

We now explore this decomposition with a number of geometric examples, beginning with the simplest case of decomposing a point on a line into a sum of two weighted points in the line.

Decomposition in a 2-space

Decomposition of a vector in a vector 2-space

Suppose we have a vector 2-space defined by the bivector $\alpha \wedge \beta$. We wish to decompose a vector $\mathbf{x}$ in the space to give one component in α and the other in β. Applying the decomposition formula gives:

$$\mathbf{x}_\alpha == \frac{\alpha \vee (\mathbf{x} \wedge \beta)}{\alpha \vee \beta} == \left(\frac{\mathbf{x} \vee \beta}{\alpha \vee \beta}\right) \alpha$$

$$\mathbf{x}_\beta == \frac{\beta \vee (\mathbf{x} \wedge \alpha)}{\beta \vee \alpha} == \left(\frac{\mathbf{x} \vee \alpha}{\beta \vee \alpha}\right) \beta$$

Graphic of two vectors α and β in a plane with a third vector x between them, showing its decomposition in the directions of α and β.

The coefficients of α and β are scalars showing that $\mathbf{x}_\alpha$ is congruent to α and $\mathbf{x}_\beta$ is congruent to β. If each of the three vectors is expressed in basis form, we can determine these scalars more specifically. For example:

$$\mathbf{x} == x_1\, e_1 + x_2\, e_2; \quad \alpha == a_1\, e_1 + a_2\, e_2; \quad \beta == b_1\, e_1 + b_2\, e_2$$

$$\frac{\mathbf{x} \vee \beta}{\alpha \vee \beta} == \frac{(x_1\, e_1 + x_2\, e_2) \vee (b_1\, e_1 + b_2\, e_2)}{(a_1\, e_1 + a_2\, e_2) \vee (b_1\, e_1 + b_2\, e_2)} ==$$
$$\frac{(x_1\, b_2 - x_2\, b_1)\, e_1 \vee e_2}{(a_1\, b_2 - a_2\, b_1)\, e_1 \vee e_2} == \frac{(x_1\, b_2 - x_2\, b_1)}{(a_1\, b_2 - a_2\, b_1)}$$

Finally then we can express the original vector $\mathbf{x}$ as the required sum of two components, one in α and one in β.

$$\mathbf{x} == \frac{(x_1\, b_2 - x_2\, b_1)}{(a_1\, b_2 - a_2\, b_1)}\, \alpha + \frac{(x_1\, a_2 - x_2\, a_1)}{(b_1\, a_2 - b_2\, a_1)}\, \beta$$

Decomposition of a point in a line

These same calculations apply to decomposing a point $\mathbf{P}$ into two component weighted points congruent to two given points $\mathbf{Q}$ and $\mathbf{R}$ in a line. Let:

$$\mathbf{P} == \mathbb{O} + p\, e_1; \quad \mathbf{Q} == \mathbb{O} + q\, e_1; \quad \mathbf{R} == \mathbb{O} + r\, e_1;$$

Graphic of two points Q and R in a line with a third point P between them. Q and R are shown with weights attached.

The decomposition may be obtained by following the process *mutatis mutandis*. Alternatively we may substitute directly into the formula above by making the correspondence: $\mathbf{e_1} \to \mathbb{O}$, $\mathbf{e_2} \to \mathbf{e_1}$, whence:

$$x_1 \to 1,\ x_2 \to p,\ a_1 \to 1,\ a_2 \to q,\ b_1 \to 1,\ b_2 \to r$$

The resulting formula gives the point **P** as the sum of two weighted points, scalar multiples of the points **Q** and **R**.

$$P == \frac{(r - p)}{(r - q)} Q + \frac{(q - p)}{(q - r)} R$$

Decomposition of a vector in a line

We can also decompose a vector **x** into two component weighted points congruent to two given points **Q** and **R** in a line. Let:

$$x == a\, e_1;\quad Q == \mathbb{O} + q\, e_1;\quad R == \mathbb{O} + r\, e_1;$$

The decomposition formula then shows that the vector **x** may be expressed as the difference of two points, each with the same weight representing the ratio of the vector **x** to the parallel vector **Q−R**.

$$x == \left(\frac{a}{q - r}\right) Q - \left(\frac{a}{q - r}\right) R$$

Graphic of two points Q and R in a line with a vector x between them. Q and R are shown with weights attached.

Decomposition in a 3-space

Decomposition of a vector in a vector 3-space

Suppose we have a vector 3-space represented by the trivector $\alpha \wedge \beta \wedge \gamma$. We wish to decompose a vector **x** in this 3-space to give one component in $\alpha \wedge \beta$ and the other in γ. Applying the decomposition formula gives:

$$x_{\alpha\wedge\beta} == \frac{(\alpha \wedge \beta) \vee (x \wedge \gamma)}{(\alpha \wedge \beta) \vee \gamma}$$

$$x_\gamma == \frac{\gamma \vee (x \wedge \alpha \wedge \beta)}{\gamma \vee (\alpha \wedge \beta)}$$

Graphic of two vectors α and β in a planar direction with a third vector γ in a third direction. A vector x is shown with two components, one in the planar direction and the other in the direction of γ.

Because $x\wedge\alpha\wedge\beta$ is a 3-element it can be seen immediately that the component x_γ can be written as a scalar multiple of γ where the scalar is expressed either as a ratio of regressive products (scalars) or exterior products (*n*-elements).

$$x_\gamma == \left(\frac{x \vee (\alpha \wedge \beta)}{\gamma \vee (\alpha \wedge \beta)}\right) \gamma == \left(\frac{\alpha \wedge \beta \wedge x}{\alpha \wedge \beta \wedge \gamma}\right) \gamma$$

The component $\mathbf{x}_{\alpha\wedge\beta}$ will be a linear combination of α and β. To show this we can expand the expression above for $\mathbf{x}_{\alpha\wedge\beta}$ using the Common Factor Axiom.

$$\mathbf{x}_{\alpha\wedge\beta} == \frac{(\alpha\wedge\beta)\vee(\mathbf{x}\wedge\gamma)}{(\alpha\wedge\beta)\vee\gamma} == \frac{(\alpha\wedge\mathbf{x}\wedge\gamma)\vee\beta}{(\alpha\wedge\beta)\vee\gamma} - \frac{(\beta\wedge\mathbf{x}\wedge\gamma)\vee\alpha}{(\alpha\wedge\beta)\vee\gamma}$$

Rearranging these two terms into a similar form as that derived for $\mathbf{x}_\gamma$ gives:

$$\mathbf{x}_{\alpha\wedge\beta} == \left(\frac{\mathbf{x}\wedge\beta\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\alpha + \left(\frac{\alpha\wedge\mathbf{x}\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\beta$$

Of course we could have obtained this decomposition directly by using the results of the decomposition formula [3.51] for decomposing a 1-element into a linear combination of the factors of an *n*-element.

$$\mathbf{x} == \left(\frac{\mathbf{x}\wedge\beta\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\alpha + \left(\frac{\alpha\wedge\mathbf{x}\wedge\gamma}{\alpha\wedge\beta\wedge\gamma}\right)\beta + \left(\frac{\alpha\wedge\beta\wedge\mathbf{x}}{\alpha\wedge\beta\wedge\gamma}\right)\gamma$$

Decomposition of a point in a plane

Suppose we have a plane represented by the bound bivector $\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}$. We wish to decompose a point $\mathbf{P}$ in this plane to give one component in the line represented by $\mathbb{O}\wedge\mathbf{Q}$ and the other as a scalar multiple of $\mathbf{R}$. Applying the decomposition formula derived in the previous section gives:

$$\mathbf{P} == \left(\frac{\mathbf{P}\wedge\mathbf{Q}\wedge\mathbf{R}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbb{O} + \left(\frac{\mathbb{O}\wedge\mathbf{P}\wedge\mathbf{R}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{Q} + \left(\frac{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{P}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{R}$$

From this we can read off immediately the components $\mathbf{P}_{\mathbb{O}\wedge\mathbf{Q}}$ and $\mathbf{P}_\mathbf{R}$ we seek. $\mathbf{P}_{\mathbb{O}\wedge\mathbf{Q}}$ is the weighted point:

$$\mathbf{P}_{\mathbb{O}\wedge\mathbf{Q}} == \left(\frac{\mathbf{P}\wedge\mathbf{Q}\wedge\mathbf{R}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\left(\mathbb{O} + \left(\frac{\mathbb{O}\wedge\mathbf{P}\wedge\mathbf{R}}{\mathbf{P}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{Q}\right)$$

whilst the weighted point located at $\mathbf{R}$ is:

$$\mathbf{P}_\mathbf{R} == \left(\frac{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{P}}{\mathbb{O}\wedge\mathbf{Q}\wedge\mathbf{R}}\right)\mathbf{R}$$

Graphic of a line and points P and R external to it. A line joins R and P and intersects L.

Decomposition in a 4-space

Decomposition of a vector in a vector 4-space

Suppose we have a vector 4-space represented by the trivector $\alpha\wedge\beta\wedge\gamma\wedge\delta$. We wish to decompose a vector $\mathbf{x}$ in this 4-space. Applying the decomposition formula [3.51] gives:

$$\mathbf{x} ==$$
$$\left(\frac{\mathbf{x}\wedge\beta\wedge\gamma\wedge\delta}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\alpha + \left(\frac{\alpha\wedge\mathbf{x}\wedge\gamma\wedge\delta}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\beta + \left(\frac{\alpha\wedge\beta\wedge\mathbf{x}\wedge\delta}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\gamma + \left(\frac{\alpha\wedge\beta\wedge\gamma\wedge\mathbf{x}}{\alpha\wedge\beta\wedge\gamma\wedge\delta}\right)\delta$$

For simplicity we write $\mathbf{x}$ as:

$$\mathbf{x} == \mathbf{a}\,\alpha + \mathbf{b}\,\beta + \mathbf{c}\,\gamma + \mathbf{d}\,\delta$$

where the coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are defined by the first equation above.

We can rearrange these components in whatever combinations we require. For example if we wanted the decomposition of $\mathbf{x}$ into three components, one parallel to α, one parallel to β, and one parallel to $\gamma \wedge \delta$ we would simply write:

$$\mathbf{x} == \mathbf{x}_\alpha + \mathbf{x}_\beta + \mathbf{x}_{\gamma\wedge\delta} == (\mathbf{a}\,\alpha) + (\mathbf{b}\,\beta) + (\mathbf{c}\,\gamma + \mathbf{d}\,\delta)$$

Of course, as we have seen in many of the examples above, decomposition of a point or vector in a 4-space whose elements are interpreted as points or vectors follows the same process: only the interpretation of the symbols differs.

Decomposition of a point or vector in an *n*-space

Decomposition of a point or vector in a space of *n* dimensions is generally most directly accomplished by using the decomposition formula [3.51] derived in Chapter 3: The Regressive Product. For example the decomposition of a point can be written:

$$\mathbf{P} == \sum_{i=1}^{n} \left(\frac{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \mathbf{P} \wedge \cdots \wedge \alpha_n}{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_i \wedge \cdots \wedge \alpha_n} \right) \alpha_i \qquad 4.5$$

As we have already seen for the 4-space example above, the components are simply arranged in whatever combinations are required to achieve the decomposition.

4.6 Geometrically Interpreted Spaces

Vector and point spaces

The space of a simple non-zero *m*-element $\underset{m}{\alpha}$ has been defined in Section 2.3 as the set of 1-elements $\mathbf{x}$ whose exterior product with $\underset{m}{\alpha}$ is zero: $\left\{\mathbf{x} : \underset{m}{\alpha} \wedge \mathbf{x} == 0\right\}$.

If $\mathbf{x}$ is interpreted as a vector $\mathbf{v}$, then the *vector space* of $\underset{m}{\alpha}$ is defined as $\left\{\mathbf{v} : \underset{m}{\alpha} \wedge \mathbf{v} == 0\right\}$.

If $\mathbf{x}$ is interpreted as a point $\mathbf{P}$, then the *point space* of $\underset{m}{\alpha}$ is defined as $\left\{\mathbf{P} : \underset{m}{\alpha} \wedge \mathbf{P} == 0\right\}$.

The vector space of a simple *m*-vector is an *m*-dimensional vector space. Conversely, the *m*-dimensional vector space may be said to be *defined* by the *m*-vector.

The point space of a bound simple *m*-vector is called an *m-plane* (sometimes *multiplane*). Thus the point space of a bound vector is a 1-plane (or line) and the point space of a bound simple bivector is a 2-plane (or, simply, a plane). The *m*-plane will be said to be *defined* by the bound simple *m*-vector.

The geometric interpretation for the notion of set inclusion is taken as 'to lie in'. Thus for example, a point may be said *to lie in* an *m*-plane.

The point and vector spaces for a bound simple *m*-element are tabulated below.

m	$\mathbf{P} \wedge \underset{m}{\alpha}$	**Point space**	**Vector space**
0	bound scalar	point	
1	bound vector	line	1 –dimensional vector space
2	bound simple bivector	plane	2 –dimensional vector space
m	bound simple *m*–vector	*m*–plane	m –dimensional vector space
$n-1$	bound $(n-1)$ –vector	hyperplane	$(n-1)$ –dimensional vector space
n	bound *n*–vector	*n*–plane	*n* –dimensional vector space

Two congruent bound simple *m*-vectors $\mathbf{P} \wedge \underset{m}{\alpha}$ and $\mathbf{a}\,\mathbf{P} \wedge \underset{m}{\alpha}$ define the same *m*-plane. Thus, for example, the point $\mathbb{O}+\mathbf{x}$ and the weighted point $2\mathbb{O}+2\mathbf{x}$ define the same point.

Bound simple *m*-vectors as *m*-planes

We have defined an *m*-plane as a set of points *defined by* a bound simple *m*-vector. It will often turn out however to be more convenient and conceptually fruitful to work with *m*-planes *as if* they were the bound simple *m*-vectors which define them. This is in fact the approach taken by Grassmann and the early workers in the Grassmannian tradition (for example, [Whitehead], [Hyde] and [Forder]). This will be satisfactory provided that the equality relationship we define for *m*-planes is that of *congruence* rather than the more specific *equals*.

Thus we may, when speaking in a geometric context, refer to a bound simple *m*-vector as an *m*-plane and *vice versa*. Hence in saying Π is an *m*-plane, we are also saying that all $\mathbf{a}\,\Pi$ (where $\mathbf{a}$ is a scalar factor, not zero) is the same *m*-plane.

Coordinate spaces

The *coordinate spaces* of a Grassmann algebra are the spaces defined by the basis elements.

The *coordinate m-spaces* are the spaces defined by the basis elements of $\underset{m}{\Lambda}$. For example, if $\underset{1}{\Lambda}$ has basis $\{\mathbb{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, that is, we are working in the bound vector 3-space $\mathbb{P}_3$, then the Grassmann algebra it generates has basis:

```
ℙ₃; BasisΛ[]
```

$\{1, \mathbb{O}, e_1, e_2, e_3, \mathbb{O}\wedge e_1, \mathbb{O}\wedge e_2, \mathbb{O}\wedge e_3, e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3, \mathbb{O}\wedge e_1\wedge e_2, \mathbb{O}\wedge e_1\wedge e_3, \mathbb{O}\wedge e_2\wedge e_3, e_1\wedge e_2\wedge e_3, \mathbb{O}\wedge e_1\wedge e_2\wedge e_3\}$

Each one of these basis elements defines a coordinate space. Most familiar are the coordinate *m*-planes. The coordinate 1-planes $\mathbb{O}\wedge\mathbf{e}_1$, $\mathbb{O}\wedge\mathbf{e}_2$, $\mathbb{O}\wedge\mathbf{e}_3$ define the coordinate axes, while the coordinate 2-planes $\mathbb{O}\wedge\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbb{O}\wedge\mathbf{e}_1\wedge\mathbf{e}_3$, $\mathbb{O}\wedge\mathbf{e}_2\wedge\mathbf{e}_3$ define the coordinate planes. Additionally however there are the coordinate vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the coordinate bivectors $\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbf{e}_1\wedge\mathbf{e}_3$, $\mathbf{e}_2\wedge\mathbf{e}_3$.

Perhaps less familiar is the fact that there are no coordinate *m*-planes in a vector space, but rather simply coordinate *m*-vectors.

Geometric dependence

In Chapter 2 the notion of dependence was discussed for elements of a linear space. Non-zero 1-elements are said to be dependent if and only if their exterior product is zero.

If the elements concerned have been endowed with a geometric interpretation, the notion of dependence takes on an additional geometric interpretation, as the following table shows.

$x_1 \wedge x_2 == 0$ $P_1 \wedge P_2 == 0$	x_1, x_2 are parallel (co–directional) P_1, P_2 are coincident
$x_1 \wedge x_2 \wedge x_3 == 0$ $P_1 \wedge P_2 \wedge P_3 == 0$	x_1, x_2, x_3 are co–2–directional (or parallel) P_1, P_2, P_3 are collinear (or coincident)
$x_1 \wedge \cdots \wedge x_m == 0$ $P_1 \wedge \cdots \wedge P_m == 0$	$x_1, \cdots, x_m$ are co–k–directional, $k < m$ $P_1, \cdots, P_m$ are co–k–planar, $k < m - 1$

Geometric duality

The concept of duality introduced in Chapter 3 is most striking when interpreted geometrically. Suppose:

P defines a point
L defines a line
π defines a plane
V defines a 3-plane

In what follows we tabulate the dual relationships of these entities to each other.

Duality in a plane

In a plane there are just three types of geometric entity: points, lines and planes. In the table below we can see that in the plane, points and lines are 'dual' entities, and planes and scalars are 'dual' entities, because their definitions convert under the application of the Duality Principle.

$L \equiv P_1 \wedge P_2$	$P \equiv L_1 \vee L_2$
$\pi \equiv P_1 \wedge P_2 \wedge P_3$	$1 \equiv L_1 \vee L_2 \vee L_3$
$\pi \equiv L \wedge P$	$1 \equiv P \vee L$

Duality in a 3-plane

In the 3-plane there are just four types of geometric entity: points, lines, planes and 3-planes. In the table below we can see that in the 3-plane, lines are self-dual, points and planes are now dual, and scalars are now dual to 3-planes.

$L \equiv P_1 \wedge P_2$	$L \equiv \pi_1 \vee \pi_2$
$\pi \equiv P_1 \wedge P_2 \wedge P_3$	$P \equiv \pi_1 \vee \pi_2 \vee \pi_3$
$V \equiv P_1 \wedge P_2 \wedge P_3 \wedge P_4$	$1 \equiv \pi_1 \vee \pi_2 \vee \pi_3 \vee \pi_4$
$\pi \equiv L \wedge P$	$P \equiv L \vee \pi$
$V \equiv L_1 \wedge L_2$	$1 \equiv L_1 \vee L_2$
$V \equiv \pi \wedge P$	$1 \equiv P \vee \pi$

Duality in an *n*-plane

From these cases the types of relationships in higher dimensions may be composed straightforwardly. For example, if P defines a point and H defines a hyperplane ((*n*–1)-plane), then we have the dual formulations:

$H \equiv P_1 \wedge P_2 \wedge \cdots \wedge P_{n-1}$	$P \equiv H_1 \vee H_2 \vee \cdots \vee H_{n-1}$

4.7 *m*-planes

In the previous section *m*-planes have been defined as point spaces of bound simple *m*-vectors. In this section *m*-planes will be considered from three other aspects: the first in terms of a simple exterior product of points, the second as an *m*-vector and the third as an exterior quotient.

m-planes defined by points

Grassmann and those who wrote in the style of the *Ausdehnungslehre* considered the point more fundamental than the vector for exploring geometry. This approach indeed has its merits. An *m*-plane is quite straightforwardly defined and expressed as the (space of the) exterior product of *m*+1 points.

$$\Pi == P_0 \wedge P_1 \wedge P_2 \wedge \cdots \wedge P_m$$

m-planes defined by *m*-vectors

Consider a bound simple *m*-vector $P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m$. Its *m*-plane is the set of points P such that:

$$P \wedge P_0 \wedge x_1 \wedge x_2 \wedge \cdots \wedge x_m == 0$$

This equation is equivalent to the statement: *there exist scalars* a, a_0, a_i, *not all zero, such that*:

$$a\, P + a_0\, P_0 + \Sigma\, a_i\, x_i == 0$$

And since this is only possible if $a == -a_0$ (since for the sum to be zero, it must be a sum of vectors) then:

$$a\, (P - P_0) + \Sigma\, a_i\, x_i == 0$$

or equivalently:

$$(\mathbf{P} - \mathbf{P_0}) \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m} == 0$$

We are thus lead to the following alternative definition of an *m*-plane: an *m*-plane defined by the bound simple *m*-vector $\mathbf{P_0} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m}$ is the set of points:

$$\{\mathbf{P} : (\mathbf{P} - \mathbf{P_0}) \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m} == 0\}$$

This is of course equivalent to the usual definition of an *m*-plane. That is, since the vectors $(\mathbf{P} - \mathbf{P_0})$, $\mathbf{x_1}$, $\mathbf{x_2}$, $\cdots$, $\mathbf{x_m}$ are dependent, then for scalar parameters $\mathbf{t_i}$:

$$(\mathbf{P} - \mathbf{P_0}) == \mathbf{t_1}\, \mathbf{x_1} + \mathbf{t_2}\, \mathbf{x_2} + \cdots + \mathbf{t_m}\, \mathbf{x_m} \qquad 4.6$$

m-planes as exterior quotients

The alternative definition of an *m*-plane developed above shows that an *m*-plane may be defined as the set of points $\mathbf{P}$ such that:

$$\mathbf{P} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m} == \mathbf{P_0} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m}$$

'Solving' for $\mathbf{P}$ and noting from Section 2.11 that the quotient of an (*m*+1)-element by an *m*-element contained in it is a 1-element with *m* arbitrary scalar parameters, we can write:

$$\mathbf{P} == \frac{\mathbf{P_0} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m}}{\mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m}} == \mathbf{P_0} + \mathbf{t_1}\, \mathbf{x_1} + \mathbf{t_2}\, \mathbf{x_2} + \cdots + \mathbf{t_m}\, \mathbf{x_m}$$

$$\mathbf{P} == \frac{\mathbf{P_0} \wedge \mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m}}{\mathbf{x_1} \wedge \mathbf{x_2} \wedge \cdots \wedge \mathbf{x_m}} \qquad 4.7$$

The operator ∂

We can define an operator ∂ which takes a simple (*m*+1)-element of the form $\mathbf{P_0} \wedge \mathbf{P_1} \wedge \mathbf{P_2} \wedge \cdots \wedge \mathbf{P_m}$ and converts it to an *m*-element of the form $(\mathbf{P_1} - \mathbf{P_0}) \wedge (\mathbf{P_2} - \mathbf{P_0}) \wedge \cdots \wedge (\mathbf{P_m} - \mathbf{P_0})$.

The interesting property of this operation is that when it is applied twice, the result is zero. Operationally, $\partial^2 == 0$. For example:

$$\partial(\mathbf{P}) == 1$$

$$\partial(\mathbf{P_0} \wedge \mathbf{P_1}) == \mathbf{P_1} - \mathbf{P_0}$$
$$\partial(\mathbf{P_1} - \mathbf{P_0}) == 1 - 1 == 0$$

$$\partial(\mathbf{P_0} \wedge \mathbf{P_1} \wedge \mathbf{P_2}) == \mathbf{P_1} \wedge \mathbf{P_2} + \mathbf{P_2} \wedge \mathbf{P_0} + \mathbf{P_0} \wedge \mathbf{P_1}$$
$$\partial(\mathbf{P_1} \wedge \mathbf{P_2} + \mathbf{P_2} \wedge \mathbf{P_0} + \mathbf{P_0} \wedge \mathbf{P_1}) == \mathbf{P_2} - \mathbf{P_1} + \mathbf{P_0} - \mathbf{P_2} + \mathbf{P_1} - \mathbf{P_0} == 0$$

Remember that $\mathbf{P_1} \wedge \mathbf{P_2} + \mathbf{P_2} \wedge \mathbf{P_0} + \mathbf{P_0} \wedge \mathbf{P_1}$ is simple since it may be expressed as

$$(P_1 - P_0) \wedge (P_2 - P_0) == (P_2 - P_1) \wedge (P_0 - P_1) == (P_0 - P_2) \wedge (P_1 - P_2)$$

This property of nilpotence is shared by the boundary operator of algebraic topology and the exterior derivative. Furthermore, if a product with a given 1-element is considered an operation, then the exterior, regressive and interior products are all likewise nilpotent.

4.8 Line Coordinates

We have already seen that lines are defined by bound vectors independent of the dimension of the space. We now look at the types of coordinate descriptions we can use to define lines in bound spaces (multiplanes) of various dimensions.

For simplicity of exposition we refer to a bound vector as 'a line', rather than as 'defining a line'.

Lines in a plane

To explore lines in a plane, we first declare the basis of the plane: $\mathbb{P}_2$.

$$\mathbb{P}_2$$

$$\{\mathbb{O}, e_1, e_2\}$$

A line in a plane can be written in several forms. The most intuitive form perhaps is as a product of two points **𝕆+x** and **𝕆+y** where **x** and **y** are position vectors.

$$L \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y)$$

Graphic of a line through two points specified by position vectors.

We can automatically generate a basis form for each of the position vectors **x** and **y** by using the *GrassmannAlgebra* `CreateVector` function.

```
{X = CreateVector[x], Y = CreateVector[y]}
```

$$\{e_1 x_1 + e_2 x_2, e_1 y_1 + e_2 y_2\}$$

$$L \equiv (\mathbb{O} + X) \wedge (\mathbb{O} + Y)$$

$$L \equiv (\mathbb{O} + e_1 x_1 + e_2 x_2) \wedge (\mathbb{O} + e_1 y_1 + e_2 y_2)$$

Or, we can express the line as the product of any point in it and a vector parallel to it. For example:

```
L ≡ (𝕆 + X) ∧ (Y - X) ≡ (𝕆 + Y) ∧ (Y - X) // Simplify
```

$$L \equiv (\mathbb{O} + e_1 x_1 + e_2 x_2) \wedge (e_1 (-x_1 + y_1) + e_2 (-x_2 + y_2)) \equiv (\mathbb{O} + e_1 y_1 + e_2 y_2) \wedge (e_1 (-x_1 + y_1) + e_2 (-x_2 + y_2))$$

Graphic of a line through a point parallel to the difference between two position vectors.

Alternatively, we can express **L** without specific reference to points in it. For example:

$$\mathbf{L} \equiv \mathbb{O} \wedge (a\, e_1 + b\, e_2) + c\, e_1 \wedge e_2$$

The first term $\mathbb{O} \wedge (a\, e_1 + b\, e_2)$ is a vector bound through the origin, and hence defines a line through the origin. The second term $c\, e_1 \wedge e_2$ is a bivector whose addition represents a shift in the line parallel to itself, away from the origin.

Graphic showing a line through the origin plus a shift due to the addition of a bivector. Don't draw the basis vectors at right angles.

We know that this can indeed represent a line since we can factorize it into any of the forms:

- A line of gradient $\frac{b}{a}$ through the point with coordinate $\frac{c}{b}$ on the $\mathbb{O} \wedge e_1$ axis.

$$\mathbf{L} \equiv \left(\mathbb{O} + \frac{c}{b}\, e_1\right) \wedge (a\, e_1 + b\, e_2)$$

Graphic of a line through a point on the axis with given gradient.

- A line of gradient $\frac{b}{a}$ through the point with coordinate $-\frac{c}{a}$ on the $\mathbb{O} \wedge e_2$ axis.

$$\mathbf{L} \equiv \left(\mathbb{O} - \frac{c}{a}\, e_2\right) \wedge (a\, e_1 + b\, e_2)$$

Graphic of a line through a point on the axis with given gradient.

- Or, a line through both points.

$$\mathbf{L} \equiv \frac{a\, b}{c} \left(\mathbb{O} - \frac{c}{a}\, e_2\right) \wedge \left(\mathbb{O} + \frac{c}{b}\, e_1\right)$$

Of course the scalar factor $\frac{a\, b}{c}$ is inessential so we can just as well say:

$$\mathbf{L} \equiv \left(\mathbb{O} - \frac{c}{a}\, e_2\right) \wedge \left(\mathbb{O} + \frac{c}{b}\, e_1\right)$$

Graphic of a line through a point on each axis.

◆ Information required to express a line in a plane

The expression of the line above in terms of a pair of points requires the four coordinates of the points. Expressed without specific reference to points, we seem to need three parameters. However, the last expression shows, as expected, that it is really only two parameters that are necessary (*viz* $y = m\, x + c$).

Lines in a 3-plane

Lines in a 3-plane $\mathbb{P}_3$ have the same form when expressed in coordinate-free notation as they do in a plane $\mathbb{P}_2$. Remember that a 3-plane is a bound vector 3-space whose basis may be chosen as 3 independent vectors and a point, or equivalently as 4 independent points. For example, we can still express a line in a 3-plane in any of the following equivalent forms.

$$\mathbf{L} \equiv (\mathbb{O} + \mathbf{x}) \wedge (\mathbb{O} + \mathbf{y})$$
$$\mathbf{L} \equiv (\mathbb{O} + \mathbf{x}) \wedge (\mathbf{y} - \mathbf{x})$$
$$\mathbf{L} \equiv (\mathbb{O} + \mathbf{y}) \wedge (\mathbf{y} - \mathbf{x})$$
$$\mathbf{L} \equiv \mathbb{O} \wedge (\mathbf{y} - \mathbf{x}) + \mathbf{x} \wedge \mathbf{y}$$

Here, $\mathbf{x}$ and $\mathbf{y}$ are independent vectors in the 3-plane.

The coordinate form however will appear somewhat different to that in the 2-plane case. To explore this, we redeclare the basis as $\mathbb{P}_3$.

$$\mathbb{P}_3$$

$$\{\mathbb{O},\ e_1,\ e_2,\ e_3\}$$

```
{X = CreateVector[x], Y = CreateVector[y]}
```

$$\{e_1\, x_1 + e_2\, x_2 + e_3\, x_3,\ e_1\, y_1 + e_2\, y_2 + e_3\, y_3\}$$

$$\mathbf{L} \equiv (\mathbb{O} + \mathbf{X}) \wedge (\mathbb{O} + \mathbf{Y})$$

$$L \equiv (\mathbb{O} + e_1\, x_1 + e_2\, x_2 + e_3\, x_3) \wedge (\mathbb{O} + e_1\, y_1 + e_2\, y_2 + e_3\, y_3)$$

Multiplying out this expression gives:

$$\mathbf{L} \equiv \mathcal{G}[(\mathbb{O} + e_1\, x_1 + e_2\, x_2 + e_3\, x_3) \wedge (\mathbb{O} + e_1\, y_1 + e_2\, y_2 + e_3\, y_3)]$$

$$\begin{aligned} L \equiv\ & \mathbb{O} \wedge (e_1\, (-x_1 + y_1) + e_2\, (-x_2 + y_2) + e_3\, (-x_3 + y_3)) + \\ & (-x_2\, y_1 + x_1\, y_2)\, e_1 \wedge e_2 + \\ & (-x_3\, y_1 + x_1\, y_3)\, e_1 \wedge e_3 + (-x_3\, y_2 + x_2\, y_3)\, e_2 \wedge e_3 \end{aligned}$$

The scalar coefficients in this expression are sometimes called the *Plücker coordinates* of the line.

Alternatively, we can express $\mathbf{L}$ in terms of basis elements, but without specific reference to points or vectors in it. For example:

$$\mathbf{L} \equiv \mathbb{O} \wedge (\mathbf{a}\, e_1 + \mathbf{b}\, e_2 + \mathbf{c}\, e_3) + \mathbf{d}\, e_1 \wedge e_2 + \mathbf{e}\, e_2 \wedge e_3 + \mathbf{f}\, e_1 \wedge e_3$$

The first term $\mathbb{O} \wedge (\mathbf{a}\, e_1 + \mathbf{b}\, e_2 + \mathbf{c}\, e_3)$ is a vector bound through the origin, and hence defines a line through the origin. The second term $\mathbf{d}\, e_1 \wedge e_2 + \mathbf{e}\, e_2 \wedge e_3 + \mathbf{f}\, e_1 \wedge e_3$ is a bivector whose addition represents a shift in the line parallel to itself, away from the origin. In order to effect this shift, however, it is necessary that the bivector contain the vector $(\mathbf{a}\, e_1 + \mathbf{b}\, e_2 + \mathbf{c}\, e_3)$. Hence there will be some constraint on the coefficients $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$. To determine this we only need to determine the condition that the exterior product of the vector and the bivector is zero.

$$\mathcal{G}[(a\,e_1 + b\,e_2 + c\,e_3) \wedge (d\,e_1 \wedge e_2 + e\,e_2 \wedge e_3 + f\,e_1 \wedge e_3) == 0]$$

$$(c\,d + a\,e - b\,f)\;e_1 \wedge e_2 \wedge e_3 == 0$$

Alternatively, this constraint amongst the coefficients could have been obtained by noting that in order to be a line, **L** must be simple, hence the exterior product with itself must be zero.

$$L = \mathbb{O} \wedge (a\,e_1 + b\,e_2 + c\,e_3) + d\,e_1 \wedge e_2 + e\,e_2 \wedge e_3 + f\,e_1 \wedge e_3;$$

$$\mathcal{G}[L \wedge L == 0]$$

$$2\,(c\,d + a\,e - b\,f)\;\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 == 0$$

Thus the constraint that the coefficients must obey in order for a general bound vector of the form **L** to be a line in a 3-plane is that:

$$c\,d + a\,e - b\,f == 0$$

This constraint is sometimes referred to as the *Plücker identity*.

Given this constraint, and supposing neither **a**, **b** or **c** is zero, we can factorize the line into any of the following forms:

$$L == \left(\mathbb{O} + \frac{f}{c}\,e_1 + \frac{e}{c}\,e_2\right) \wedge (a\,e_1 + b\,e_2 + c\,e_3)$$

$$L == \left(\mathbb{O} - \frac{d}{a}\,e_2 - \frac{f}{a}\,e_3\right) \wedge (a\,e_1 + b\,e_2 + c\,e_3)$$

$$L == \left(\mathbb{O} + \frac{d}{b}\,e_1 - \frac{e}{b}\,e_3\right) \wedge (a\,e_1 + b\,e_2 + c\,e_3)$$

Each of these forms represents a line in the direction of $a\,e_1 + b\,e_2 + c\,e_3$ and intersecting a coordinate plane. For example, the first form intersects the $\mathbb{O} \wedge e_1 \wedge e_2$ coordinate plane in the point $\mathbb{O} + \frac{f}{c}\,e_1 + \frac{e}{c}\,e_2$ with coordinates ($\frac{f}{c}$, $\frac{e}{c}$, **0**).

The most compact form, in terms of the number of scalar parameters used, is when **L** is expressed as the product of two points, each of which lies in a coordinate plane.

$$L = \frac{a\,c}{f}\left(\mathbb{O} - \frac{d}{a}\,e_2 - \frac{f}{a}\,e_3\right) \wedge \left(\mathbb{O} + \frac{f}{c}\,e_1 + \frac{e}{c}\,e_2\right);$$

We can verify that this formulation gives us the original form of the line by expanding the product and substituting the constraint relation previously obtained.

$$\mathcal{G}[L]\;/.\;(c\,d + a\,e \to f\,b)$$

$$\mathbb{O} \wedge (a\,e_1 + b\,e_2 + c\,e_3) + d\,e_1 \wedge e_2 + f\,e_1 \wedge e_3 + e\,e_2 \wedge e_3$$

Similar expressions may be obtained for **L** in terms of points lying in the other coordinate planes. To summarize, there are three possibilities in a 3-plane, corresponding to the number of different pairs of coordinate 2-planes in the 3-plane.

$$\mathbf{L} \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2) \wedge (\mathbb{O} + y_2\, e_2 + y_3\, e_3)$$
$$\mathbf{L} \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2) \wedge (\mathbb{O} + z_1\, e_1 + z_3\, e_3) \qquad 4.8$$
$$\mathbf{L} \equiv (\mathbb{O} + y_2\, e_2 + y_3\, e_3) \wedge (\mathbb{O} + z_1\, e_1 + z_3\, e_3)$$

Graphic of a line through 3 points, one in each coordinate plane.

Information required to express a line in a 3-plane

As with a line in a 2-plane, we find that a line in a 3-plane is expressed with the minimum number of parameters by expressing it as the product of two points, each in one of the coordinate planes. In this form, there are just 4 independent scalar parameters (coordinates) required to express the line.

Checking the invariance of the description of the line

We can use *GrassmannAlgebra* to explore the invariance of how a line is expressed. Again, suppose we are in a 3-plane.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

Define a line $\mathbf{L}$ as the exterior product of two points, one in the $\mathbb{O} \wedge e_1 \wedge e_2$ coordinate plane and one in the $\mathbb{O} \wedge e_2 \wedge e_3$ coordinate plane.

$\mathbf{L} = (\mathbb{O} + x_1\, e_1 + x_2\, e_2) \wedge (\mathbb{O} + y_2\, e_2 + y_3\, e_3);$

Declare the coordinates as scalars.

DeclareExtraScalars$[\{x_1, x_2, y_2, y_3\}];$

Verify that the intersection of the line with the first coordinate plane does indeed give a point congruent to the first point.

$P_1 = \mathcal{G}[\mathbf{L} \vee (\mathbb{O} \wedge e_1 \wedge e_2)]$ // **ToCongruenceForm**

$$\frac{(\mathbb{O} + e_1\, x_1 + e_2\, x_2)\, y_3}{\mathbb{k}}$$

Next determine the (weighted) point in the line in the *third* coordinate plane (the coordinate plane which did not figure in the original definition of the line).

$P_2 = \mathcal{G}[\mathbf{L} \vee (\mathbb{O} \wedge e_1 \wedge e_3)]$ // **ToCongruenceForm**

$$\frac{\mathbb{O}\,(x_2 - y_2) - e_1\, x_1\, y_2 + e_3\, x_2\, y_3}{\mathbb{k}}$$

We can now confirm that the exterior product of these two points is still congruent to the original specification of the line by expanding the quotient of the two expressions and showing that it reduces to a scalar.

$$\text{ScalarQ}\left[\text{Simplify}\left[\text{FactorScalars}\left[\text{ExpandProducts}\left[\frac{\mathcal{G}[P_1 \wedge P_2]}{\mathcal{G}[L]}\right]\right]\right]\right]$$

True

Lines in a 4-plane

Lines in a 4-plane $\mathbb{P}_4$ have the same form when expressed in coordinate-free notation as they do in any multiplane.

To obtain the Plücker coordinates of a line in a 4-plane, express the line as the exterior product of two points and multiply it out. The resulting coefficients of the basis elements are the Plücker coordinates of the line.

Additionally, from the results above, we can expect that a line in a 4-plane may be expressed with the least number of scalar parameters as the exterior product of two points, each point lying in one of the coordinate 3-planes. For example, the expression for the line as the product of the points in the coordinate 3-planes $\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3$ and $\mathbb{O} \wedge e_2 \wedge e_3 \wedge e_4$ is

$$L = (\mathbb{O} + x_1\, e_1 + x_2\, e_2 + x_3\, e_3) \wedge (\mathbb{O} + y_2\, e_2 + y_3\, e_3 + y_4\, e_4)$$

Lines in an *m*-plane

The formulae below summarize some of the expressions for defining a line, valid in a multiplane of any dimension.

Coordinate-free expressions may take any of a number of forms. For example:

$$\begin{gathered} L \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y) \\ L \equiv (\mathbb{O} + x) \wedge (y - x) \\ L \equiv (\mathbb{O} + y) \wedge (y - x) \\ L \equiv \mathbb{O} \wedge (y - x) + x \wedge y \end{gathered} \qquad 4.9$$

A line can be expressed in terms of the 2*m* coordinates of any two points on it.

$$L \equiv (\mathbb{O} + x_1\, e_1 + x_2\, e_2 + \cdots + x_m\, e_m) \wedge (\mathbb{O} + y_1\, e_1 + y_2\, e_2 + \cdots + y_m\, e_m) \qquad 4.10$$

When multiplied out, the expression for the line takes a form explicitly displaying the Plücker coordinates of the line.

$$\begin{gathered} L \equiv \mathbb{O} \wedge ((y_1 - x_1)\, e_1 + (y_2 - x_2)\, e_2 + \cdots + (y_m - x_m)\, e_m) \\ + (x_1\, y_2 - x_2\, y_1)\, e_1 \wedge e_2 + (x_1\, y_3 - x_3\, y_1)\, e_1 \wedge e_3 + \\ (x_1\, y_4 - x_4\, y_1)\, e_1 \wedge e_4 + \cdots + (x_{m-1}\, y_m - x_m\, y_{m-1})\, e_{m-1} \wedge e_m \end{gathered} \qquad 4.11$$

Alternatively, a line in an m-plane $\mathbb{O} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_m$ can be expressed in terms of its intersections with two of its coordinate (m–1)-planes, $\mathbb{O} \wedge e_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge e_m$ and $\mathbb{O} \wedge e_1 \wedge \cdots \wedge \square_j \wedge \cdots \wedge e_m$ say. The notation $\square_i$ means that the ith element or term is missing.

$$L \equiv (\mathbb{O} + x_1\, e_1 + \cdots + \square_i + \cdots + x_m\, e_m) \wedge (\mathbb{O} + y_1\, e_1 + \cdots + \square_j + \cdots + y_m\, e_m) \qquad 4.12$$

This formulation indicates that a line in m-space has at most 2(m–1) independent parameters required to describe it.

It also implies that in the special case when the line lies in one of the coordinate (m–1)-spaces, it can be even more economically expressed as the product of two points, each lying in one of the coordinate (m–2)-spaces contained in the (m–1)-space. And so on.

4.9 Plane Coordinates

We have already seen that planes are defined by simple bound bivectors independent of the dimension of the space. We now look at the types of coordinate descriptions we can use to define planes in bound spaces (multiplanes) of various dimensions.

Planes in a 3-plane

A plane Π in a 3-plane can be written in several forms. The most intuitive form perhaps is as a product of three non-collinear points $\mathbb{O}+x$, $\mathbb{O}+y$ and $\mathbb{O}+z$, where x, y and z are vectors.

$$\Pi \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y) \wedge (\mathbb{O} + z)$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any two different points in it and a vector parallel to it (but not in the direction of the line joining the two points). For example:

$$\Pi \equiv (\mathbb{O} + x) \wedge (\mathbb{O} + y) \wedge (z - x)$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any point in it and any two independent vectors parallel to it. For example:

$$\Pi \equiv (\mathbb{O} + x) \wedge (y - x) \wedge (z - x)$$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any line in it and any point in it not in the line. For example:

$\Pi \equiv \mathbf{L} \wedge (\mathbb{O} + \mathbf{x})$

Graphic of a plane with the preceding definition.

Or, we can express it as the product of any line in it and any vector parallel to it (but not parallel to the line). For example:

$\Pi \equiv \mathbf{L} \wedge (\mathbf{z} - \mathbf{x})$

Graphic of a plane with the preceding definition.

Given a basis, we can always express the plane in terms of the coordinates of the points or vectors in the expressions above. However the form which requires the least number of coordinates is that which expresses the plane as the exterior product of its three points of intersection with the coordinate axes.

$\Pi \equiv (\mathbb{O} + a\, e_1) \wedge (\mathbb{O} + b\, e_2) \wedge (\mathbb{O} + c\, e_3)$

Graphic of a plane with the preceding definition.

If the plane is parallel to one of the coordinate axes, say $\mathbb{O} \wedge e_3$, it may be expressed as:

$\Pi \equiv (\mathbb{O} + a\, e_1) \wedge (\mathbb{O} + b\, e_2) \wedge e_3$

Whereas, if it is parallel to two of the coordinate axes, say $\mathbb{O} \wedge e_2$ and $\mathbb{O} \wedge e_3$, it may be expressed as:

$\Pi \equiv (\mathbb{O} + a\, e_1) \wedge e_2 \wedge e_3$

If we wish to express a plane as the exterior product of its intersection points with the coordinate axes, we first determine its points of intersection with the axes and then take the exterior product of the resulting points. This leads to the following identity:

$\Pi \equiv (\Pi \vee (\mathbb{O} \wedge e_1)) \wedge (\Pi \vee (\mathbb{O} \wedge e_2)) \wedge (\Pi \vee (\mathbb{O} \wedge e_3))$

■ Example: To express a plane in terms of its intersections with the coordinate axes

Suppose we have a plane in a 3-plane defined by three points.

$\mathbb{P}_3;\ \Pi = (\mathbb{O} + e_1 + 2\, e_2 + 5\, e_3) \wedge (\mathbb{O} - e_1 + 9\, e_2) \wedge (\mathbb{O} - 7\, e_1 + 6\, e_2 + 4\, e_3)$

$(\mathbb{O} + e_1 + 2\, e_2 + 5\, e_3) \wedge (\mathbb{O} - e_1 + 9\, e_2) \wedge (\mathbb{O} - 7\, e_1 + 6\, e_2 + 4\, e_3)$

To express this plane in terms of its intersections with the coordinate axes we calculate the intersection points with the axes.

$\mathcal{G}[\Pi \vee (\mathbb{O} \wedge e_1)]$

$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (13\, \mathbb{O} + 329\, e_1)$

$$\mathcal{G}[\Pi \vee (\mathbb{O} \wedge e_2)]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (38\,\mathbb{O} + 329\,e_2)$$

$$\mathcal{G}[\Pi \vee (\mathbb{O} \wedge e_3)]$$

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (48\,\mathbb{O} + 329\,e_3)$$

We then take the product of these points (ignoring the weights) to form the plane.

$$\Pi \equiv \left(\mathbb{O} + \frac{329\,e_1}{13}\right) \wedge \left(\mathbb{O} + \frac{329\,e_2}{38}\right) \wedge \left(\mathbb{O} + \frac{329\,e_3}{48}\right)$$

To verify that this is indeed the same plane, we can check to see if these points are in the original plane. For example:

$$\mathcal{G}\left[\Pi \wedge \left(\mathbb{O} + \frac{329\,e_1}{13}\right)\right]$$

$$0$$

Planes in a 4-plane

From the results above, we can expect that a plane in a 4-plane is most economically expressed as the product of three points, each point lying in one of the coordinate 2-planes. For example:

$$\Pi \equiv (\mathbb{O} + x_1\,e_1 + x_2\,e_2) \wedge (\mathbb{O} + y_2\,e_2 + y_3\,e_3) \wedge (\mathbb{O} + z_3\,e_3 + z_4\,e_4)$$

If a plane is expressed in any other form, we can express it in the form above by first determining its points of intersection with the coordinate planes and then taking the exterior product of the resulting points. This leads to the following identity:

$$\Pi \equiv (\Pi \vee (\mathbb{O} \wedge e_1 \wedge e_2)) \wedge (\Pi \vee (\mathbb{O} \wedge e_2 \wedge e_3)) \wedge (\Pi \vee (\mathbb{O} \wedge e_3 \wedge e_4))$$

Planes in an *m*-plane

A plane in an *m*-plane is most economically expressed as the product of three points, each point lying in one of the coordinate (*m*–2)-planes.

$$\begin{aligned}\Pi \equiv\ & \left(\mathbb{O} + x_1\,e_1 + \cdots + \underbrace{x_{i_1}\,e_{i_1}} + \cdots + \underbrace{x_{i_2}\,e_{i_2}} + \cdots + x_m\,e_m\right)\\ \wedge\ & \left(\mathbb{O} + y_1\,e_1 + \cdots + \underbrace{y_{i_3}\,e_{i_3}} + \cdots + \underbrace{y_{i_4}\,e_{i_4}} + \cdots + y_m\,e_m\right)\\ \wedge\ & \left(\mathbb{O} + z_1\,e_1 + \cdots + \underbrace{z_{i_5}\,e_{i_5}} + \cdots + \underbrace{z_{i_6}\,e_{i_6}} + \cdots + z_m\,e_m\right)\end{aligned}$$

Here the notation $\underbrace{x_i\,e_i}$ means that the term is *missing* from the sum.

This formulation indicates that a plane in an *m*-plane has at most 3(*m*–2) independent scalar parameters required to describe it.

4.10 Calculation of Intersections

The intersection of two lines in a plane

Suppose we wish to find the point **P** of intersection of two lines $\mathbf{L_1}$ and $\mathbf{L_2}$ in a plane. We have seen in the previous section how we could express a line in a plane as the exterior product of two points, and that these points could be taken as the points of intersection of the line with the coordinate axes.

First declare the basis of the plane, and then define the lines.

$\mathbb{P}_2$

$\{\mathbb{O}, e_1, e_2\}$

$$L_1 = (\mathbb{O} + a\, e_1) \wedge (\mathbb{O} + b\, e_2);$$
$$L_2 = (\mathbb{O} + c\, e_1) \wedge (\mathbb{O} + d\, e_2);$$

Next, take the regressive product of the two lines and simplify it.

$\mathcal{G}[L_1 \vee L_2]$

$\mathbb{O} \wedge e_1 \wedge e_2 \vee ((b\,c - a\,d)\,\mathbb{O} + a\,c\,(b - d)\,e_1 + b\,(-a + c)\,d\,e_2)$

This shows that the intersection point **P** is given by:

$$P = \mathbb{O} + \frac{a\,c\,(b - d)}{b\,c - a\,d}\,e_1 + \frac{b\,d\,(c - a)}{b\,c - a\,d}\,e_2;$$

To verify that this point lies in both lines, we can take its exterior product with each of the lines and show the result to be zero.

$\mathcal{G}[\{L_1 \wedge P, L_2 \wedge P\}]$

$\{0, 0\}$

In the special case in which the lines are parallel, that is b c - a d = 0, their intersection is no longer a point, but a *vector defining their common direction*.

The intersection of a line and a plane in a 3-plane

Suppose we wish to find the point **P** of intersection of a line **L** and a plane **Π** in a 3-plane. We express the line as the exterior product of two points in two coordinate planes, and the plane as the exterior product of the points of intersection of the plane with the coordinate axes.

First declare the basis of the 3-plane, and then define the line and plane.

$\mathbb{P}_3$

$\{\mathbb{O}, e_1, e_2, e_3\}$

```
L = (𝕆 + a e1 + b e2) ∧ (𝕆 + c e2 + d e3);
Π = (𝕆 + e e1) ∧ (𝕆 + f e2) ∧ (𝕆 + g e3);
```

Next, take the regressive product of the line and the plane and simplify it.

```
𝒢[L ∨ Π]
```

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee ((d\,e\,f - (b\,e - c\,e + a\,f)\,g)\,\mathbb{O} + a\,e\,(d\,f + (c - f)\,g)\,e_1 + f\,(b\,e\,(d - g) + c\,(-a + e)\,g)\,e_2 - d\,(b\,e + (a - e)\,f)\,g\,e_3)$$

This shows that the intersection point **P** is given by:

$$P = \mathbb{O} + \frac{a\,e\,(d\,f + (c - f)\,g)}{d\,e\,f - (b\,e - c\,e + a\,f)\,g}\,e_1 + \frac{f\,(b\,e\,(d - g) + c\,(-a + e)\,g)}{d\,e\,f - (b\,e - c\,e + a\,f)\,g}\,e_2 - \frac{d\,(b\,e + (a - e)\,f)\,g}{d\,e\,f - (b\,e - c\,e + a\,f)\,g}\,e_3;$$

To verify that this point lies in both the line and the plane, we can take its exterior product with each of the lines and show the result to be zero.

```
𝒢[{L ∧ P, Π ∧ P}]
```

{0, 0}

In the special case in which the line is parallel to the plane, their intersection is no longer a point, but a vector defining their common direction. When the line lies in the plane, the result from the calculation will be zero.

The intersection of two planes in a 3-plane

Suppose we wish to find the line **L** of intersection of two planes Π_1 and Π_2 in a 3-plane.

First declare the basis of the 3-plane, and then define the planes.

```
ℙ3
```

$\{\mathbb{O}, e_1, e_2, e_3\}$

```
Π1 = (𝕆 + a e1) ∧ (𝕆 + b e2) ∧ (𝕆 + c e3);
Π2 = (𝕆 + e e1) ∧ (𝕆 + f e2) ∧ (𝕆 + g e3);
```

Next, take the regressive product of the two lines and simplify it.

```
𝒢[Π1 ∨ Π2]
```

$$\mathbb{O} \wedge e_1 \wedge e_2 \wedge e_3 \vee (\mathbb{O} \wedge (a\,e\,(c\,f - b\,g)\,e_1 + b\,f\,(-c\,e + a\,g)\,e_2 + c\,(b\,e - a\,f)\,g\,e_3) + a\,b\,e\,f\,(-c + g)\,e_1 \wedge e_2 + a\,c\,e\,(b - f)\,g\,e_1 \wedge e_3 + b\,c\,(-a + e)\,f\,g\,e_2 \wedge e_3)$$

This shows that the line of intersection **L** is given by:

$$\mathbf{L} = \mathbb{O} \wedge (\mathbf{a\,e\,(c\,f - b\,g)}\, e_1 + \mathbf{b\,f\,(a\,g - c\,e)}\, e_2 + \mathbf{c\,g\,(b\,e - a\,f)}\, e_3) \\ +\mathbf{a\,b\,e\,f\,(g - c)}\, e_1 \wedge e_2 \\ +\mathbf{a\,c\,e\,g\,(b - f)}\, e_1 \wedge e_3 \\ +\mathbf{b\,c\,f\,g\,(e - a)}\, e_2 \wedge e_3;$$

To verify that this line lies in both planes, we can take its exterior product with each of the planes and show the result to be zero.

$\mathcal{G}[\{\Pi_1 \wedge \mathbf{L}, \Pi_2 \wedge \mathbf{L}\}]$

$\{0, 0\}$

In the special case in which the planes are parallel, their intersection is no longer a line, but a *bivector defining their common 2-direction*.

Example: The osculating plane to a curve

◆ The problem

Show that the osculating planes at any three points to the curve defined by:

$$\mathbf{P} == \mathbb{O} + \mathbf{u}\, e_1 + \mathbf{u}^2\, e_2 + \mathbf{u}^3\, e_3$$

intersect at a point coplanar with these three points.

◆ The solution

The osculating plane Π to the curve at the point $\mathbf{P}$ is given by $\Pi == \mathbf{P} \wedge \mathbf{P}' \wedge \mathbf{P}''$, where $\mathbf{u}$ is a scalar parametrizing the curve, and $\mathbf{P}'$ and $\mathbf{P}''$ are the first and second derivatives of $\mathbf{P}$ with respect to $\mathbf{u}$.

$\Pi = \mathbf{P} \wedge \mathbf{P}' \wedge \mathbf{P}'';$

$\mathbf{P} = \mathbb{O} + \mathbf{u}\, e_1 + \mathbf{u}^2\, e_2 + \mathbf{u}^3\, e_3;$

$\mathbf{P}' = e_1 + 2\,\mathbf{u}\, e_2 + 3\,\mathbf{u}^2\, e_3;$

$\mathbf{P}'' = 2\, e_2 + 6\,\mathbf{u}\, e_3;$

We can declare the space to be a 3-plane and the parameter $\mathbf{u}$ (and subscripts of it) to be a scalar, and then use `GrassmannSimplify` to derive the expression for the osculating plane as a function of $\mathbf{u}$.

$\mathbb{P}_3$; DeclareExtraScalars[{u, u_}]; $\mathcal{G}[\Pi]$

$2\, \mathbb{O} \wedge e_1 \wedge e_2 + 6\, u\, \mathbb{O} \wedge e_1 \wedge e_3 + 6\, u^2\, \mathbb{O} \wedge e_2 \wedge e_3 + 2\, u^3\, e_1 \wedge e_2 \wedge e_3$

Now select any three points on the curve $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$.

$$P_1 = \mathbb{O} + u_1\, e_1 + u_1{}^2\, e_2 + u_1{}^3\, e_3;$$
$$P_2 = \mathbb{O} + u_2\, e_1 + u_2{}^2\, e_2 + u_2{}^3\, e_3;$$
$$P_3 = \mathbb{O} + u_3\, e_1 + u_3{}^2\, e_2 + u_3{}^3\, e_3;$$

The osculating planes at these three points are:

$$\Pi_1 = \mathbb{O}\wedge e_1\wedge e_2 + 3\, u_1\, \mathbb{O}\wedge e_1\wedge e_3 + 3\, u_1{}^2\, \mathbb{O}\wedge e_2\wedge e_3 + u_1{}^3\, e_1\wedge e_2\wedge e_3;$$
$$\Pi_2 = \mathbb{O}\wedge e_1\wedge e_2 + 3\, u_2\, \mathbb{O}\wedge e_1\wedge e_3 + 3\, u_2{}^2\, \mathbb{O}\wedge e_2\wedge e_3 + u_2{}^3\, e_1\wedge e_2\wedge e_3;$$
$$\Pi_3 = \mathbb{O}\wedge e_1\wedge e_2 + 3\, u_3\, \mathbb{O}\wedge e_1\wedge e_3 + 3\, u_3{}^2\, \mathbb{O}\wedge e_2\wedge e_3 + u_3{}^3\, e_1\wedge e_2\wedge e_3;$$

The point of intersection of these three planes may be obtained by calculating their regressive product.

$$\mathcal{G}[\Pi_1 \vee \Pi_2 \vee \Pi_3]$$

$$\begin{aligned}&\mathbb{O}\wedge e_1\wedge e_2\wedge e_3 \vee \mathbb{O}\wedge e_1\wedge e_2\wedge e_3 \vee (-9\,\mathbb{O}\,(u_1-u_2)\,(u_1-u_3)\,(u_2-u_3) +\\ &\quad 3\, e_1\,(u_1-u_2)\,(u_1-u_3)\,(-u_1-u_2-u_3)\,(u_2-u_3) -\\ &\quad 9\, e_3\, u_1\,(u_1-u_2)\, u_2\,(u_1-u_3)\,(u_2-u_3)\, u_3 +\\ &\quad 3\, e_2\,(u_1-u_2)\,(u_1-u_3)\,(u_2-u_3)\,(-u_2\, u_3 - u_1\,(u_2+u_3)))\end{aligned}$$

This expression is congruent to the point of intersection which we write more simply as:

$$Q = \mathbb{O} + \frac{1}{3}\,(u_1+u_2+u_3)\, e_1 + \frac{1}{3}\,(u_1\, u_2 + u_2\, u_3 + u_3\, u_1)\, e_2 + (u_1\, u_2\, u_3)\, e_3;$$

Finally, to show that this point of intersection Q is coplanar with the points P_1, P_2, and P_3, we compute their exterior product.

$$\mathcal{G}[P_1 \wedge P_2 \wedge P_3 \wedge Q]$$

$$0$$

This proves the original assertion.

5 The Complement

5.1 Introduction

Up to this point various linear spaces and the dual exterior and regressive product operations have been introduced. The elements of these spaces were incommensurable unless they were congruent, that is, nowhere was there involved the concept of measure or magnitude of an element by which it could be compared with any other element. Thus the subject of the last chapter on Geometric Interpretations was explicitly non-metric geometry; or, to put it another way, it was what geometry is before the ability to compare or measure is added.

The question then arises, how do we associate a measure with the elements of a Grassmann algebra in a consistent way? Of course, we already know the approach that has developed over the last century, that of defining a metric tensor. In this book we will indeed define a metric tensor, but we will take an approach which develops from the concepts of the exterior product and the duality operations which are the foundations of the Grassmann algebra. This will enable us to see how the metric tensor on the underlying linear space generates metric tensors on the exterior linear spaces of higher grade; the implications of the symmetry of the metric tensor; and how consistently to generalize the notion of inner product to elements of arbitrary grade.

One of the consequences of the anti-symmetry of the exterior product of 1-elements is that the exterior linear space of *m*-elements has the same dimension as the exterior linear space of (*n*–*m*)-elements. We have already seen this property evidenced in the notions of duality and cobasis element. And one of the consequences of the notion of duality is that the regressive product of an *m*-element and an (*n*–*m*)-element is a *scalar*. Thus there is the opportunity of defining for each *m*-element a corresponding 'co-*m*-element' of grade *n*–*m* such that the regressive product of these two elements gives a scalar. We will see that this scalar measures the square of the 'magnitude' of the *m*-element or the (*n*–*m*)-element, and corresponds to the inner product of either of them with itself. We will also see that the notion of orthogonality is *defined* by the correspondence between *m*-elements and their 'co-*m*-elements'. But most importantly, the definition of this inner product as a regressive product of an *m*-element with a 'co-*m*-element' is immediately generalizable to elements of arbitrary grade, thus permitting a theory of interior products to be developed which is consistent with the exterior and regressive product axioms and which, via the notion of 'co-*m*-element', leads to explicit and easily derived formulae between elements of arbitrary (and possibly different) grade.

The foundation of the notion of measure or metric then is the notion of 'co-*m*-element'. In this book we use the term *complement* rather than 'co-*m*-element'. In this chapter we will develop the notion of complement in preparation for the development of the notions of interior product and orthogonality in the next chapter.

The complement of an element is denoted with a horizontal bar over the element. For example $\underset{m}{\alpha}$, $x+y$ and $x \wedge y$ are denoted $\overline{\underset{m}{\alpha}}$, $\overline{x+y}$ and $\overline{x \wedge y}$.

Finally, it should be noted that the term 'complement' may be used either to refer to an operation (the operation of taking the complement of an element), or to the element itself (which is the result of the operation).

Historical Note

Grassmann introduced the notion of complement (*Ergänzung*) into the *Ausdehnungslehre* of 1862 [Grassmann 1862]. He denoted the complement of an element x by preceding it with a vertical bar, *viz* $|x$. For mnemonic reasons which will become apparent later, the notation for the complement used in this book is rather the horizontal bar: $\overline{x}$. In discussing the complement, Grassmann defines the product of the *n* basis elements (the basis *n*-element) to be unity. That is $[e_1\, e_2\, \cdots\, e_n] == 1$ or, in the present notation, $e_1 \wedge e_2 \wedge \cdots \wedge e_n == 1$. Since Grassmann discussed only the *Euclidean* complement (equivalent to imposing a Euclidean metric $g_{ij} == \delta_{ij}$), this statement in the present notation is equivalent to $\overline{1} == 1$. The introduction of such an identity, however, destroys the essential duality between $\underset{m}{\Lambda}$ and $\underset{n-m}{\Lambda}$ which requires rather the identity $\overline{\overline{1}} == 1$. In current terminology, equating $e_1 \wedge e_2 \wedge \cdots \wedge e_n$ to 1 is equivalent to equating *n*-elements (or pseudo-scalars) and scalars. All other writers in the Grassmannian tradition (for example, Hyde, Whitehead and Forder) followed Grassmann's approach. This enabled them to use the same notation for both the progressive and regressive products. While being an attractive approach in a Euclidean system, it is not tenable for general metric spaces. The tenets upon which the *Ausdehnungslehre* are based are so geometrically fundamental however, that it is readily extended to more general metrics.

5.2 Axioms for the Complement

The grade of a complement

◆ $\overline{1}$: **The complement of an *m*-element is an (*n–m)*-element.**

$$\underset{m}{\alpha} \in \underset{m}{\Lambda} \Longrightarrow \overline{\underset{m}{\alpha}} \in \underset{n-m}{\Lambda} \qquad 5.1$$

The grade of the complement of an element is the complementary grade of the element.

The linearity of the complement operation

◆ $\overline{2}$: **The complement operation is linear.**

$$\overline{\mathbf{a}\,\underset{m}{\alpha} + \mathbf{b}\,\underset{k}{\beta}} == \overline{\mathbf{a}\,\underset{m}{\alpha}} + \overline{\mathbf{b}\,\underset{k}{\beta}} == \mathbf{a}\,\overline{\underset{m}{\alpha}} + \mathbf{b}\,\overline{\underset{k}{\beta}} \qquad 5.2$$

For scalars **a** and **b**, the complement of a sum of elements (perhaps of different grades) is the sum of the complements of the elements. The complement of a scalar multiple of an element is the scalar multiple of the complement of the element.

The complement axiom

◆ $\overline{3}$: **The complement of a product is the dual product of the complements.**

$$\overline{\underset{m}{\alpha} \wedge \underset{k}{\beta}} == \overline{\underset{m}{\alpha}} \vee \overline{\underset{k}{\beta}} \qquad 5.3$$

$$\overline{\underset{m}{\alpha} \vee \underset{k}{\beta}} == \overline{\underset{m}{\alpha}} \wedge \overline{\underset{k}{\beta}} \qquad 5.4$$

Note that for the terms on each side of the expression 5.3 to be non-zero we require $m+k \leq n$, while in expression 5.4 we require $m+k \geq n$.

Expressions 5.3 and 5.4 are duals of each other. We call these dual expressions the *complement axiom*. Note its enticing similarity to de Morgan's law in Boolean algebra.

This axiom is of central importance in the development of the properties of the complement and interior, inner and scalar products, and formulae relating these with exterior and regressive products. In particular, it permits us to be able to consistently generate the complements of basis *m*-elements from the complements of basis 1-elements, and hence via the linearity axiom, the complements of arbitrary elements.

We may verify that these are dual formulae by applying the *GrassmannAlgebra* `Dual` function to either of them. For example:

$$\mathbf{Dual}\left[\overline{\underset{m}{\alpha}\wedge\underset{k}{\beta}} == \overline{\underset{m}{\alpha}}\vee\overline{\underset{k}{\beta}}\right]$$

$$\overline{\underset{m}{\alpha}\vee\underset{k}{\beta}} == \overline{\underset{m}{\alpha}}\wedge\overline{\underset{k}{\beta}}$$

The forms 5.3 and 5.4 may be written for any number of elements. To see this, let $\underset{k}{\beta} == \underset{p}{\gamma}\wedge\underset{q}{\delta}$ and substitute for $\underset{k}{\beta}$ in expression 5.3:

$$\overline{\underset{m}{\alpha}\wedge\underset{p}{\gamma}\wedge\underset{q}{\delta}} == \overline{\underset{m}{\alpha}}\vee\overline{\underset{p}{\gamma}\wedge\underset{q}{\delta}} == \overline{\underset{m}{\alpha}}\vee\overline{\underset{p}{\gamma}}\vee\overline{\underset{q}{\delta}}$$

In general then, expressions 5.3 and 5.4 may be stated in the equivalent forms:

$$\overline{\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\cdots\wedge\underset{p}{\gamma}} == \overline{\underset{m}{\alpha}}\vee\overline{\underset{k}{\beta}}\vee\cdots\vee\overline{\underset{p}{\gamma}} \qquad 5.5$$

$$\overline{\underset{m}{\alpha}\vee\underset{k}{\beta}\vee\cdots\vee\underset{p}{\gamma}} == \overline{\underset{m}{\alpha}}\wedge\overline{\underset{k}{\beta}}\wedge\cdots\wedge\overline{\underset{p}{\gamma}} \qquad 5.6$$

In Grassmann's work, this axiom was hidden in his notation. However, since modern notation explicitly distinguishes the progressive and regressive products, this axiom needs to be explicitly stated.

The complement of a complement axiom

◆ **$\overline{4}$: The complement of the complement of an element is equal (apart from a possible sign) to the element itself.**

$$\overline{\overline{\underset{m}{\alpha}}} == (-1)^{\phi}\,\underset{m}{\alpha} \qquad 5.7$$

Axiom $\overline{1}$ says that the complement of an *m*-element is an (*n*–*m*)-element. Clearly then the complement of an (*n*–*m*)-element is an *m*-element. Thus the complement of the complement of an *m*-element is itself an *m*-element.

In the interests of symmetry and simplicity we will require that the complement of the complement of an element is equal (apart from a possible sign) to the element itself. Although consistent algebras could no doubt be developed by rejecting this axiom, it will turn out to be an essential underpinning to the development of the standard metric concepts to which we are accustomed. For example, its satisfaction will require the metric tensor to be symmetric.

It will turn out that, again for compliance with standard results, the index ϕ is $m(n-m)$, but this result is more in the nature of a theorem than an axiom.

The complement of unity

◆ $\overline{5}$: **The complement of unity is equal to the unit *n*-element.**

$$\overline{1} == \underset{n}{1} == \mathbb{k}\ e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 5.8$$

This is the axiom which finally enables us to define the unit *n*-element. This axiom requires that in a metric space the unit *n*-element be identical to the complement of unity. The hitherto unspecified scalar constant $\mathbb{k}$ may now be determined from the specific complement mapping or metric under consideration.

The dual of this axiom is:

$$\text{Dual}\left[\overline{\underset{0}{1}} == \underset{n}{1}\right]$$

$$\overline{\underset{n}{1}} == \underset{0}{1}$$

Hence taking the complement of expression 5.8 and using this dual axiom tells us that the complement of the complement of unity is unity. This result clearly complies with axiom $\overline{4}$.

$$\overline{\overline{1}} == \overline{\underset{n}{1}} == 1 \qquad 5.9$$

We can now write the unit *n*-element as $\overline{1}$ instead of $\underset{n}{1}$ in any Grassmann algebras that have a metric. For example in a metric space, $\overline{1}$ becomes the unit for the regressive product.

$$\underset{m}{\alpha} == \overline{1} \vee \underset{m}{\alpha} \qquad 5.10$$

5.3 Defining the Complement

The complement of an *m*-element

To define the complement of a general *m*-element, we need only define the complement of basis elements, since by the linearity axiom $\overline{2}$, we have that for $\underset{m}{\alpha}$ a general *m*-element expressed as a linear combination of basis *m*-elements, the complement of $\underset{m}{\alpha}$ is the corresponding linear combination of the complements of the basis *m*-elements.

$$\underset{m}{\alpha} == \sum a_i \underset{m}{e_i} \quad \Leftrightarrow \quad \overline{\underset{m}{\alpha}} == \sum a_i \overline{\underset{m}{e_i}} \qquad 5.11$$

The complement of a basis *m*-element

To define the complement of an *m*-element we need to define the complements of the basis *m*-elements. The complement of a basis *m*-element however, cannot be defined independently of the complements of basis elements in exterior linear spaces of other grades, since they are related by the complement axiom. For example, the complement of a basis 4-element may also be expressed as the regressive product of two basis 2-elements, or as the regressive product of a basis 3-element and a basis 1-element.

$$\overline{e_1 \wedge e_2 \wedge e_3 \wedge e_4} == \overline{e_1 \wedge e_2} \vee \overline{e_3 \wedge e_4} == \overline{e_1 \wedge e_2 \wedge e_3} \vee \overline{e_4}$$

The complement axiom enables us to define the complement of a basis *m*-element in terms *only* of the complements of basis 1-elements.

$$\overline{e_1 \wedge e_2 \wedge \cdots \wedge e_m} == \overline{e_1} \vee \overline{e_2} \vee \cdots \vee \overline{e_m} \qquad 5.12$$

Thus in order to define the complement of any element in a Grassmann algebra, we only need to define the complements of the basis 1-elements, that is, the correspondence between basis 1-elements and basis (*n*–1)-elements.

Defining the complement of a basis 1-element

The most general definition we could devise for the complement of a basis 1-element is a linear combination of basis (*n*–1)-elements. For example in a 3-space we could define the complement of e_1 as a linear combination of the three basis 2-elements.

$$\overline{e_1} == a_{12}\, e_1 \wedge e_2 + a_{13}\, e_1 \wedge e_3 + a_{23}\, e_2 \wedge e_3$$

However, it will be much more notationally convenient to define the complements of basis 1-elements as a linear combination of their cobasis elements. Hence in a space of any number of dimensions we can write:

$$\overline{e_i} == \sum_{j=1}^{n} a_{ij}\, \underline{e_j}$$

For reasons which the reader will perhaps discern from the choice of notation, we extract the scalar factor $\Bbbk$ as an explicit factor from the coefficients of the complement mapping.

$$\boxed{\overline{e_i} == \Bbbk \sum_{j=1}^{n} g_{ij}\, \underline{e_j}} \qquad 5.13$$

This then is the *form* in which we will define the complement of basis 1-elements.

The scalar $\Bbbk$ and the scalars g_{ij} are *at this point* entirely arbitrary. However, we must still ensure that our definition satisfies the complement of a complement axiom $\overline{4}$. We will see that in order to satisfy $\overline{4}$, some constraints will need to be imposed on $\Bbbk$ and g_{ij}. Therefore, our task now in the ensuing sections is to determine these constraints. We begin by determining $\Bbbk$ in terms of the g_{ij}.

Determining the value of $\Bbbk$

A consideration of axiom $\overline{5}$ in the form $1 == \overline{\overline{1}}$ enables us to determine a relationship between the scalar $\Bbbk$ and the g_{ij}.

$$1 == \overline{\overline{1}} == \overline{\Bbbk\;\, e_1 \wedge e_2 \wedge \cdots \wedge e_n}$$

$$== \Bbbk\;\, \overline{e_1 \wedge e_2 \wedge \cdots \wedge e_n}$$

$$== \Bbbk\;\, \overline{e_1} \vee \overline{e_2} \vee \cdots \vee \overline{e_n}$$

$$== \Bbbk\;\, \Bbbk^n\, |g_{ij}|\;\, \underline{e_1} \vee \underline{e_2} \vee \cdots \vee \underline{e_n}$$

$$== \Bbbk\;\, \Bbbk^n\, |g_{ij}|\;\, \frac{1}{\Bbbk^{n-1}}\, \underline{(e_1 \wedge e_2 \wedge \cdots \wedge e_n)}$$

$$== \Bbbk^2\, |g_{ij}|\;\, 1$$

Here, we have used (in order) axioms $\overline{5}$, $\overline{2}$, $\overline{3}$, the definition of the complement [5.13], and formula 3.35. The symbol $|g_{ij}|$ denotes the determinant of the g_{ij}.

Thus we have shown that the scalar $\Bbbk$ is related to the coefficients g_{ij} by:

$$\Bbbk == \pm \frac{1}{\sqrt{|g_{ij}|}}$$

However, it is not the sign of $\Bbbk$ that is important, since we can always redefine the ordering of the basis elements to ensure that $\Bbbk$ is positive (or in a 1-space change the sign of the only basis element). For simplicity then we will retain only the positive sign of $\Bbbk$.

When the determinant $|g_{ij}|$ is negative, $\Bbbk$ will be imaginary. This is an important case, which can be developed in either of two ways. Either we allow the implications of an imaginary $\Bbbk$ to propagate through the algebra, and in consequence maintain a certain simplicity of structure in the resulting formulae, or we can reformulate our approach to allow $\overline{\overline{1}} == (-1)^s\, 1$, where the parity of *s* is even if $|g_{ij}|$ is positive, and odd if it is negative. This latter approach would add a factor $(-1)^s$ to many of the axioms and formulae developed from this chapter on. Because of the introductory nature of the book however, we have chosen to adopt the former approach, allowing readers less versed in the algebra to be undistracted by the extra complexities of pseudo-Riemannian metrics.

From this point on then, we take the value of $\Bbbk$ to be the reciprocal of the positive square root of the determinant of the coefficients of the complement mapping g_{ij}.

$$\Bbbk == \frac{1}{\sqrt{|g_{ij}|}} \qquad 5.14$$

This relation clearly also implies that the array $[g_{ij}]$ is required to be non-singular.

5.4 The Euclidean Complement

Tabulating Euclidean complements of basis elements

The *Euclidean complement* of a basis *m*-element may be defined as its cobasis element. Conceptually this is the simplest correspondence we can define between basis *m*-elements and basis (*n*–1)-elements. In this case the matrix of the metric tensor is the identity matrix, with the result that the 'volume' $\Bbbk$ of the basis *n*-element is unity. We tabulate the basis-complement pairs for spaces of two, three and four dimensions.

Basis elements and their Euclidean complements in 2-space

Λ	BASIS	COMPLEMENT
$\underset{0}{\Lambda}$	1	$e_1 \wedge e_2$
$\underset{1}{\Lambda}$	e_1	e_2
$\underset{1}{\Lambda}$	e_2	$-e_1$
$\underset{2}{\Lambda}$	$e_1 \wedge e_2$	1

Basis elements and their Euclidean complements in 3-space

Λ	BASIS	COMPLEMENT
Λ_0	1	$e_1 \wedge e_2 \wedge e_3$
Λ_1	e_1	$e_2 \wedge e_3$
Λ_1	e_2	$-(e_1 \wedge e_3)$
Λ_1	e_3	$e_1 \wedge e_2$
Λ_2	$e_1 \wedge e_2$	e_3
Λ_2	$e_1 \wedge e_3$	$-e_2$
Λ_2	$e_2 \wedge e_3$	e_1
Λ_3	$e_1 \wedge e_2 \wedge e_3$	1

Basis elements and their Euclidean complements in 4-space

Λ	BASIS	COMPLEMENT
Λ_0	1	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$
Λ_1	e_1	$e_2 \wedge e_3 \wedge e_4$
Λ_1	e_2	$-(e_1 \wedge e_3 \wedge e_4)$
Λ_1	e_3	$e_1 \wedge e_2 \wedge e_4$
Λ_1	e_4	$-(e_1 \wedge e_2 \wedge e_3)$
Λ_2	$e_1 \wedge e_2$	$e_3 \wedge e_4$
Λ_2	$e_1 \wedge e_3$	$-(e_2 \wedge e_4)$
Λ_2	$e_1 \wedge e_4$	$e_2 \wedge e_3$
Λ_2	$e_2 \wedge e_3$	$e_1 \wedge e_4$
Λ_2	$e_2 \wedge e_4$	$-(e_1 \wedge e_3)$
Λ_2	$e_3 \wedge e_4$	$e_1 \wedge e_2$
Λ_3	$e_1 \wedge e_2 \wedge e_3$	e_4
Λ_3	$e_1 \wedge e_2 \wedge e_4$	$-e_3$
Λ_3	$e_1 \wedge e_3 \wedge e_4$	e_2
Λ_3	$e_2 \wedge e_3 \wedge e_4$	$-e_1$
Λ_4	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	1

Formulae for the Euclidean complement of basis elements

In this section we summarize some of the formulae involving Euclidean complements of basis elements. In Section 5.10 below these formulae are generalized for the general metric case.

The complement of a basis 1-element is its cobasis (n–1)-element. For a basis 1-element e_i then we have:

$$\overline{e_i} == \underline{e_i} == (-1)^{i-1}\, e_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge e_n \qquad 5.15$$

This simple relationship extends naturally to complements of basis elements of any grade.

$$\overline{\underset{m}{e_i}} == \underline{\underset{m}{e_i}} \qquad 5.16$$

This may also be written:

$$\overline{e_{i_1} \wedge \cdots \wedge e_{i_m}} == (-1)^{K_m}\, e_1 \wedge \cdots \wedge \square_{i_1} \wedge \cdots \wedge \square_{i_m} \wedge \cdots \wedge e_n \qquad 5.17$$

where $K_m = \sum_{\gamma=1}^{m} i_\gamma + \frac{1}{2}\, m\, (m+1)$ and the symbol $\square$ means the corresponding element is missing from the product.

In particular, the unit n-element is now the basis n-element:

$$\overline{1} == e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 5.18$$

and the complement of the basis n-element is just unity.

$$1 == \overline{e_1 \wedge e_2 \wedge \cdots \wedge e_n} \qquad 5.19$$

This simple correspondence is that of a Euclidean complement, defined by a Euclidean metric, simply given by $g_{ij} == \delta_{ij}$, with consequence that $\mathbb{k} == 1$.

$$g_{ij} == \delta_{ij} \qquad\qquad \mathbb{k} == 1 \qquad 5.20$$

Finally, we can see that exterior and regressive products of basis elements with complements take on particularly simple forms.

$$\underset{m}{e_i} \wedge \overline{\underset{m}{e_j}} == \delta_{ij}\, \overline{1} \qquad 5.21$$

$$\underset{m}{e_i} \vee \overline{\underset{m}{e_j}} == \delta_{ij} \qquad 5.22$$

These forms will be the basis for the definition of the inner product in the next chapter.

We note that the concept of cobasis which we introduced in Chapter 2: The Exterior Product, despite its formal similarity to the Euclidean complement, is only a notational convenience. We do not define it for linear combinations of elements as the definition of a complement requires.

5.5 Complementary Interlude

Alternative forms for complements

As a consequence of the complement axiom and the fact that multiplication by a scalar is equivalent to exterior multiplication (see Section [2.4]) we can write the complement of a scalar, an *m*-element, and a scalar multiple of an *m*-element in several alternative forms.

Alternative forms for the complement of a scalar

From the linearity axiom $\overline{2}$ we have $\overline{a\,\underset{m}{\alpha}} == a\,\overline{\underset{m}{\alpha}}$, hence:

$$\overline{a} == \overline{a\,1} == a\,\overline{1} == a \wedge \overline{1}$$

The complement of a scalar can then be expressed in any of the following forms:

$$\overline{a} == \overline{1 \wedge a} == \overline{1} \vee \overline{a} == 1 \wedge \overline{a} == 1\,\overline{a} == a \wedge \overline{1} == a\,\overline{1} \qquad 5.23$$

Alternative forms for the complement of an *m*-element

$$\overline{\underset{m}{\alpha}} == \overline{1 \wedge \underset{m}{\alpha}} == \overline{1} \vee \overline{\underset{m}{\alpha}} == 1 \wedge \overline{\underset{m}{\alpha}} == 1\,\overline{\underset{m}{\alpha}} \qquad 5.24$$

Alternative forms for the complement of a scalar multiple of an *m*-element

$$\overline{a\,\underset{m}{\alpha}} == \overline{a \wedge \underset{m}{\alpha}} == \overline{a} \vee \overline{\underset{m}{\alpha}} == a \wedge \overline{\underset{m}{\alpha}} == a\,\overline{\underset{m}{\alpha}} \qquad 5.25$$

Orthogonality

The specific complement mapping that we impose on a Grassmann algebra will *define* the notion of orthogonality for that algebra. A simple element and its complement will be referred to as being *orthogonal* to each other. In standard linear space terminology, the *space* of a simple element and the *space* of its complement are said to be *orthogonal complements* of each other.

This orthogonality is total. That is, every 1-element in a given simple *m*-element is orthogonal to every 1-element in the complement of the *m*-element.

Visualizing the complement axiom

We can use this notion of orthogonality to visualize the complement axiom geometrically.

Consider the bivector $x \wedge y$. Then $\overline{x \wedge y}$ is orthogonal to $x \wedge y$. But since $\overline{x \wedge y} == \overline{x} \vee \overline{y}$ this also means that the intersection of the two (n–1)-spaces defined by $\overline{x}$ and $\overline{y}$ is orthogonal to $x \wedge y$. We can depict it in 3-space as follows:

Graphic showing three pictures
1) A bivector labeled $\overline{x}$ with vector x orthogonal to it.
2) A bivector labeled $\overline{y}$ with vector y orthogonal to it.
3) The bivectors shown intersecting in a vector that is orthogonal to the bivector $x \wedge y$.
Ref JMB p60

The regressive product in terms of complements

All the operations in the Grassmann algebra can be expressed in terms *only* of the exterior product and the complement operations. It is this fact that makes the complement so important for an understanding of the algebra.

In particular the regressive product (discussed in Chapter 3) and the interior product (to be discussed in the next chapter) have simple representations in terms of the exterior and complement operations.

We have already introduced the complement axiom as part of the definition of the complement.

Taking the complement of both sides, noting that the degree of $\underset{m}{\alpha} \vee \underset{k}{\beta}$ is $m+k-n$, and using the complement of a complement axiom gives:

$$\overline{\overline{\underset{m}{\alpha} \vee \underset{k}{\beta}}} == \overline{\overline{\underset{m}{\alpha}} \wedge \overline{\underset{k}{\beta}}} == (-1)^{(m+k-n)\,(n-(m+k-n))}\,\underset{m}{\alpha} \vee \underset{k}{\beta}$$

Or finally:

$$\underset{m}{\alpha} \vee \underset{k}{\beta} == (-1)^{(m+k)\,(m+k-n)}\,\overline{\overline{\underset{m}{\alpha}} \wedge \overline{\underset{k}{\beta}}} \qquad 5.26$$

Relating exterior and regressive products

The exterior product of an *m*-element with the complement of another *m*-element is an *n*-element, and hence must be a scalar multiple of the unit *n*-element $\overline{1}$. Hence:

$$\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}} == a\,\overline{1} \quad \Leftrightarrow \quad \overline{\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}}} == a\,\overline{\overline{1}} == a$$

$$\Leftrightarrow \quad (-1)^{m\,(n-m)}\,\overline{\overline{\underset{m}{\beta}} \wedge \underset{m}{\alpha}} == a \quad \Leftrightarrow \quad (-1)^{m\,(n-m)}\,\overline{\overline{\underset{m}{\beta}}} \wedge \overline{\underset{m}{\alpha}} == a$$

$$\Leftrightarrow \quad \underset{m}{\beta} \vee \overline{\underset{m}{\alpha}} == a$$

$$\boxed{\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}} == a\,\overline{1} \qquad \Leftrightarrow \qquad \underset{m}{\beta} \vee \overline{\underset{m}{\alpha}} == a} \qquad 5.27$$

5.6 The Complement of a Complement

The complement of a cobasis element

In order to determine any conditions on the g_{ij} required to satisfy the complement of a complement axiom in Section 5.2, we only need to compute the complement of a complement of basis 1-elements and compare the result to the form given by the axiom.

The complement of the complement of this basis element is obtained by taking the complement of expression 5.13 for the case *i* = 1.

$$\overline{\overline{e_i}} == \Bbbk \sum_{j=1}^{n} g_{1\,j}\,\overline{\underline{e_j}}$$

In order to determine $\overline{\overline{e_i}}$, we need to obtain an expression for the complement of a cobasis element of a 1-element $\underline{e_j}$. Note that whereas the complement of a cobasis element is defined, the cobasis element of a complement is not. To simplify the development we consider, without loss of generality, a specific basis element e_1.

First we express the cobasis element as a basis (*n*–1)-element, and then use the complement axiom to express the right-hand side as a regressive product of complements of 1-elements.

$$\overline{\underline{e_1}} == \overline{e_2 \wedge e_3 \wedge \cdots \wedge e_n} == \overline{e_2} \vee \overline{e_3} \vee \cdots \vee \overline{e_n}$$

Substituting for the $\overline{e_i}$ from the definition [5.13]:

$$\begin{aligned} \overline{\underline{e_1}} == {} & \Bbbk\,(g_{21}\,\underline{e_1} + g_{22}\,\underline{e_2} + \cdots + g_{2\,n}\,\underline{e_n}) \\ & \vee \Bbbk\,(g_{31}\,\underline{e_1} + g_{32}\,\underline{e_2} + \cdots + g_{3\,n}\,\underline{e_n}) \\ & \vee \;\cdots \\ & \vee \Bbbk\,(g_{n1}\,\underline{e_1} + g_{n2}\,\underline{e_2} + \cdots + g_{nn}\,\underline{e_n}) \end{aligned}$$

Expanding this expression and collecting terms gives:

$$\overline{\underline{e_1}} == \mathbb{k}^{n-1} \sum_{j=1}^{n} \underline{g_{1\,j}}\ (-1)^{j-1}\ \underline{e_1} \vee \underline{e_2} \vee \cdots \vee \square_j \vee \cdots \vee \underline{e_n}$$

In this expression, the notation $\square_j$ means that the *j*th factor has been omitted. The scalars $\underline{g_{1\,j}}$ are the cofactors of $g_{1\,j}$ in the array g_{ij}. Further discussion of cofactors, and how they result from such products of (*n*–1) elements may be found in Chapter 2. The results for regressive products are identical *mutatis mutandis* to those for the exterior product.

By equation 3.36, regressive products of the form above simplify to a scalar multiple of the missing element.

$$(-1)^{j-1}\ \underline{e_1} \vee \underline{e_2} \vee \cdots \vee \square_j \vee \cdots \vee \underline{e_n} == \frac{1}{\mathbb{k}^{n-2}}\ (-1)^{n-1}\ e_j$$

The expression for $\overline{\underline{e_1}}$ then simplifies to:

$$\overline{\underline{e_1}} == (-1)^{n-1}\ \mathbb{k} \sum_{j=1}^{n} \underline{g_{1\,j}}\ e_j$$

The final step is to see that the actual basis element chosen (e_1) was, as expected, of no significance in determining the final *form* of the formula. We thus have the more general result:

$$\overline{\underline{e_i}} == (-1)^{n-1}\ \mathbb{k} \sum_{j=1}^{n} \underline{g_{ij}}\ e_j \qquad 5.28$$

Thus we have determined the complement of the cobasis element of a basis 1-element as a specific 1-element. The coefficients of this 1-element are the products of the scalar $\mathbb{k}$ with cofactors of the g_{ij}, and a possible sign depending on the dimension of the space.

Our major use of this formula is to derive an expression for the complement of a complement of a basis element. This we do below.

The complement of the complement of a basis 1-element

Consider again the basis element e_1. The complement of e_1 is, by definition:

$$\overline{e_i} == \mathbb{k} \sum_{j=1}^{n} g_{ij}\ \underline{e_j}$$

Taking the complement a second time gives:

$$\overline{\overline{e_i}} == \mathbb{k} \sum_{j=1}^{n} g_{ij}\ \overline{\underline{e_j}}$$

We can now substitute for the $\overline{\underline{e_j}}$ from the formula derived in the previous section to get:

$$\overline{\overline{e_i}} == (-1)^{n-1}\,\mathbb{k}^2 \sum_{j=1}^{n} g_{ij} \sum_{k=1}^{n} \underline{g_{jk}}\, e_k$$

In Chapter 2 that the sum over *j* of the terms $g_{ij}\ \underline{g_{kj}}$ was shown to be equal to the determinant $|g_{ij}|$ whenever *i* equals *k* and zero otherwise. That is:

$$\sum_{j=1}^{n} g_{ij}\ \underline{g_{kj}} == |g_{ij}|\ \delta_{ik}$$

Note that in this sum, the order of the subscripts is reversed in the cofactor term compared to that in the expression for $\overline{\overline{e_i}}$.

Thus we conclude that *if and only if the array* g_{ij} *is symmetric*, that is, $g_{ij} == g_{ji}$, can we express the complement of a complement of a basis 1-element in terms only of itself and no other basis element.

$$\overline{\overline{e_i}} == (-1)^{n-1}\,\mathbb{k}^2\ |g_{ij}|\ \delta_{ik}\, e_k == (-1)^{n-1}\,\mathbb{k}^2\ |g_{ij}|\ e_i$$

Furthermore, since we have already shown in Section 5.3 that $\mathbb{k}^2\ |g_{ij}| == 1$, we also have that:

$$\overline{\overline{e_i}} == (-1)^{n-1}\, e_i$$

In sum: In order to satisfy the complement of a complement axiom for *m*-elements it is necessary that $g_{ij} == g_{ji}$. For 1-elements it is also sufficient.

Below we shall show that the symmetry of the g_{ij} is also sufficient for *m*-elements.

The complement of the complement of a basis *m*-element

Consider the basis *m*-element $e_1 \wedge e_2 \wedge \cdots \wedge e_m$. By taking the complement of the complement of this element and applying the complement axiom twice, we obtain:

$$\overline{\overline{e_1 \wedge e_2 \wedge \cdots \wedge e_m}} == \overline{\overline{e_1} \vee \overline{e_2} \vee \cdots \vee \overline{e_m}} == \overline{\overline{e_1}} \wedge \overline{\overline{e_2}} \wedge \cdots \wedge \overline{\overline{e_m}}$$

But since $\overline{\overline{e_i}} == (-1)^{n-1}\, e_i$ we obtain immediately that:

$$\overline{\overline{e_1 \wedge e_2 \wedge \cdots \wedge e_m}} == (-1)^{m\,(n-1)}\, e_1 \wedge e_2 \wedge \cdots \wedge e_m$$

But of course the form of this result is valid for any basis element $\underset{m}{e_i}$. Writing $(-1)^{m\,(n-1)}$ in the equivalent form $(-1)^{m\,(n-m)}$ we obtain that for *any* basis element of *any* grade:

$$\boxed{\overline{\overline{\underset{m}{e_i}}} == (-1)^{m\,(n-m)}\, \underset{m}{e_i}} \qquad 5.29$$

The complement of the complement of an *m*-element

Consider the general *m*-element $\underset{m}{\alpha} == \Sigma\, a_i \underset{m}{e_i}$. By taking the complement of the complement of this element and substituting from equation 5.18, we obtain:

$$\overline{\overline{\underset{m}{\alpha}}} == \sum_i a_i \overline{\overline{\underset{m}{e_i}}} == \sum_i a_i\, (-1)^{m\,(n-m)} \underset{m}{e_i} == (-1)^{m\,(n-m)} \underset{m}{\alpha}$$

Finally then we have shown that, provided the complement mapping g_{ij} is symmetric and the constant $\Bbbk$ is such that $\Bbbk^2\, |g_{ij}| == 1$, the complement of a complement axiom is satisfied by an otherwise arbitrary mapping, with sign $(-1)^{m\,(n-m)}$.

$$\overline{\overline{\underset{m}{\alpha}}} == (-1)^{m\,(n-m)} \underset{m}{\alpha} \qquad 5.30$$

Special cases

The complement of the complement of a scalar is the scalar itself.

$$\overline{\overline{a}} == a \qquad 5.31$$

The complement of the complement of an *n*-element is the *n*-element itself.

$$\overline{\overline{\underset{n}{\alpha}}} == \underset{n}{\alpha} \qquad 5.32$$

The complement of the complement of any element in a 3-space is the element itself, since $(-1)^{m\,(3-m)}$ is positive for *m* equal to 0, 1, 2, or 3.

Alternatively, we can say that $\overline{\overline{\underset{m}{\alpha}}} == \underset{m}{\alpha}$ *except when* $\underset{m}{\alpha}$ *is of odd degree in an even-dimensional space*.

5.7 Working with Metrics

The default metric

In order to calculate complements and interior products, and any products defined in terms of them (for example, Clifford and hypercomplex products), *GrassmannAlgebra* needs to know what metric has been imposed on the underlying linear space $\underset{1}{\Lambda}$. Grassmann and all those writing in the tradition of the *Ausdehnungslehre* tacitly assumed a Euclidean metric; that is, one in which $g_{ij} == \delta_{ij}$. This metric is also the one tacitly assumed in beginning presentations of the three-dimensional vector calculus, and is most evident in the definition of the cross product as a vector *normal* to the factors of the product.

The default metric assumed by *GrassmannAlgebra* is the Euclidean metric. In the case that the *GrassmannAlgebra* package has just been loaded, entering `Metric` will show the components of the default Euclidean metric tensor.

```
Metric
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

These components are arranged as the elements of a 3×3 matrix because the default basis is 3-dimensional.

```
Basis
```

$\{e_1, e_2, e_3\}$

However, if the dimension of the basis is changed, the default metric changes accordingly.

```
DeclareBasis[{𝕆, i, j, k}]
{𝕆, i, j, k}
Metric
{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

We will take the liberty of referring to this matrix as *the* metric.

Declaring a metric

GrassmannAlgebra permits us to declare a metric as a matrix with any numeric or symbolic components. There are just three conditions to which a valid matrix must conform in order to be a metric:
1) It must be symmetric (and hence square).
2) Its order must be the same as the dimension of the declared linear space.
3) Its components must be scalars.

It is up to the user to ensure the first two conditions. The third is handled by *GrassmannAlgebra* automatically adding any symbols in the matrix not already declared as scalars to the list of declared scalars.

Example: Declaring a metric

We return to 3-space (by entering $\mathbb{V}_3$), create a 3×3 matrix **M**, and then declare it as the metric.

```
𝕍₃; M = {{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}}; MatrixForm[M]
```

$$\begin{pmatrix} 1 & 0 & \nu \\ 0 & -1 & 0 \\ \nu & 0 & -1 \end{pmatrix}$$

```
DeclareMetric[M]
{{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}}
```

We can verify that this is indeed the metric by entering `Metric`.

```
Metric
```

```
{{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}}
```

And verify that ν has indeed been added to the list of declared scalars by entering `Scalars`.

```
Scalars
```

$\{a, b, c, d, e, f, g, h, \mathbb{k}, \nu, (\_ \ominus \_)\text{?InnerProductQ}, \underset{0}{\_}\}$

Declaring a general metric

For theoretical calculations it is sometimes useful to be able to quickly declare a metric of general symbolic elements. We can do this as described in the previous section, or we can use the *GrassmannAlgebra* function `DeclareMetric[g]` where **g** is a symbol. We will often use the 'double-struck' symbol $\mathbb{g}$ for the kernel symbol of the metric components.

$\mathbb{V}_3$**; G = DeclareMetric[**$\mathbb{g}$**]; MatrixForm[G]**

$$\begin{pmatrix} \mathbb{g}_{1,1} & \mathbb{g}_{1,2} & \mathbb{g}_{1,3} \\ \mathbb{g}_{1,2} & \mathbb{g}_{2,2} & \mathbb{g}_{2,3} \\ \mathbb{g}_{1,3} & \mathbb{g}_{2,3} & \mathbb{g}_{3,3} \end{pmatrix}$$

Note that *GrassmannAlgebra* has automatically declared the *pattern* for these components to be scalars (rather than the components themselves). This means that *any* symbol of the form $\mathbb{g}_{i,j}$ is considered a scalar (for example $\mathbb{g}_{A,B}$).

ScalarQ[$\mathbb{g}_{A,B}$**]**

```
True
```

You can test to see if a symbol is a component of the currently declared metric by using `MetricQ`.

MetricQ[{$\mathbb{g}_{2,3}$**,** $\mathbb{g}_{23}$**,** $\mathbb{g}_{3,4}$**}]**

```
{True, False, False}
```

Calculating induced metrics

The *GrassmannAlgebra* function for calculating the metric induced on $\underset{m}{\Lambda}$ by the metric in $\underset{1}{\Lambda}$ is `MetricΛ[m]`.

Example: Induced general metrics

Suppose we are working in 3-space with the general metric defined above. Then the metrics on $\underset{0}{\Lambda}, \underset{1}{\Lambda}, \underset{2}{\Lambda}, \underset{3}{\Lambda}$ can be calculated by entering:

```
𝕍3; DeclareMetric[𝕘];
```

```
MetricΛ[0]
```

1

```
M1 = MetricΛ[1]; MatrixForm[M1]
```

$$\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{1,2} & g_{2,2} & g_{2,3} \\ g_{1,3} & g_{2,3} & g_{3,3} \end{pmatrix}$$

```
M2 = MetricΛ[2]; MatrixForm[M2]
```

$$\begin{pmatrix} -g_{1,2}^2 + g_{1,1}\, g_{2,2} & -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3} & -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3} \\ -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3} & -g_{1,3}^2 + g_{1,1}\, g_{3,3} & -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3} \\ -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3} & -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3} & -g_{2,3}^2 + g_{2,2}\, g_{3,3} \end{pmatrix}$$

```
MetricΛ[3]
```

$$-g_{1,3}^2\, g_{2,2} + 2\, g_{1,2}\, g_{1,3}\, g_{2,3} - g_{1,1}\, g_{2,3}^2 - g_{1,2}^2\, g_{3,3} + g_{1,1}\, g_{2,2}\, g_{3,3}$$

The metric in $\underset{3}{\Lambda}$ is of course just the determinant of the metric in $\underset{1}{\Lambda}$.

We can also give `MetricΛ` several grades for which to calculate the metric. For example:

```
MetricΛ[{1, 3}]
```

$$\{\{\{g_{1,1}, g_{1,2}, g_{1,3}\}, \{g_{1,2}, g_{2,2}, g_{2,3}\}, \{g_{1,3}, g_{2,3}, g_{3,3}\}\},$$
$$-g_{1,3}^2\, g_{2,2} + 2\, g_{1,2}\, g_{1,3}\, g_{2,3} - g_{1,1}\, g_{2,3}^2 - g_{1,2}^2\, g_{3,3} + g_{1,1}\, g_{2,2}\, g_{3,3}\}$$

■ Example: Induced specific metrics

We return to the metric discussed in a previous example.

```
𝕍3; M = {{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}};
DeclareMetric[M]; MatrixForm[M]
```

$$\begin{pmatrix} 1 & 0 & \nu \\ 0 & -1 & 0 \\ \nu & 0 & -1 \end{pmatrix}$$

```
MetricΛ[{1, 2, 3}]
```

$$\{\{\{1, 0, \nu\}, \{0, -1, 0\}, \{\nu, 0, -1\}\},$$
$$\{\{-1, 0, \nu\}, \{0, -1 - \nu^2, 0\}, \{\nu, 0, 1\}\}, 1 + \nu^2\}$$

Verifying the symmetry of the induced metrics

It is easy to verify the symmetry of the induced metrics in any particular case by inspection. However to automate this we need only ask *Mathematica* to compare the metric to its transpose. For example we can verify the symmetry of the metric induced on $\underset{3}{\Lambda}$ by a general metric in 4-space.

```
𝕍₄; DeclareMetric[𝕘]; M₃ = MetricΛ[3]; M₃ == Transpose[M₃]
True
```

The metric for a cobasis

In the sections above we have shown how to calculate the metric induced on $\underset{m}{\Lambda}$ by the metric defined on $\underset{1}{\Lambda}$. Because of the way in which the cobasis elements of $\underset{m}{\Lambda}$ are naturally ordered alphanumerically, their ordering and signs will not generally correspond to that of the basis elements of $\underset{n-m}{\Lambda}$. The arrangement of the elements of the metric tensor for the cobasis elements of $\underset{m}{\Lambda}$ will therefore differ from the arrangement of the elements of the metric tensor for the basis elements of $\underset{n-m}{\Lambda}$. The metric tensor of a cobasis is the tensor of *cofactors* of the metric tensor of the basis. Hence we can obtain the tensor of cofactors of the elements of the metric tensor of $\underset{m}{\Lambda}$ by reordering and resigning the elements of the metric tensor induced on $\underset{n-m}{\Lambda}$.

As an example, take the general metric in 3-space discussed in the previous section. We will show that we can obtain the tensor of the cofactors of the elements of the metric tensor of $\underset{1}{\Lambda}$ by reordering and resigning the elements of the metric tensor induced on $\underset{2}{\Lambda}$.

The metric on $\underset{1}{\Lambda}$ is given as a correspondence $\mathbf{G_1}$ between the basis elements and cobasis elements of $\underset{1}{\Lambda}$.

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} == \mathbb{k}\, G_1 \begin{pmatrix} e_2 \wedge e_3 \\ -(e_1 \wedge e_3) \\ e_1 \wedge e_2 \end{pmatrix}; \qquad G_1 = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{1,2} & g_{2,2} & g_{2,3} \\ g_{1,3} & g_{2,3} & g_{3,3} \end{pmatrix}$$

The metric on $\underset{2}{\Lambda}$ is given as a correspondence $\mathbf{G_2}$ between the basis elements and cobasis elements of $\underset{2}{\Lambda}$.

$$\begin{pmatrix} e_1 \wedge e_2 \\ e_1 \wedge e_3 \\ e_2 \wedge e_3 \end{pmatrix} == \mathbb{k}\, G_2 \begin{pmatrix} e_3 \\ -e_2 \\ e_1 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} -g_{1,2}^2 + g_{1,1}\, g_{2,2} & -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3} & -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3} \\ -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3} & -g_{1,3}^2 + g_{1,1}\, g_{3,3} & -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3} \\ -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3} & -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3} & -g_{2,3}^2 + g_{2,2}\, g_{3,3} \end{pmatrix}$$

We can transform the columns of basis elements in this equation into the form of the preceding one with the transformation $\mathbf{T}$ which is simply determined as:

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$\mathbf{T} \begin{pmatrix} e_1 \wedge e_2 \\ e_1 \wedge e_3 \\ e_2 \wedge e_3 \end{pmatrix} == \begin{pmatrix} e_2 \wedge e_3 \\ -(e_1 \wedge e_3) \\ e_1 \wedge e_2 \end{pmatrix}; \quad \mathbf{T} \begin{pmatrix} e_3 \\ -e_2 \\ e_1 \end{pmatrix} == \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix};$$

And since this transformation is its own inverse, we can also transform $\mathbf{G_2}$ to $\mathbf{T\ G_2\ T}$. We now expect this transformed array to be the array of cofactors of the metric tensor $\mathbf{G_1}$. We can easily check this in *Mathematica* by entering the predicate

```
Simplify[G1.(T.G2.T) == Det[G1] IdentityMatrix[3]]
```

```
True
```

Creating tables of induced metrics

You can create a table of all the induced metrics by declaring a metric and then entering `MetricTable`.

```
𝕍3; M = {{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}};
DeclareMetric[M]; MetricTable
```

Λ	METRIC
$\underset{0}{\Lambda}$	1
$\underset{1}{\Lambda}$	$\begin{pmatrix} 1 & 0 & \nu \\ 0 & -1 & 0 \\ \nu & 0 & -1 \end{pmatrix}$
$\underset{2}{\Lambda}$	$\begin{pmatrix} -1 & 0 & \nu \\ 0 & -1-\nu^2 & 0 \\ \nu & 0 & 1 \end{pmatrix}$
$\underset{3}{\Lambda}$	$1+\nu^2$

```
V₄; MetricTable
```

$\wedge$	METRIC
$\underset{0}{\wedge}$	1
$\underset{1}{\wedge}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\underset{2}{\wedge}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$\underset{3}{\wedge}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$\underset{4}{\wedge}$	1

Creating palettes of induced metrics

You can create a palette of all the induced metrics by declaring a metric and then entering `MetricPalette`.

```
V₃; DeclareMetric[g]; MetricPalette
```

$\wedge$	METRIC
$\underset{0}{\wedge}$	1
$\underset{1}{\wedge}$	$\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{1,2} & g_{2,2} & g_{2,3} \\ g_{1,3} & g_{2,3} & g_{3,3} \end{pmatrix}$
$\underset{2}{\wedge}$	$\begin{pmatrix} -g_{1,2}^2 + g_{1,1} g_{2,2} & -g_{1,2} g_{1,3} + g_{1,1} g_{2,3} & -g_{1,3} g_{2,2} + g_{1,2} g_{2,3} \\ -g_{1,2} g_{1,3} + g_{1,1} g_{2,3} & -g_{1,3}^2 + g_{1,1} g_{3,3} & -g_{1,3} g_{2,3} + g_{1,2} g_{3,3} \\ -g_{1,3} g_{2,2} + g_{1,2} g_{2,3} & -g_{1,3} g_{2,3} + g_{1,2} g_{3,3} & -g_{2,3}^2 + g_{2,2} g_{3,3} \end{pmatrix}$
$\underset{3}{\wedge}$	$-g_{1,3}^2 g_{2,2} + 2 g_{1,2} g_{1,3} g_{2,3} - g_{1,1} g_{2,3}^2 - g_{1,2}^2 g_{3,3} + g_{1,1} g_{2,2} g_{3,3}$

Remember that the difference between a table and a palette is that a table can be edited, whereas a palette can be clicked on to paste the contents of the cell. For example, clicking on the metric for $\underset{3}{\wedge}$ gives:

$$-g_{1,3}^2\, g_{2,2} + 2\, g_{1,2}\, g_{1,3}\, g_{2,3} - g_{1,1}\, g_{2,3}^2 - g_{1,2}^2\, g_{3,3} + g_{1,1}\, g_{2,2}\, g_{3,3}$$

5.8 Calculating Complements

Entering a complement

To enter the expression for the complement of a Grassmann expression **X** in *GrassmannAlgebra*, enter `GrassmannComplement[X]`. For example to enter the expression for the complement of **x∧y**, enter:

```
GrassmannComplement[x ∧ y]
```

$$\overline{x \wedge y}$$

Or, you can simply select the expression **x∧y** and click the ■ button on the *GrassmannAlgebra* palette.

Note that an expression with a bar over it symbolizes another expression (the complement of the original expression), not a *GrassmannAlgebra* operation on the expression. Hence no conversions will occur by entering an overbarred expression.

`GrassmannComplement` will also work for lists, matrices, or tensors of elements. For example, here is a matrix:

```
M = {{a e1 ∧ e2, b e2}, {-b e2, c e3 ∧ e1}}; MatrixForm[M]
```

$$\begin{pmatrix} a\, e_1 \wedge e_2 & b\, e_2 \\ -b\, e_2 & c\, e_3 \wedge e_1 \end{pmatrix}$$

And here is its complement:

$\overline{\mathbf{M}}$ **// MatrixForm**

$$\begin{pmatrix} \overline{a\, e_1 \wedge e_2} & \overline{b\, e_2} \\ \overline{-b\, e_2} & \overline{c\, e_3 \wedge e_1} \end{pmatrix}$$

Converting to complement form

In order to see what a Grassmann expression, or list of Grassmann expressions, looks like when expressed only in terms of the exterior product and complement operations, we can use the *GrassmannAlgebra* operation `ToComplementForm`. Essentially, this operation takes an expression and repeatedly uses rules to replace any regressive products by exterior products and complements. This operation also works on other products, like the interior product, which will be defined in later chapters. Note that the signs will be calculated using the dimension of the currently declared basis.

For example here is a pair of expressions in 3-space:

$\mathbb{V}_3$;
$X = \{(e_1 \wedge e_2) \vee (e_2 \wedge e_3) \vee (e_3 \wedge e_1),\ (e_1 \wedge e_2 \wedge e_3) \vee (e_1 \wedge e_2 \wedge e_3) \vee e_1\}$;

ToComplementForm[X]

$\{\overline{\overline{e_1 \wedge e_2} \wedge \overline{e_2 \wedge e_3} \wedge \overline{e_3 \wedge e_1}},\ \overline{\overline{e_1 \wedge e_2 \wedge e_3} \wedge \overline{e_1 \wedge e_2 \wedge e_3} \wedge \overline{e_1}}\}$

Converting symbolic expressions

`ToComplementForm` may also be used with elements of symbolic grade. Note that the sign of the result will depend on the dimension of the space, and for it to calculate correctly the grades must be declared scalars.

For example, here is a symbolic expression in 3-space. The grades **a**, **b**, and **c** are by default declared scalars. Since the sign $(-1)^{m\,(3-m)}$ is positive for all elements in a 3-space we have a result independent of the grades of the elements.

$\mathbb{V}_3$; **ToComplementForm**$\left[\underset{a}{\alpha} \vee \underset{b}{\beta} \vee \underset{c}{\gamma}\right]$

$\overline{\underset{a}{\bar{\alpha}} \wedge \underset{b}{\bar{\beta}} \wedge \underset{c}{\bar{\gamma}}}$

However, in a 4-space, the sign depends on the grades.

$\mathbb{V}_4$; **ToComplementForm**$\left[\underset{a}{\alpha} \vee \underset{b}{\beta} \vee \underset{c}{\gamma}\right]$

$(-1)^{a+b+c}\ \overline{\underset{a}{\bar{\alpha}} \wedge \underset{b}{\bar{\beta}} \wedge \underset{c}{\bar{\gamma}}}$

Converting to complement form in an arbitrary dimension

It is possible to do calculations of this type in spaces of arbitrary dimension by declaring a symbolic dimension with `DeclareSymbolicDimension[n]` (make sure **n** has been declared a scalar). However, be aware that few functions (particularly those involving basis elements) will give a correct result because the linear spaces in *GrassmannAlgebra* are by default assumed finite. Remember to redeclare a finite basis when you have finished using the arbitrary dimension!

DeclareExtraScalars[ω]; DeclareSymbolicDimension[ω]

ω

ToComplementForm$\left[\underset{a}{\alpha} \vee \underset{b}{\beta} \vee \underset{c}{\gamma}\right]$

$(-1)^{(a+b+c)\,(1+\omega)}\ \overline{\underset{a}{\bar{\alpha}} \wedge \underset{b}{\bar{\beta}} \wedge \underset{c}{\bar{\gamma}}}$

Simplifying complements

The *GrassmannAlgebra* function for converting the complements in an expression to a simpler form (wherever possible) is `GrassmannSimplify` (or its alias $\mathcal{G}$). 'Simpler' here means that the complement operation appears less frequently in the result. Remember that the complement depends on the metric.

Example: Simplifying complements with a Euclidean metric

`GrassmannSimplify` will take any Grassmann expression and convert any complements of basis elements in it. Suppose we are in a 3-space with a Euclidean metric.

$\mathbb{V}_3$; **DeclareDefaultMetric[]**

{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

$\mathcal{G}[\overline{\overline{\overline{e_2} \wedge e_2 \wedge \overline{\overline{e_3}}}}]$

$e_1 \wedge e_2$

`GrassmannSimplify` works on lists and matrices of Grassmann expressions to convert any complements occurring in any of the elements. Here is a matrix whose elements we have complemented:

$\mathbf{M = \{\{a\, e_1,\, b\, e_2\},\, \{-b\, e_2,\, c\, e_3\}\};\ MatrixForm[\overline{M}]}$

$$\begin{pmatrix} \overline{a\, e_1} & \overline{b\, e_2} \\ \overline{-b\, e_2} & \overline{c\, e_3} \end{pmatrix}$$

Applying $\mathcal{G}$ to this matrix of complements simplifies each element.

$\mathbf{Mc = \mathcal{G}[\overline{M}];\ MatrixForm[Mc]}$

$$\begin{pmatrix} a\, e_2 \wedge e_3 & -b\, e_1 \wedge e_3 \\ b\, e_1 \wedge e_3 & c\, e_1 \wedge e_2 \end{pmatrix}$$

It also works on complements of complements of symbolic expressions. In a 3-space, the complement of the complement of any element is the element itself.

$\mathcal{G}[\overline{\overline{\underset{m}{\alpha} \vee \underset{k}{\beta}}}]$

$\underset{m}{\alpha} \vee \underset{k}{\beta}$

In a 4-space however, the sign of the result will depend on the grades involved.

$\mathbb{V}_4;\ \mathcal{G}[\overline{\overline{\underset{m}{\alpha} \vee \underset{k}{\beta}}}]$

$(-1)^{k+m}\, \underset{m}{\alpha} \vee \underset{k}{\beta}$

Example: Simplifying complements with any metric

`GrassmannSimplify` will convert complements of basis elements in any declared metric.

```
V3; G = {{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}};
DeclareMetric[G]; MatrixForm[G]
```

$$\begin{pmatrix} 1 & 0 & \nu \\ 0 & -1 & 0 \\ \nu & 0 & -1 \end{pmatrix}$$

$$A = \mathcal{G}[\{\overline{1}, \overline{e_1}, \overline{e_1 \wedge e_2}, \overline{e_1 \wedge e_2 \wedge e_3}\}]$$

$$\left\{\frac{e_1 \wedge e_2 \wedge e_3}{\sqrt{1+\nu^2}}, \frac{\nu\, e_1 \wedge e_2}{\sqrt{1+\nu^2}} + \frac{e_2 \wedge e_3}{\sqrt{1+\nu^2}}, \frac{\nu\, e_1}{\sqrt{1+\nu^2}} - \frac{e_3}{\sqrt{1+\nu^2}}, \sqrt{1+\nu^2}\right\}$$

Here, $\mathbb{k}$ is equal to the inverse of $\sqrt{1+\nu^2}$.

Of course, converting a complement of a complement gives a result independent of the metric, which we can verify by taking the complement of the previous result and applying $\mathcal{G}$ again.

$$\text{GrassmannSimplify}[\overline{A}]$$

$$\{1, e_1, e_1 \wedge e_2, e_1 \wedge e_2 \wedge e_3\}$$

Note that it is only because the space is three-dimensional in this example that there are no sign differences between any of the basis elements in the list and the complement of its complement.

Creating tables and palettes of complements of basis elements

GrassmannAlgebra has a function `ComplementTable` for tabulating basis elements and their complements in the currently declared space and the currently declared metric. Once the table is generated you can edit it, or copy from it.

Without further explanation we show how the functions are applied in some commonly occurring cases.

Euclidean metric

Here we repeat the construction of the complement table of Section 5.4. Tables of complements for other bases are obtained by declaring the bases, then entering the command `ComplementTable`.

```
V2; DeclareDefaultMetric[] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
ComplementTable
```

Λ	BASIS	COMPLEMENT
$\underset{0}{\Lambda}$	1	$e_1 \wedge e_2$
$\underset{1}{\Lambda}$	e_1	e_2
$\underset{1}{\Lambda}$	e_2	$-e_1$
$\underset{2}{\Lambda}$	$e_1 \wedge e_2$	1

Non-Euclidean metric

```
𝕍3; DeclareMetric[{{1, 0, ν}, {0, -1, 0}, {ν, 0, -1}}] //
 MatrixForm
```

$$\begin{pmatrix} 1 & 0 & \nu \\ 0 & -1 & 0 \\ \nu & 0 & -1 \end{pmatrix}$$

```
ComplementTable
```

Λ	BASIS	COMPLEMENT
$\underset{0}{\Lambda}$	1	$\frac{e_1 \wedge e_2 \wedge e_3}{\sqrt{1+\nu^2}}$
$\underset{1}{\Lambda}$	e_1	$\frac{\nu\, e_1 \wedge e_2}{\sqrt{1+\nu^2}} + \frac{e_2 \wedge e_3}{\sqrt{1+\nu^2}}$
$\underset{1}{\Lambda}$	e_2	$\frac{e_1 \wedge e_3}{\sqrt{1+\nu^2}}$
$\underset{1}{\Lambda}$	e_3	$-\frac{e_1 \wedge e_2}{\sqrt{1+\nu^2}} + \frac{\nu\, e_2 \wedge e_3}{\sqrt{1+\nu^2}}$
$\underset{2}{\Lambda}$	$e_1 \wedge e_2$	$\frac{\nu\, e_1}{\sqrt{1+\nu^2}} - \frac{e_3}{\sqrt{1+\nu^2}}$
$\underset{2}{\Lambda}$	$e_1 \wedge e_3$	$\sqrt{1+\nu^2}\; e_2$
$\underset{2}{\Lambda}$	$e_2 \wedge e_3$	$\frac{e_1}{\sqrt{1+\nu^2}} + \frac{\nu\, e_3}{\sqrt{1+\nu^2}}$
$\underset{3}{\Lambda}$	$e_1 \wedge e_2 \wedge e_3$	$\sqrt{1+\nu^2}$

Creating palettes of complements of basis elements

GrassmannAlgebra also has a function `ComplementPalette` which creates a palette of basis elements and their complements in the currently declared space and the currently declared metric. You can click on a cell in the palette to paste the contents at the insertion point.

```
𝕍2; DeclareMetric[{{g11, g12}, {g12, g22}}]
```

$\{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}$

ComplementPalette

Λ	BASIS	COMPLEMENT
$\underset{0}{\Lambda}$	1	$\frac{e_1 \wedge e_2}{\sqrt{-g_{12}^2 + g_{11}\, g_{22}}}$
$\underset{1}{\Lambda}$	e_1	$\frac{e_2\, g_{11}}{\sqrt{-g_{12}^2 + g_{11}\, g_{22}}} - \frac{e_1\, g_{12}}{\sqrt{-g_{12}^2 + g_{11}\, g_{22}}}$
$\underset{1}{\Lambda}$	e_2	$\frac{e_2\, g_{12}}{\sqrt{-g_{12}^2 + g_{11}\, g_{22}}} - \frac{e_1\, g_{22}}{\sqrt{-g_{12}^2 + g_{11}\, g_{22}}}$
$\underset{2}{\Lambda}$	$e_1 \wedge e_2$	$\sqrt{-g_{12}^2 + g_{11}\, g_{22}}$

5.9 Geometric Interpretations

The Euclidean complement in a vector 2-space

Consider a vector $\mathbf{x}$ in a 2-dimensional vector space expressed in terms of basis vectors $\mathbf{e_1}$ and $\mathbf{e_2}$.

$$\mathbf{x} == \mathbf{a}\, \mathbf{e_1} + \mathbf{b}\, \mathbf{e_2}$$

Since this is a *Euclidean* vector space, we can depict the basis vectors at right angles to each other. But note that since it is a *vector* space, we do not depict an origin.

Graphic of $\mathbf{x}$ and two orthogonal basis vectors at right angles to each other.

The complement of $\mathbf{x}$ is given by:

$$\overline{\mathbf{x}} == \mathbf{a}\, \overline{\mathbf{e_1}} + \mathbf{b}\, \overline{\mathbf{e_2}} == \mathbf{a}\, \mathbf{e_2} - \mathbf{b}\, \mathbf{e_1}$$

Remember, the Euclidean complement of a basis element is its cobasis element, and a basis element and its cobasis element are defined by their exterior product being the basis *n*-element, in this case $\mathbf{e_1} \wedge \mathbf{e_2}$.

It is clear from simple geometry that $\mathbf{x}$ and $\overline{\mathbf{x}}$ are at right angles to each other, thus verifying our geometric interpretation of the algebraic notion of orthogonality: *a simple element and its complement are orthogonal*.

Taking the complement of $\overline{\mathbf{x}}$ gives $-\mathbf{x}$:

$$\overline{\overline{\mathbf{x}}} == \mathbf{a}\, \overline{\mathbf{e_2}} - \mathbf{b}\, \overline{\mathbf{e_1}} == -\mathbf{a}\, \mathbf{e_1} - \mathbf{b}\, \mathbf{e_2} == -\mathbf{x}$$

Or, we could have used the complement of a complement axiom:

$$\overline{\overline{\mathbf{x}}} == (-1)^{1\,(2-1)}\; \mathbf{x} == -\mathbf{x}$$

Continuing to take complements we find that we eventually return to the original element.

$$\overline{\overline{\overline{\mathbf{x}}}} == -\overline{\mathbf{x}}$$

$$\overline{\overline{\overline{\overline{\mathbf{x}}}}} == \mathbf{x}$$

In a vector 2-space, taking the complement of a vector is thus equivalent to *rotating the vector by one right angle counterclockwise*.

The Euclidean complement in a plane

Now suppose we are working in a Euclidean plane with a basis of one origin point $\mathbb{O}$ and two basis vectors $\mathbf{e_1}$ and $\mathbf{e_2}$. Let us declare this basis then call for a palette of basis elements and their complements.

$\mathbb{P}_2$; ComplementPalette

Λ	BASIS	COMPLEMENT
$\underset{0}{\Lambda}$	1	$\mathbb{O} \wedge e_1 \wedge e_2$
$\underset{1}{\Lambda}$	$\mathbb{O}$	$e_1 \wedge e_2$
$\underset{1}{\Lambda}$	e_1	$-(\mathbb{O} \wedge e_2)$
$\underset{1}{\Lambda}$	e_2	$\mathbb{O} \wedge e_1$
$\underset{2}{\Lambda}$	$\mathbb{O} \wedge e_1$	e_2
$\underset{2}{\Lambda}$	$\mathbb{O} \wedge e_2$	$-e_1$
$\underset{2}{\Lambda}$	$e_1 \wedge e_2$	$\mathbb{O}$
$\underset{3}{\Lambda}$	$\mathbb{O} \wedge e_1 \wedge e_2$	1

The complement table tells us that each basis vector is orthogonal to the *axis* involving the other basis vector. Suppose now we take a general vector $\mathbf{x}$ as in the previous example. The complement of this vector is:

$$\overline{\mathbf{x}} == \mathbf{a}\,\overline{\mathbf{e_1}} + \mathbf{b}\,\overline{\mathbf{e_2}} == -\,\mathbf{a}\,\mathbb{O} \wedge \mathbf{e_2} + \mathbf{b}\,\mathbb{O} \wedge \mathbf{e_1} == \mathbb{O} \wedge (\mathbf{b}\,\mathbf{e_1} - \mathbf{a}\,\mathbf{e_2})$$

Again, this is an axis (bound vector) through the origin at right angles to the vector $\mathbf{x}$.

Now let us take a general point $\mathbf{P} == \mathbb{O} + \mathbf{x}$, and explore what element is orthogonal to this point.

$$\overline{\mathbf{P}} == \overline{\mathbb{O}} + \overline{\mathbf{x}} == \mathbf{e_1} \wedge \mathbf{e_2} + \mathbb{O} \wedge (\mathbf{b}\,\mathbf{e_1} - \mathbf{a}\,\mathbf{e_2})$$

The effect of adding the bivector $\mathbf{e_1} \wedge \mathbf{e_2}$ is to shift the bound vector $\mathbb{O} \wedge (\mathbf{b}\,\mathbf{e_1} - \mathbf{a}\,\mathbf{e_2})$ parallel to itself. We can factor $\mathbf{e_1} \wedge \mathbf{e_2}$ into the exterior product of two vectors, one parallel to $\mathbf{b}\,\mathbf{e_1} - \mathbf{a}\,\mathbf{e_2}$.

$$\mathbf{e_1} \wedge \mathbf{e_2} == \mathbf{z} \wedge (\mathbf{b}\,\mathbf{e_1} - \mathbf{a}\,\mathbf{e_2})$$

Now we can write $\overline{\mathbf{P}}$ as:

$$\overline{\mathbf{P}} == (\mathbb{O} + \mathbf{z}) \wedge (\mathbf{b}\,\mathbf{e_1} - \mathbf{a}\,\mathbf{e_2})$$

The vector $\mathbf{z}$ is the position vector of *any* point on the line defined by the bound vector $\overline{\mathbf{P}}$. A particular point of interest is the point on the line closest to the point $\mathbf{P}$ or the origin. The position vector $\mathbf{z}$ would then be a vector at right angles to the direction of the line. Thus we can write $\mathbf{e_1 \wedge e_2}$ as:

$$\mathbf{e_1 \wedge e_2 == -\frac{(a\,e_1 + b\,e_2) \wedge (b\,e_1 - a\,e_2)}{a^2 + b^2}}$$

The final expression for the complement of the point $\mathbf{P == \mathbb{O} + a\,e_1 + b\,e_2}$ can then be written as a bound vector in a direction perpendicular to the position vector of $\mathbf{P}$ through a point $\mathbf{P^*}$.

$$\overline{\mathbf{P}} == \left(\mathbb{O} - \frac{(\mathbf{a\,e_1 + b\,e_2})}{\mathbf{a^2 + b^2}}\right) \wedge (\mathbf{b\,e_1 - a\,e_2}) == \mathbf{P^*} \wedge (\mathbf{b\,e_1 - a\,e_2})$$

Graphic of $\mathbf{P}$, $\mathbf{P^*}$, their position vectors, the line joining them, and the line and bound vector perpendicular to this line through $\mathbf{P^*}$.

The point $\mathbf{P^*}$ is called the *inverse point* to $\mathbf{P}$. Inverse points are situated on the same line through the origin, and on opposite sides of it. The product of their distances from the origin is unity.

The Euclidean complement in a vector 3-space

Now we come to the classical 3-dimensional vector space which is the usual geometric interpretation of a 3-dimensional linear space. We start our explorations by generating a palette of basis elements and their complements.

```
𝕍3; ComplementPalette
```

Λ	BASIS	COMPLEMENT
Λ_0	1	$e_1 \wedge e_2 \wedge e_3$
Λ_1	e_1	$e_2 \wedge e_3$
Λ_1	e_2	$-(e_1 \wedge e_3)$
Λ_1	e_3	$e_1 \wedge e_2$
Λ_2	$e_1 \wedge e_2$	e_3
Λ_2	$e_1 \wedge e_3$	$-e_2$
Λ_2	$e_2 \wedge e_3$	e_1
Λ_3	$e_1 \wedge e_2 \wedge e_3$	1

First, we consider a vector $\mathbf{x}$ and take its complement.

$$\overline{\mathbf{x}} == \mathbf{a\,\overline{e_1} + b\,\overline{e_2} + c\,\overline{e_3} == a\,e_2 \wedge e_3 - b\,e_1 \wedge e_3 + c\,e_1 \wedge e_2}$$

The bivector $\overline{\mathbf{x}}$ is thus a sum of components, each in one of the coordinate bivectors. We have already shown in Chapter 2: The Exterior Product that a bivector in a 3-space is simple, and hence can be factored into the exterior product of two vectors. It is in this form that the bivector will be most easily interpreted as a geometric entity. There is an infinity of vectors that are orthogonal to the vector $\mathbf{x}$, but they are all contained in the bivector $\overline{\mathbf{x}}$.

We can use the *GrassmannAlgebra* function `FindFactoredForm` discussed in Chapter 3 to give us one possible factorization.

$$\mathbf{FindFactoredForm}[a\,e_2 \wedge e_3 - b\,e_1 \wedge e_3 + c\,e_1 \wedge e_2]$$

$$\frac{(c\,e_2 - b\,e_3) \wedge (-c\,e_1 + a\,e_3)}{c}$$

The first factor is a vector in the $\mathbf{e_2 \wedge e_3}$ coordinate bivector, while the second is in the $\mathbf{e_1 \wedge e_3}$ coordinate bivector.

Graphic of x showing its orthogonality to each of $c\,e_2 - b\,e_3$, and $-c\,e_1 + a\,e_3$.

The complements of each of these vectors will be bivectors which contain the original vector $\mathbf{x}$. We can verify this easily with `GrassmannSimplify`.

$$\mathcal{G}[\{(\overline{c\,e_2 - b\,e_3}) \wedge (a\,e_1 + b\,e_2 + c\,e_3),\; (\overline{-c\,e_1 + a\,e_3}) \wedge (a\,e_1 + b\,e_2 + c\,e_3)\}]$$

$$\{0,\ 0\}$$

5.10 Complements in a vector subspace of a multiplane

Metrics in a multiplane

If we want to interpret one element of a linear space as an origin point, we need to consider which forms of metric make sense, or are useful in some degree. We have seen above that a Euclidean metric makes sense both in vector spaces (as we expected) but also in the plane where one basis element is interpreted as the origin point and the other two as basis vectors.

More general metrics make sense in vector spaces, because the entities all have the same interpretation. This leads us to consider hybrid metrics on *n*-planes in which the vector subspace has a general metric but the origin is orthogonal to all vectors. We can therefore adopt a Euclidean metric for the origin, and a more general metric for the vector subspace.

$$G_{ij} == \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{11} & \cdots & g_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & g_{n1} & \cdots & g_{nn} \end{pmatrix} \qquad 5.33$$

These hybrid metrics will be useful later when we discuss screw algebra in Chapter 7 and Mechanics in Chapter 8. Of course the permitted transformations on a space with such a metric must be restricted to those which maintain the orthogonality of the origin point to the vector subspace.

All the formulae for complements in the vector subspace still hold. In an *n*-plane (which is of dimension *n*+1), the vector subspace has dimension *n*, and hence the complement operation for the *n*-plane and its vector subspace will not be the same. We will denote the complement operation in the vector subspace by using an *overvector* $\vec{\square}$ instead of an overbar $\overline{\square}$.

Such a complement is called a *free complement* because it is a complement in a vector space in which all elements are 'free', that is, not bound through points.

The constant $\Bbbk$ will be the same in both spaces and still equate to the inverse of the square root of the determinant $\mathbf{g}$ of the metric tensor. Hence it is easy to show from the metric tensor above that:

$$\overline{1} == \frac{1}{\sqrt{g}}\, \mathbb{O} \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 5.34$$

$$\vec{1} == \frac{1}{\sqrt{g}}\, e_1 \wedge e_2 \wedge \cdots \wedge e_n \qquad 5.35$$

$$\overline{1} == \mathbb{O} \wedge \vec{1} \qquad 5.36$$

$$\overline{\mathbb{O}} == \vec{1} \qquad 5.37$$

In this interpretation $\overline{1}$ is called the unit *n*-plane, while $\vec{1}$ is called the unit *n*-vector. The unit *n*-plane is the unit *n*-vector bound through the origin.

The complement of an *m*-vector

If we define $\underbrace{e_i}$ as the cobasis element of e_i in the vector basis of the vector subspace (note that this is denoted by an 'underbracket' rather than an 'underbar'), then the formula for the complement of a basis vector in the vector subspace is:

$$\vec{e_i} == \frac{1}{\sqrt{g}} \sum_{j=1}^{n} g_{ij}\, \underbrace{e_j} \qquad 5.38$$

In the *n*-plane, the complement $\overline{e_i}$ of a basis vector e_i is given by the metric of the *n*-plane as:

$$\overline{e_i} == \frac{1}{\sqrt{g}} \sum_{j=1}^{n} g_{ij} \left(-\mathbb{O} \wedge \underbrace{e_j}\right)$$

Hence by using formula 5.31 we have that:

$$\overline{e_i} = -\mathbb{O} \wedge \overrightarrow{e_i} \tag{5.39}$$

More generally we have that for a basis *m*-vector, its complement in the *n*-plane is related to its complement in the vector subspace by:

$$\overline{\underset{m}{e_i}} = (-1)^m \, \mathbb{O} \wedge \overrightarrow{\underset{m}{e_i}} \tag{5.40}$$

And for a general *m*-vector:

$$\overline{\underset{m}{\alpha}} = (-1)^m \, \mathbb{O} \wedge \overrightarrow{\underset{m}{\alpha}} \tag{5.41}$$

The free complement of any exterior product involving the origin $\mathbb{O}$ is undefined.

The complement of an element bound through the origin

To express the complement of a bound element in the *n*-plane in terms of the complement in the vector subspace we can use the complement axiom.

$$\overline{\mathbb{O} \wedge e_i} = \overline{\mathbb{O}} \vee \overline{e_i} = \overrightarrow{1} \vee (-\mathbb{O} \wedge \overrightarrow{e_i})$$

$$= \left(\frac{1}{\sqrt{g}} e_1 \wedge e_2 \wedge \cdots \wedge e_n\right) \vee \left(-\mathbb{O} \wedge \frac{1}{\sqrt{g}} \sum_{j=1}^{n} g_{ij} \, \underline{e_j}\right)$$

$$= -\frac{1}{g} \sum_{j=1}^{n} g_{ij} \, \left(e_j \wedge \underline{e_j}\right) \vee \left(\mathbb{O} \wedge \underline{e_j}\right)$$

$$= -\frac{1}{g} \sum_{j=1}^{n} g_{ij} \, \left(e_j \wedge \mathbb{O} \wedge \underline{e_j}\right) \vee \underline{e_j}$$

$$= \frac{1}{\sqrt{g}} \sum_{j=1}^{n} g_{ij} \, \overline{1} \vee \underline{e_j}$$

$$= \overline{1} \vee \left(\frac{1}{\sqrt{g}} \sum_{j=1}^{n} g_{ij} \, \underline{e_j}\right) = \overline{1} \vee \overrightarrow{e_i} = \overrightarrow{e_i}$$

$$\overline{\mathbb{O} \wedge e_i} = \overrightarrow{e_i} \tag{5.42}$$

More generally, we can show that for a general *m*-vector, the complement of the *m*-vector bound through the origin in an *n*-plane is simply the complement of the *m*-vector in the vector subspace of the *n*-plane.

$$\overline{\mathbb{O}\wedge\underset{m}{\alpha}} = \overrightarrow{\underset{m}{\alpha}} \qquad 5.43$$

The complement of the complement of an *m*-vector

We can now use this relationship to relate the complement of the complement of an *m*-vector in the *n*-plane to the complement of the complement in the vector subspace.

$$\overline{\underset{m}{\alpha}} = (-1)^m\, \mathbb{O}\wedge\overrightarrow{\underset{m}{\alpha}}$$

$$\overline{\overline{\underset{m}{\alpha}}} = (-1)^m\, \overline{\mathbb{O}\wedge\overrightarrow{\underset{m}{\alpha}}} = (-1)^m\, \overrightarrow{\overrightarrow{\underset{m}{\alpha}}}$$

$$\overline{\overline{\underset{m}{\alpha}}} = (-1)^m\, \overrightarrow{\overrightarrow{\underset{m}{\alpha}}} \qquad 5.44$$

The complement of a bound element

We now come to the point where we can determine formulae which express the complement of a general bound element in an *n*-plane in terms of the origin and complements in the vector subspace of the *n*-plane. We have already introduced a specific case of this in Section 5.9 above. In Chapter 6: The Interior Product, we will depict the concepts graphically.

Consider a bound *m*-vector $P\wedge\underset{m}{\alpha}$ in an *n*-plane.

$$P\wedge\underset{m}{\alpha} = (\mathbb{O}+x)\wedge\underset{m}{\alpha} = \mathbb{O}\wedge\underset{m}{\alpha} + x\wedge\underset{m}{\alpha}$$

The complement of this bound *m*-vector is:

$$\overline{P\wedge\underset{m}{\alpha}} = \overline{\mathbb{O}\wedge\underset{m}{\alpha}} + \overline{x\wedge\underset{m}{\alpha}}$$

$$= \overrightarrow{\underset{m}{\alpha}} + (-1)^{m+1}\, \mathbb{O}\wedge\overrightarrow{x\wedge\underset{m}{\alpha}}$$

$$= \overrightarrow{\underset{m}{\alpha}} - \mathbb{O}\wedge\overrightarrow{\underset{m}{\alpha}\wedge x}$$

$$\overline{P\wedge\underset{m}{\alpha}} = -\mathbb{O}\wedge\overrightarrow{\underset{m}{\alpha}\wedge x} + \overrightarrow{\underset{m}{\alpha}} \qquad P = \mathbb{O}+x \qquad 5.45$$

The simplest bound element is the point. Putting $\underset{m}{\alpha} = 1$ in the formula above gives:

$$\overline{P} = -\mathbb{O}\wedge\vec{x} + \vec{1} \qquad P = \mathbb{O}+x \qquad 5.46$$

The complement of a bound vector is:

$$\overline{\mathbf{P}\wedge\alpha} == -\mathbb{O}\wedge\overrightarrow{\alpha\wedge\mathbf{x}} + \overrightarrow{\alpha} \qquad\qquad \mathbf{P} == \mathbb{O} + \mathbf{x} \tag{5.47}$$

Calculating with free complements

Entering a free complement

To enter a vector subspace complement of a Grassmann expression **X** in *GrassmannAlgebra* you can either use the *GrassmannAlgebra* palette by selecting the expression **X** and clicking the button $\overrightarrow{\blacksquare}$, or simply enter `OverVector[X]` directly.

```
OverVector[e1 ∧ e2]
```

$\overrightarrow{e_1 \wedge e_2}$

Simplifying a free complement

If the basis of your currently declared space does not contain the origin $\mathbb{O}$, then the vector space complement (`OverVector`) operation is equivalent to the normal (`OverBar`) operation, and `GrassmannSimplify` will treat them as the same.

$\mathbb{V}_3;\ \mathcal{G}[\{\overrightarrow{e_1},\ \vec{x},\ \overline{e_1},\ \overline{x}\}]$

$\{e_2 \wedge e_3,\ \bar{x},\ e_2 \wedge e_3,\ \bar{x}\}$

If, on the other hand, the basis of your currently declared space does contain the origin $\mathbb{O}$, then `GrassmannSimplify` will convert any expressions containing `OverVector` complements to their equivalent `OverBar` forms.

$\mathbb{P}_3;\ \mathcal{G}[\{\overrightarrow{e_1},\ \vec{x},\ \overline{e_1},\ \overline{x}\}]$

$\{e_2 \wedge e_3,\ \overline{\mathbb{O}\wedge x},\ -(\mathbb{O}\wedge e_2 \wedge e_3),\ \bar{x}\}$

The free complement of the origin

Note that the free complement of the origin or any exterior product of elements involving the origin is undefined, and will be left unevaluated by `GrassmannSimplify`.

$\mathbb{P}_3;\ \mathcal{G}\left[\{\vec{\mathbb{O}},\ \overrightarrow{\mathbb{O}\wedge e_1},\ \overrightarrow{\mathbb{O}+x},\ \overrightarrow{\mathbb{O}+e_1}\}\right]$

$\{\vec{\mathbb{O}},\ \overrightarrow{\mathbb{O}\wedge e_1},\ \overline{\mathbb{O}\wedge x}+\vec{\mathbb{O}},\ \vec{\mathbb{O}}+e_2\wedge e_3\}$

Example: The complement of a screw

Consider an interpreted 2-element **S** in a 3-plane which is the sum of a bivector **x∧y** and a vector bound through the origin perpendicular to the bivector. We can represent the vector as congruent to the free complement of the bivector. In the simplest case when the vector is equal to the free complement of the bivector we have:

Graphic of a vector bound through the origin perpendicular to a bivector x∧y.

$$S == \mathbb{O}\wedge\overrightarrow{\overline{x\wedge y}} + x\wedge y$$

One of the interesting properties of such an entity is that *it is equal to its own complement*. We can show this by simple transformations using the formulae derived in the sections above. The complement of each term turns into the other term.

$$\overline{S} == \overline{\mathbb{O}\wedge\overrightarrow{\overline{x\wedge y}}} + \overline{x\wedge y}$$

$$== \overrightarrow{\overline{\overline{x\wedge y}}} + (-1)^2\,\mathbb{O}\wedge\overrightarrow{\overline{x\wedge y}}$$

$$== x\wedge y + \mathbb{O}\wedge\overrightarrow{\overline{x\wedge y}} == S$$

We can verify this by expressing **x** and **y** in basis form and using `GrassmannSimplify`.

```
ℙ3; x = CreateVector[a]; y = CreateVector[b];
```

```
𝒢[S == S̄]
```

```
True
```

This entity is a simple case of a *screw*. We shall explore its properties further in Chapter 7: Exploring Screw Algebra, and see applications to mechanics in Chapter 8: Exploring Mechanics.

5.11 Reciprocal Bases

Contravariant and covariant bases

Up to this point we have only considered one basis for $\underset{1}{\Lambda}$, which we now call the *contravariant* basis. Introducing a second basis called the *covariant* basis enables us to write the formulae for complements in a more symmetric way. Contravariant basis elements were denoted with subscripted indices, for example e_i. Following standard practice, we denote covariant basis elements with superscripts, for example e^i. The two bases are said to be *reciprocal.*

This section will summarize formulae relating basis elements and their cobases and complements in terms of reciprocal bases. For simplicity, we adopt the Einstein summation convention.

In $\underset{1}{\Lambda}$ the metric tensor g_{ij} forms the relationship between the reciprocal bases.

$$e_i == g_{ij}\, e^j \qquad 5.48$$

This relationship induces a metric tensor $\overset{m}{g}_{i,j}$ on $\underset{m}{\Lambda}$.

$$\underset{m}{e_i} == \overset{m}{g}_{ij}\ \underset{m}{e^j} \qquad 5.49$$

The complement of a basis element

The complement of a basis 1-element of the contravariant basis has already been defined by:

$$\overline{e_i} == \frac{1}{\sqrt{g}}\ g_{ij}\ \underline{e_j} \qquad 5.50$$

$$g == |g_{ij}|$$

Here $g == |g_{ij}|$ is the determinant of the metric tensor. Taking the complement of equation 5.22 and substituting for $\overline{e_i}$ in equation 5.24 gives:

$$\overline{e^i} == \frac{1}{\sqrt{g}}\ \underline{e_i} \qquad 5.51$$

The reciprocal relation to this is:

$$\overline{e_i} == \sqrt{g}\ \underline{e^i} \qquad 5.52$$

These formulae can be extended to basis elements of $\underset{m}{\Lambda}$.

$$\overline{\underset{m}{e_i}} == \sqrt{g}\ \underline{\underset{m}{e^i}} \qquad 5.53$$

$$\overline{\underset{m}{e^i}} == \frac{1}{\sqrt{g}}\ \underline{\underset{m}{e_i}} \qquad 5.54$$

Particular cases of these formulae are:

$$\frac{1}{\sqrt{g}}\ e_1 \wedge e_2 \wedge \cdots \wedge e_n == \overline{1} == \sqrt{g}\ e^1 \wedge e^2 \wedge \cdots \wedge e^n \qquad 5.55$$

$$\overline{e_1 \wedge e_2 \wedge \cdots \wedge e_n} == \sqrt{g} \qquad 5.56$$

$$\overline{e^1 \wedge e^2 \wedge \cdots \wedge e^n} == \frac{1}{\sqrt{g}} \qquad 5.57$$

$$\overline{e_i} == \sqrt{g}\ (-1)^{i-1}\ e^1 \wedge \cdots \wedge \square^i \wedge \cdots \wedge e^n \qquad 5.58$$

$$\overline{e^i} == \frac{1}{\sqrt{g}}\ (-1)^{i-1}\ e_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge e_n \qquad 5.59$$

$$\overline{e_{i_1} \wedge \cdots \wedge e_{i_m}} == \sqrt{g}\ (-1)^{K_m}\ e^1 \wedge \cdots \wedge \square^{i_1} \wedge \cdots \wedge \square^{i_m} \wedge \cdots \wedge e^n \qquad 5.60$$

$$\overline{e^{i_1} \wedge \cdots \wedge e^{i_m}} == \frac{1}{\sqrt{g}}\ (-1)^{K_m}\ e_1 \wedge \cdots \wedge \square_{i_1} \wedge \cdots \wedge \square_{i_m} \wedge \cdots \wedge e_n \qquad 5.61$$

where $K_m = \sum_{\gamma=1}^{m} i_\gamma + \frac{1}{2} m (m+1)$ and the symbol □ means the corresponding element is missing from the product.

The complement of a cobasis element

The cobasis of a general basis element was developed in Section 2.6 as:

$$\underline{\underset{m}{e_i}} == (-1)^{K_m}\ e_1 \wedge \cdots \wedge \square_{i_1} \wedge \cdots \wedge \square_{i_m} \wedge \cdots \wedge e_n$$

where $K_m == \sum_{\gamma=1}^{m} i_\gamma + \frac{1}{2} m (m+1)$.

Taking the complement of this formula and applying equation 5.27 gives the required result.

$$\overline{\underline{\underset{m}{e_i}}} == (-1)^{K_m}\ \overline{e_1 \wedge \cdots \wedge \square_{i_1} \wedge \cdots \wedge \square_{i_m} \wedge \cdots \wedge e_n}$$

$$== \sqrt{g}\ (-1)^{K_m}\ \underline{e^1 \wedge \cdots \wedge \square^{i_1} \wedge \cdots \wedge \square^{i_m} \wedge \cdots \wedge e^n}$$

$$== \sqrt{g}\ (-1)^{m\,(n-m)}\ e^{i_1} \wedge \cdots \wedge e^{i_m}$$

$$== \sqrt{g}\ (-1)^{m\,(n-m)}\ \underset{m}{e^i}$$

$$\overline{\underline{\underset{m}{e_i}}} == \sqrt{g}\ (-1)^{m\,(n-m)}\ \underset{m}{e^i} \qquad 5.62$$

Similarly:

$$\overline{\underset{\underline{m}}{e}^{i}} == \frac{1}{\sqrt{g}} (-1)^{m(n-m)} \underset{m}{e}_{i} \qquad 5.63$$

The complement of a complement of a basis element

To verify the complement of a complement axiom, we begin with the equation for the complement of a covariant basis element, take the complement of the equation, and then substitute for $\overline{\underset{\underline{m}}{e}^{i}}$ from equation 5.37.

$$\overline{\overline{\underset{m}{e}_{i}}} == \sqrt{g}\ \overline{\underset{\underline{m}}{e}^{i}}$$

$$== \left(\sqrt{g}\right) \left(\frac{1}{\sqrt{g}}\right) (-1)^{m(n-m)} \underset{m}{e}_{i}$$

$$== (-1)^{m(n-m)} \underset{m}{e}_{i}$$

A similar result is obtained for the complement of the complement of a contravariant basis element.

The exterior product of basis elements

The exterior product of a basis element with the complement of the corresponding basis element in the reciprocal basis is equal to the unit n-element. The exterior product with the complement of any other basis element in the reciprocal basis is zero.

$$\underset{m}{e}_{i} \wedge \overline{\underset{m}{e}^{j}} == \underset{m}{e}_{i} \wedge \left(\frac{1}{\sqrt{g}}\right) \underset{\underline{m}}{e}_{j} == \delta_{i}^{j}\ \overline{1}$$

$$\underset{m}{e}^{i} \wedge \overline{\underset{m}{e}_{j}} == \underset{m}{e}^{i} \wedge \left(\sqrt{g}\right) \underset{\underline{m}}{e}^{j} == \delta_{j}^{i}\ \overline{1}$$

$$\underset{m}{e}_{i} \wedge \overline{\underset{m}{e}^{j}} == \delta_{i}^{j}\ \overline{1} \qquad\qquad \underset{m}{e}^{i} \wedge \overline{\underset{m}{e}_{j}} == \delta_{j}^{i}\ \overline{1} \qquad 5.64$$

The exterior product of a basis element with the complement of any other basis element in the same basis is equal to the corresponding component of the metric tensor times the unit n-element.

$$\underset{m}{e}_{i} \wedge \overline{\underset{m}{e}_{j}} == \underset{m}{e}_{i} \wedge \left(\overset{m}{g}_{jk}\ \overline{\underset{m}{e}^{k}}\right) == \underset{m}{e}_{i} \wedge \overset{m}{g}_{jk} \left(\frac{1}{\sqrt{g}}\right) \underset{\underline{m}}{e}_{k} == \overset{m}{g}_{ij}\ \overline{1}$$

$$\underset{m}{e}^{i} \wedge \overline{\underset{m}{e}^{j}} == \underset{m}{e}^{i} \wedge \left(\overset{m}{g}^{jk}\ \overline{\underset{m}{e}_{k}}\right) == \underset{m}{e}^{i} \wedge \overset{m}{g}^{jk}\ \left(\sqrt{g}\right)\ \underline{\underset{m}{e}^{k}} == \overset{m}{g}^{ij}\ \overline{1}$$

$$\underset{m}{e}_{i} \wedge \overline{\underset{m}{e}_{j}} == \overset{m}{g}_{ij}\ \overline{1} \qquad \underset{m}{e}^{i} \wedge \overline{\underset{m}{e}^{j}} == \overset{m}{g}^{ij}\ \overline{1} \tag{5.65}$$

In $\underset{1}{\wedge}$ these reduce to:

$$e_{i} \wedge \overline{e^{j}} == \delta_{i}^{j}\ \overline{1} \qquad e^{i} \wedge \overline{e_{j}} == \delta_{j}^{i}\ \overline{1} \tag{5.66}$$

$$e_{i} \wedge \overline{e_{j}} == g_{ij}\ \overline{1} \qquad e^{i} \wedge \overline{e^{j}} == g^{ij}\ \overline{1} \tag{5.67}$$

The regressive product of basis elements

In the series of formulae below we have repeated the formulae derived for the exterior product and shown its equivalent for the regressive product in the same box for comparison. We have used the fact that δ_{j}^{i} and $\overset{m}{g}^{ij}$ are symmetric to interchange the indices.

$$\underset{m}{e}_{i} \wedge \overline{\underset{m}{e}^{j}} == \delta_{i}^{j}\ \overline{1} \qquad \underset{m}{e}_{i} \vee \overline{\underset{m}{e}^{j}} == \delta_{i}^{j} \tag{5.68}$$

$$\underset{m}{e}^{i} \wedge \overline{\underset{m}{e}_{j}} == \delta_{j}^{i}\ \overline{1} \qquad \underset{m}{e}^{i} \vee \overline{\underset{m}{e}_{j}} == \delta_{j}^{i} \tag{5.69}$$

$$\underset{m}{e}_{i} \wedge \overline{\underset{m}{e}_{j}} == \overset{m}{g}_{ij}\ \overline{1} \qquad \underset{m}{e}_{i} \vee \overline{\underset{m}{e}_{j}} == \overset{m}{g}_{ij} \tag{5.70}$$

$$\underset{m}{e}^{i} \wedge \overline{\underset{m}{e}^{j}} == \overset{m}{g}^{ij}\ \overline{1} \qquad \underset{m}{e}^{i} \vee \overline{\underset{m}{e}^{j}} == \overset{m}{g}^{ij} \tag{5.71}$$

The complement of a simple element is simple

The expression of the complement of a basis element in terms of its cobasis element in the reciprocal basis allows us a straightforward proof that *the complement of a simple element is simple*.

To show this, suppose a simple element $\underset{m}{\alpha}$ expressed in terms of its factors α_{i}.

$$\underset{m}{\alpha} == \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$$

Take any other *n–m* 1-elements $\alpha_{m+1}, \alpha_{m+2}, \cdots, \alpha_n$ such that $\alpha_1, \alpha_2, \cdots, \alpha_m, \alpha_{m+1}, \cdots, \alpha_n$ form an independent set, and thus a basis of $\underset{1}{\Lambda}$. Then by equation 5.34 we have:

$$\overline{\underset{m}{\alpha}} == \overline{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m} == \sqrt{g_\alpha}\; \alpha^{m+1} \wedge \alpha^{m+2} \wedge \cdots \wedge \alpha^n$$

where g_α is the determinant of the metric tensor in the α basis.

The complement of $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$ is thus a scalar multiple of $\alpha^{m+1} \wedge \alpha^{m+2} \wedge \cdots \wedge \alpha^n$. Since this is evidently simple, the assertion is proven.

5.12 Summary

In this chapter we have shown that by defining the complement operation on a basis of $\underset{1}{\Lambda}$ as $\overline{e_i} == \mathbb{k} \sum_{j=1}^{n} g_{ij}\, \underline{e_j}$, and accepting the complement axiom $\overline{\underset{m}{\alpha} \wedge \underset{k}{\beta}} == \overline{\underset{m}{\alpha}} \vee \overline{\underset{k}{\beta}}$ as the mechanism for extending the complement to higher grade elements, the requirement that $\overline{\overline{\underset{m}{\alpha}}} == \pm \underset{m}{\alpha}$ is true, constrains g_{ij} to be symmetric and $\mathbb{k}$ to have the value $\pm \frac{1}{\sqrt{g}}$,where g is the determinant of the g_{ij}.

The metric tensor g_{ij} was introduced as a mapping between the two linear spaces $\underset{1}{\Lambda}$ and $\underset{n-1}{\Lambda}$. To this point has not yet been related to notions of interior, inner or scalar products. This will be addressed in the next chapter, where we will also see that the constraint $\mathbb{k} == \pm \frac{1}{\sqrt{g}}$ is equivalent to requiring that the magnitude of the unit *n*-element is unity.

Once we have constrained g_{ij} to be symmetric, we are able to introduce the standard notion of a reciprocal basis. The formulae for complements of basis elements in any of the $\underset{m}{\Lambda}$ then become much simpler to express.

6 The Interior Product

6.1 Introduction

To this point we have defined three important operations associated with a linear space: the *exterior product,* the *regressive product*, and the *complement*.

In this chapter we introduce the *interior product*, the fourth operation of fundamental importance to Grassmann algebra. The interior product of two elements is defined as the regressive product of one element with the complement of the other. Whilst the exterior product of an *m*-element and a *k*-element generates an ($m+k$)-element, the interior product of an *m*-element and a *k*-element ($m \geq k$) generates an ($m-k$)-element. This means that the interior product of two elements of equal grade is a 0-element (or scalar). The interior product of an element with itself is a scalar, and it is this scalar that is used to define the *measure* of the element.

The interior product of two 1-elements corresponds to the usual notion of *inner*, *scalar*, or *dot product*. But we will see that the notion of measure is not restricted to 1-elements. Just as one may associate the measure of a vector with a *length*, the measure of a bivector may be associated with an *area*, and the measure of a trivector with a *volume*.

If the exterior product of an *m*-element and a 1-element is zero, then it is known that the 1-element is contained in the *m*-element. If the interior product of an *m*-element and a 1-element is zero, then this means that the 1-element is contained in the *complement* of the *m*-element. In this case it may be said that the 1-element is *orthogonal* to the *m*-element.

The basing of the notion of interior product on the notions of regressive product and complement follows here the Grassmannian tradition rather than that of the current literature which introduces the inner product onto a linear space as an arbitrary extra definition. We do this in the belief that it is the most straightforward way to obtain consistency within the algebra and to see and exploit the relationships between the notions of exterior product, regressive product, complement and interior product, and to discover and prove formulae relating them.

We use the term 'interior' in addition to 'inner' to signal that the products are not quite the same. In traditional usage the inner product has resulted in a scalar. The interior product is however more general, being able to operate on two elements of any and perhaps different grades. We reserve the term *inner product* for the interior product of two elements of the same grade. An inner product of two elements of grade 1 is called a *scalar product*. In sum: inner products are scalar whilst, in general, interior products are not.

We denote the interior product with a small circle with a 'bar' through it, thus $\ominus$. This is to signify that it has a more extended meaning than the inner product. Thus the interior product of $\underset{m}{\alpha}$ with $\underset{k}{\beta}$ becomes $\underset{m}{\alpha} \ominus \underset{k}{\beta}$. This product is zero if $m < k$. Thus the order is important: the element of higher degree should be written on the left. The interior product has the same left associativity as the negation operator, or minus sign. It is possible to define both left and right interior products, but in practice the added complexity is not rewarded by an increase in utility.

We will see in Chapter 10: The Generalized Product that the interior product of two elements can be expressed as a certain generalized product, independent of the order of the factors.

6.2 Defining the Interior Product

Definition of the interior product

The *interior product* of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is denoted $\underset{m}{\alpha} \ominus \underset{k}{\beta}$ and is defined by:

$$\boxed{\underset{m}{\alpha} \ominus \underset{k}{\beta} == \underset{m}{\alpha} \vee \overline{\underset{k}{\beta}} \quad \in \underset{m-k}{\Lambda} \qquad m \geq k} \qquad 6.1$$

If $m < k$, then $\underset{m}{\alpha} \vee \overline{\underset{k}{\beta}}$ is necessarily zero (otherwise the grade of the product would be negative), hence:

$$\boxed{\underset{m}{\alpha} \ominus \underset{k}{\beta} == 0 \qquad m < k} \qquad 6.2$$

■ An important convention

In order to avoid unnecessarily distracting caveats on every formula involving interior products, in the rest of this book we will suppose that *the grade of the first factor is always greater than or equal to the grade of the second factor*. The formulae will remain true even if this is not the case, but they will be trivially so by virtue of their terms reducing to zero.

An alternative definition

The definition 6.1 above is chosen rather than $\overline{\underset{m}{\alpha}} \vee \underset{k}{\beta}$ ($k \geq m$) as a possible alternative since, given the way in which the complement has been defined earlier, only the proposed definition leads to the interior product of an *m*-element with itself in Euclidean space ($g_{ij} == \delta_{ij}$) being invariably positive, independent of the grade of the element or the dimension of the space. We can see this from equation 5.69 in the last chapter. Instead of $\underset{m}{e_i} \ominus \underset{m}{e_i} == 1$, we would have rather $\underset{m}{e_i} \ominus \underset{m}{e_i} == (-1)^{m\,(n-m)}$.

Historical Note

Grassmann and workers in the Grassmannian tradition define the interior product of two elements as the product of one with the complement of the other, the product being either exterior or regressive depending on which interpretation produces a non-zero result. Furthermore, when the grades of the elements are equal, it is defined either way. This definition involves the confusion between scalars and *n*-elements discussed in Chapter 5, Section 5.1 (equivalent to assuming a Euclidean metric and identifying scalars with pseudo-scalars). It is to obviate this inconsistency and restriction on generality that the approach adopted here bases its definition of the interior product explicitly on the *regressive* exterior product.

Implications of the regressive product axioms

By expressing one or more elements as a complement, the relations of the regressive product axiom set may be rewritten in terms of the interior product, thus yielding some of its more fundamental properties.

◆ **⊖ 6: The interior product of an *m*-element and a *k*-element is an (*m*–*k*)-element.**

The grade of the interior product of two elements is the difference of their grades.

$$\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{k}{\beta} \in \underset{k}{\Lambda} \quad \Longrightarrow \quad \underset{m}{\alpha} \ominus \underset{k}{\beta} \in \underset{m-k}{\Lambda} \tag{6.3}$$

Thus, in contradistinction to the regressive product, the grade of an interior product does not depend on the dimension of the underlying linear space.

If the grade of the first factor is less than that of the second, the interior product is zero.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == 0 \qquad m < k \tag{6.4}$$

◆ **$\ominus$ 7: The interior product is not associative.**

The exterior, regressive and interior products have relations derived directly from the associativity of the regressive product.

$$\left(\underset{m}{\alpha}\ominus\underset{k}{\beta}\right)\ominus\underset{r}{\gamma} == \underset{m}{\alpha}\ominus\left(\underset{k}{\beta}\wedge\underset{r}{\gamma}\right) \qquad 6.5$$

$$\left(\underset{m}{\alpha}\vee\underset{k}{\beta}\right)\ominus\underset{r}{\gamma} == \underset{m}{\alpha}\vee\left(\underset{k}{\beta}\ominus\underset{r}{\gamma}\right) \qquad 6.6$$

By interchanging the order of the factors on the right-hand side of formula 6.5 we can derive an alternative expression for it.

$$\underset{m}{\alpha}\ominus\left(\underset{k}{\beta}\wedge\underset{r}{\gamma}\right) == (-1)^{k\,r}\,\underset{m}{\alpha}\ominus\left(\underset{r}{\gamma}\wedge\underset{k}{\beta}\right) \quad \Longrightarrow$$

$$\left(\underset{m}{\alpha}\ominus\underset{k}{\beta}\right)\ominus\underset{r}{\gamma} == (-1)^{k\,r}\left(\underset{m}{\alpha}\ominus\underset{r}{\gamma}\right)\ominus\underset{k}{\beta} \qquad 6.7$$

◆ **$\ominus$ 8: The unit scalar 1 is the identity for the interior product.**

The interior product of an element with the unit scalar **1** does not change it. The interior product of scalars is equivalent to ordinary scalar multiplication. The complement operation may be viewed as the interior product with the unit *n*-element $\overline{1}$.

$$\underset{m}{\alpha}\ominus 1 == \underset{m}{\alpha} \qquad 6.8$$

$$\underset{m}{\alpha}\ominus a == \underset{m}{\alpha}\wedge a == a\,\underset{m}{\alpha} \qquad 6.9$$

$$1\ominus 1 == 1 \qquad 6.10$$

$$\overline{\underset{m}{\alpha}} == \overline{1}\ominus\underset{m}{\alpha} \qquad 6.11$$

◆ **$\ominus$ 9: The inverse of a scalar with respect to the interior product is its complement.**

The interior product of the complement of a scalar and the reciprocal of the scalar is the unit *n*-element.

$$\overline{1} == \overline{a} \ominus \frac{1}{a} \qquad 6.12$$

- $\ominus$ **10: The interior product of two elements is congruent to the interior product of their complements in reverse order.**

The interior product of two elements is equal (apart from a possible sign) to the interior product of their complements in reverse order.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == (-1)^{(n-m)\,(m-k)}\, \overline{\underset{k}{\beta}} \ominus \overline{\underset{m}{\alpha}} \qquad 6.13$$

If the elements are of the same grade, the interior product of two elements is equal to the interior product of their complements. (It will be shown later that, because of the symmetry of the metric tensor, the interior product of two elements of the same grade is symmetric, hence the order of the factors on either side of the equation may be reversed.)

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} == \overline{\underset{m}{\beta}} \ominus \overline{\underset{m}{\alpha}} \qquad 6.14$$

- $\ominus$ **11: An interior product with zero is zero.**

Every exterior linear space has a zero element whose interior product with any other element is zero.

$$\underset{m}{\alpha} \ominus 0 == 0 == 0 \ominus \underset{m}{\alpha} \qquad 6.15$$

- $\ominus$ **12: The interior product is distributive over addition.**

The interior product is both left and right distributive over addition.

$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \ominus \underset{r}{\gamma} == \underset{m}{\alpha} \ominus \underset{r}{\gamma} + \underset{m}{\beta} \ominus \underset{r}{\gamma} \qquad 6.16$$

$$\underset{m}{\alpha} \ominus \left(\underset{r}{\beta} + \underset{r}{\gamma}\right) == \underset{m}{\alpha} \ominus \underset{r}{\beta} + \underset{m}{\alpha} \ominus \underset{r}{\gamma} \qquad 6.17$$

Orthogonality

Orthogonality is a concept generated by the complement operation.

If $\mathbf{x}$ is a 1-element and $\underset{m}{\alpha}$ is a simple *m*-element, then $\mathbf{x}$ and $\underset{m}{\alpha}$ may be said to be *linearly dependent* if and only if $\underset{m}{\alpha} \wedge \mathbf{x} == 0$, that is, $\mathbf{x}$ is contained in (the space of) $\underset{m}{\alpha}$.

Similarly, $\mathbf{x}$ and $\underset{m}{\alpha}$ are said to be *orthogonal* if and only if $\overline{\underset{m}{\alpha}} \wedge \mathbf{x} == 0$, that is, $\mathbf{x}$ is contained in $\overline{\underset{m}{\alpha}}$.

By taking the complement of $\overline{\underset{m}{\alpha}} \wedge \mathbf{x} == 0$, we can see that $\overline{\underset{m}{\alpha}} \wedge \mathbf{x} == 0$ if and only if $\underset{m}{\alpha} \vee \overline{\mathbf{x}} == 0$. Thus it may also be said that a 1-element $\mathbf{x}$ is orthogonal to a simple element $\underset{m}{\alpha}$ if and only if their interior product is zero, that is, $\underset{m}{\alpha} \ominus \mathbf{x} == 0$.

If $\underset{k}{\mathbf{x}} == \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k$ then $\underset{m}{\alpha}$ and $\underset{k}{\mathbf{x}}$ are said to be *totally orthogonal* if and only if $\underset{m}{\alpha} \ominus \mathbf{x}_i == 0$ for all $\mathbf{x}_i$ contained in $\underset{k}{\mathbf{x}}$.

However, for $\underset{m}{\alpha} \ominus (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k)$ to be zero it is only necessary that *one* of the $\mathbf{x}_i$ be orthogonal to $\underset{m}{\alpha}$. To show this, suppose it to be (without loss of generality) $\mathbf{x}_1$. Then by formula 6.5 we can write $\underset{m}{\alpha} \ominus (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k)$ as $\left(\underset{m}{\alpha} \ominus \mathbf{x}_1\right) \ominus (\mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k)$, whence it becomes immediately clear that if $\underset{m}{\alpha} \ominus \mathbf{x}_1$ is zero then so is the whole product $\underset{m}{\alpha} \ominus \underset{k}{\mathbf{x}}$.

Just as the vanishing of the exterior product of two simple elements $\underset{m}{\alpha}$ and $\underset{k}{\mathbf{x}}$ implies only that they have some 1-element in common, so the vanishing of their interior product implies only that $\overline{\underset{m}{\alpha}}$ and $\underset{k}{\mathbf{x}}$ ($m \geq k$) have some 1-element in common, and conversely. That is, there is a 1-element $\mathbf{x}$ such that the following implications hold.

$$\underset{m}{\alpha} \wedge \underset{k}{\mathbf{x}} == 0 \quad \Leftrightarrow \quad \left\{\underset{m}{\alpha} \wedge \mathbf{x} == 0,\ \underset{k}{\mathbf{x}} \wedge \mathbf{x} == 0\right\}$$

$$\underset{m}{\alpha} \ominus \underset{k}{\mathbf{x}} == 0 \quad \Leftrightarrow \quad \left\{\overline{\underset{m}{\alpha}} \wedge \mathbf{x} == 0,\ \underset{k}{\mathbf{x}} \wedge \mathbf{x} == 0\right\}$$

$$\underset{m}{\alpha} \ominus \underset{k}{\mathbf{x}} == 0 \quad \Leftrightarrow \quad \left\{\underset{m}{\alpha} \ominus \mathbf{x} == 0,\ \underset{k}{\mathbf{x}} \wedge \mathbf{x} == 0\right\}$$

$$\boxed{\underset{m}{\alpha} \ominus \underset{k}{\mathbf{x}} == 0 \quad \Leftrightarrow \quad \underset{m}{\alpha} \ominus \mathbf{x} == 0 \quad \Leftrightarrow \quad \overline{\underset{m}{\alpha}} \wedge \mathbf{x} == 0 \qquad \underset{k}{\mathbf{x}} \wedge \mathbf{x} == 0} \qquad 6.18$$

✡ The interior product of a vector with a simple bivector

Suppose $\mathbf{x}$ is a vector and $\alpha \wedge \beta$ is a simple bivector. The interior product of the bivector $\alpha \wedge \beta$ with the vector $\mathbf{x}$ is the vector ξ.

$$\xi = (\alpha \wedge \beta) \ominus \mathbf{x};$$

To explore interior products, we can use the *GrassmannAlgebra* function `ToScalarProducts`. Applying `ToScalarProducts` to expand ξ in terms of α and β gives:

```
ToScalarProducts[ξ]
```

$\beta\ (\mathbf{x}\ominus\alpha) - \alpha\ (\mathbf{x}\ominus\beta)$

We can see from this that ξ is contained in the bivector $\alpha\wedge\beta$. We can also verify this by expanding their exterior product.

```
ToScalarProducts[ξ ∧ (α ∧ β)]
```

0

The vectors ξ and $\mathbf{x}$ are also orthogonal.

```
ToScalarProducts[ξ ⊖ x]
```

0

Diagram of a vector x with a component in a bivector, and one orthogonal to it.

6.3 Properties of the Interior Product

Implications of the Complement Axiom

The Complement Axiom was introduced in Chapter 5 in the form $\overline{\underset{m}{\alpha}\wedge\underset{k}{\beta}} == \overline{\underset{m}{\alpha}}\vee\overline{\underset{k}{\beta}}$. We can recast the right-hand side into a form involving the interior product.

$$\overline{\underset{m}{\alpha}\wedge\underset{k}{\beta}} == \overline{\underset{m}{\alpha}}\ominus\underset{k}{\beta} == (-1)^{m\,k}\ \overline{\underset{k}{\beta}}\ominus\underset{m}{\alpha} \qquad 6.19$$

Further, by replacing $\underset{m}{\alpha}$ by $\overline{\underset{m}{\alpha}}$ in $\overline{\underset{m}{\alpha}\wedge\underset{k}{\beta}}$ we have that $\overline{\overline{\underset{m}{\alpha}}\wedge\underset{k}{\beta}} == \overline{\overline{\underset{m}{\alpha}}}\ominus\underset{k}{\beta}$. Putting $\overline{\overline{\underset{m}{\alpha}}} == (-1)^{m\,(n-m)}\ \underset{m}{\alpha}$ yields an alternative definition of the interior product in terms of the exterior product and the complement.

$$\underset{m}{\alpha}\ominus\underset{k}{\beta} == (-1)^{m\,(n-m)}\ \overline{\overline{\underset{m}{\alpha}}\wedge\underset{k}{\beta}} \qquad 6.20$$

Extended interior products

The interior product of an element with the exterior product of several elements may be shown straightforwardly to be a type of 'extended' interior product.

$$\underset{m}{\alpha} \ominus \left(\underset{k1}{\beta} \wedge \underset{k2}{\beta} \wedge \cdots \wedge \underset{kp}{\beta}\right) == \underset{m}{\alpha} \vee \overline{(\underset{k1}{\beta} \wedge \underset{k2}{\beta} \wedge \cdots \wedge \underset{kp}{\beta})} == \underset{m}{\alpha} \vee \left(\overline{\underset{k1}{\beta}} \vee \overline{\underset{k2}{\beta}} \vee \cdots \vee \overline{\underset{kp}{\beta}}\right)$$

$$== \left(\left(\left(\underset{m}{\alpha} \vee \overline{\underset{k1}{\beta}}\right) \vee \overline{\underset{k2}{\beta}}\right) \vee \cdots\right) \vee \overline{\underset{kp}{\beta}} == \left(\left(\left(\underset{m}{\alpha} \ominus \underset{k1}{\beta}\right) \ominus \underset{k2}{\beta}\right) \ominus \cdots\right) \ominus \underset{kp}{\beta}$$

$$\boxed{\underset{m}{\alpha} \ominus \left(\underset{k1}{\beta} \wedge \underset{k2}{\beta} \wedge \cdots \wedge \underset{kp}{\beta}\right) == \left(\left(\left(\underset{m}{\alpha} \ominus \underset{k1}{\beta}\right) \ominus \underset{k2}{\beta}\right) \ominus \cdots\right) \ominus \underset{kp}{\beta}} \qquad 6.21$$

Thus, the interior product is *left-associative*.

Interior products of elements of the same grade

The regressive product of an *m*-element and an (*n*–*m*)-element is given by formula 3.21.

$$\underset{m}{\alpha} \vee \underset{n-m}{\beta} == \left(\underset{m}{\alpha} \wedge \underset{n-m}{\beta}\right) \vee 1 \qquad \in \underset{0}{\Lambda}$$

The dual of this is given by formula 3.25. By putting $\underset{n}{1}$ equal to $\overline{1}$ (formula 5.8), we obtain

$$\underset{m}{\alpha} \wedge \underset{n-m}{\beta} == \left(\underset{m}{\alpha} \vee \underset{n-m}{\beta}\right) \overline{1} \qquad \in \underset{n}{\Lambda}$$

By putting $\overline{\underset{m}{\beta}} == \underset{n-m}{\beta}$ these may be rewritten in terms of the interior product as:

$$\boxed{\underset{m}{\alpha} \ominus \underset{m}{\beta} == \left(\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}}\right) \vee 1} \qquad 6.22$$

$$\boxed{\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}} == \left(\underset{m}{\alpha} \ominus \underset{m}{\beta}\right) \overline{1}} \qquad 6.23$$

Since the grades of the two factors of the interior product $\underset{m}{\alpha} \ominus \underset{m}{\beta}$ are the same, the product may also be called an *inner product*.

The inner product

The interior product of two elements $\underset{m}{\alpha}$ and $\underset{m}{\beta}$ *of the same grade* is (also) called their *inner product*. We proceed to show that *the inner product is symmetric*.

Taking the complement of equation 6.23 gives:

$$\overline{\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}}} == \left(\underset{m}{\alpha} \ominus \underset{m}{\beta}\right) \overline{\overline{1}} == \underset{m}{\alpha} \ominus \underset{m}{\beta}$$

On the other hand, we can also use the complement axiom [5.3] and the complement of a complement formula [5.30] to give:

$$\overline{\underset{m}{\alpha} \wedge \overline{\underset{m}{\beta}}} == \overline{\underset{m}{\alpha}} \vee \overline{\overline{\underset{m}{\beta}}} == (-1)^{m\,(n-m)}\, \overline{\underset{m}{\alpha}} \vee \underset{m}{\beta} == \underset{m}{\beta} \vee \overline{\underset{m}{\alpha}} == \underset{m}{\beta} \ominus \underset{m}{\alpha}$$

From these two results we see that the order of the factors in an inner product is immaterial, that is, the inner product is *symmetric*.

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} == \underset{m}{\beta} \ominus \underset{m}{\alpha} \qquad 6.24$$

This symmetry, is of course, a property that we would expect of an inner product. However it is interesting to remember that the major result which permits this symmetry is that the complement operation has been required to satisfy the requirement of formula 5.30, that is, that the complement of the complement of an element should be, apart from a possible sign, equal to the element itself.

We have already shown in $\ominus$ 10 of the previous section that:

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == (-1)^{(n-m)\,(m-k)}\, \overline{\underset{k}{\beta}} \ominus \overline{\underset{m}{\alpha}}$$

Putting $k = m$, and using the above result on the symmetry of the inner product shows that the inner product of two elements of the same grade is also equal to the inner product of their complements.

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} == \overline{\underset{m}{\alpha}} \ominus \overline{\underset{m}{\beta}} \qquad 6.25$$

The scalar product

A *scalar product* is simply an interior or inner product of 1-elements. If $\mathbf{x}$ and $\mathbf{y}$ are 1-elements, then $\mathbf{x} \ominus \mathbf{y}$ is a scalar product.

Example: Orthogonalizing a set of 1-elements

Suppose we have a set of independent 1-elements $\alpha_1, \alpha_2, \alpha_3, \cdots$, and we wish to create an orthogonal set $e_1, e_2, e_3, \cdots$ spanning the same space.

We begin by choosing one of the elements, α_1 say, arbitrarily and setting this to be the first element e_1 in the orthogonal set to be created.

$$e_1 == \alpha_1$$

To create a second element e_2 orthogonal to e_1 within the space concerned, we choose a second element of the space, α_2 say, and form the interior product.

$$e_2 == (e_1 \wedge \alpha_2) \ominus e_1$$

We can see that e_2 is orthogonal to e_1 by taking their interior product and using formula 6.21 to see that the product is zero:

$$e_2 \ominus e_1 == ((e_1 \wedge \alpha_2) \ominus e_1) \ominus e_1 == (e_1 \wedge \alpha_2) \ominus (e_1 \wedge e_1) == 0$$

We can also see that α_2 lies in the 2-element $e_1 \wedge e_2$ (that is, α_2 is a linear combination of e_1 and e_2) by taking their exterior product and expanding it.

$$e_1 \wedge ((e_1 \wedge \alpha_2) \ominus e_1) \wedge \alpha_2 == e_1 \wedge ((e_1 \ominus e_1)\, \alpha_2 - (e_1 \ominus \alpha_2)\, e_1) \wedge \alpha_2 == 0$$

We will develop the formula used for this expansion in Section 6.4: The Interior Common Factor Theorem. For the meantime however, all we need to observe is that e_2 must be a linear combination of e_1 and α_2, because it is expressed in no other terms.

We create the rest of the orthogonal elements in a similar manner.

$$e_3 == (e_1 \wedge e_2 \wedge \alpha_3) \ominus (e_1 \wedge e_2)$$

$$e_4 == (e_1 \wedge e_2 \wedge e_3 \wedge \alpha_4) \ominus (e_1 \wedge e_2 \wedge e_3)$$
$$\ldots$$

In general then, the (*i*+1)th element e_{i+1} of the orthogonal set e_1, e_2, e_3, $\cdots$ is obtained from the previous *i* elements and the (*i*+1)th element α_{i+1} of the original set.

$$\boxed{e_{i+1} == (e_1 \wedge e_2 \wedge \cdots \wedge e_i \wedge \alpha_{i+1}) \ominus (e_1 \wedge e_2 \wedge \cdots \wedge e_i)} \qquad 6.26$$

Following a similar procedure to the one used for the second element, we can easily show that e_{i+1} is orthogonal to $e_1 \wedge e_2 \wedge \cdots \wedge e_i$ and hence to each of the $e_1, e_2, \cdots, e_i$, and that α_i lies in $e_1 \wedge e_2 \wedge \cdots \wedge e_i$ and hence is a linear combination of $e_1, e_2, \cdots, e_i$.

This procedure is equivalent to the Gram-Schmidt orthogonalization process.

6.4 The Interior Common Factor Theorem

The Interior Common Factor Formula

The Common Factor Axiom introduced in Chapter 3 is:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \vee \left(\underset{k}{\beta} \wedge \underset{p}{\gamma}\right) == \left(\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma}\right) \vee \underset{p}{\gamma}$$

Here $\underset{p}{\gamma}$ is simple, the expression is a *p*-element and *m*+*k*+*p* = *n*.

The dual of this expression is also a p-element, but in this case *m*+*k*+*p* = *2n*.

$$\left(\underset{m}{\alpha} \vee \underset{p}{\gamma}\right) \wedge \left(\underset{k}{\beta} \vee \underset{p}{\gamma}\right) == \left(\underset{m}{\alpha} \vee \underset{k}{\beta} \vee \underset{p}{\gamma}\right) \wedge \underset{p}{\gamma}$$

We can write this dual formula with the alternative ordering:

$$\left(\underset{p}{\gamma} \vee \underset{m}{\alpha}\right) \wedge \left(\underset{p}{\gamma} \vee \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \vee \underset{m}{\alpha} \vee \underset{k}{\beta}\right) \wedge \underset{p}{\gamma}$$

Then, by replacing $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ by their complements we get:

$$\left(\underset{p}{\gamma} \vee \overline{\underset{m}{\alpha}}\right) \wedge \left(\underset{p}{\gamma} \vee \overline{\underset{k}{\beta}}\right) == \left(\underset{p}{\gamma} \vee \overline{\underset{m}{\alpha}} \vee \overline{\underset{k}{\beta}}\right) \wedge \underset{p}{\gamma}$$

Finally, by replacing $\overline{\underset{m}{\alpha}} \vee \overline{\underset{k}{\beta}}$ with $\overline{\underset{m}{\alpha} \wedge \underset{k}{\beta}}$ we get the Interior Common Factor Formula.

$$\left(\underset{p}{\gamma} \ominus \underset{m}{\alpha}\right) \wedge \left(\underset{p}{\gamma} \ominus \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \ominus \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right)\right) \underset{p}{\gamma} \qquad \mathbf{6.27}$$

In this case $\underset{p}{\gamma}$ is simple, the expression is a p-element but $p = m+k$.

The Interior Common Factor Theorem

The Common Factor Theorem enabled an explicit expression for a regressive product to be derived. We now derive the interior product version called the Interior Common Factor Theorem.

We start with the Common Factor Theorem in the form:

$$\underset{m}{\alpha} \vee \underset{s}{\beta} == \sum_{i=1}^{\nu} \left(\underset{m-p}{\alpha_i} \wedge \underset{s}{\beta}\right) \vee \underset{p}{\alpha_i}$$

where $\underset{m}{\alpha} == \underset{m-p}{\alpha_1} \wedge \underset{p}{\alpha_1} == \underset{m-p}{\alpha_2} \wedge \underset{p}{\alpha_2} == \cdots == \underset{m-p}{\alpha_\nu} \wedge \underset{p}{\alpha_\nu}$, $\underset{m}{\alpha}$ is simple, $p = m+s-n$, and $\nu = \binom{m}{p}$.

Suppose now that $\underset{s}{\beta} == \overline{\underset{k}{\beta}}$, $k = n-s = m-p$. Substituting for $\underset{s}{\beta}$ and using formula 6.23 allows us to write the term in brackets as

$$\underset{m-p}{\alpha_i} \wedge \underset{s}{\beta} == \underset{k}{\alpha_i} \wedge \overline{\underset{k}{\beta}} == \left(\underset{k}{\alpha_i} \ominus \underset{k}{\beta}\right) \overline{1}$$

so that the Common Factor Theorem becomes:

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \sum_{i=1}^{\nu} \left(\underset{k}{\alpha_i} \ominus \underset{k}{\beta}\right) \underset{m-k}{\alpha_i} \qquad \mathbf{6.28}$$

$$\underset{m}{\alpha} == \underset{k}{\alpha_1} \wedge \underset{m-k}{\alpha_1} == \underset{k}{\alpha_2} \wedge \underset{m-k}{\alpha_2} == \cdots == \underset{k}{\alpha_\nu} \wedge \underset{m-k}{\alpha_\nu}$$

where $k \le m$, and $\nu = \binom{m}{k}$, and $\underset{m}{\alpha}$ is simple.

This formula is a source of many useful relations in Grassmann algebra. It indicates that an interior product of a simple element with another, not necessarily simple, element of equal or lower grade, may be expressed in terms of the factors of the simple element of higher grade.

When $\underset{m}{\alpha}$ is not simple, it may always be expressed as the sum of simple components: $\underset{m}{\alpha} == \underset{m}{\alpha^1} + \underset{m}{\alpha^2} + \underset{m}{\alpha^3} + \cdots$ (the $\mathbf{i}$ in the $\underset{m}{\alpha^i}$ are superscripts, not powers). From the linearity of the interior product, the Interior Common Factor Theorem may then be applied to each of the simple terms.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \left(\underset{m}{\alpha^1} + \underset{m}{\alpha^2} + \underset{m}{\alpha^3} + \cdots\right) \ominus \underset{k}{\beta} == \underset{m}{\alpha^1} \ominus \underset{k}{\beta} + \underset{m}{\alpha^2} \ominus \underset{k}{\beta} + \underset{m}{\alpha^3} \ominus \underset{k}{\beta} + \cdots$$

Thus, the Interior Common Factor Theorem may be applied to the interior product of any two elements, simple or non-simple. Indeed, the application to components in the preceding expansion may be applied to the components of $\underset{m}{\alpha}$ even if it is simple.

To make the Interior Common Factor Theorem more explicit, suppose $\underset{m}{\alpha} == \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$, then:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus \underset{k}{\beta} == \sum_{i_1 \cdots i_k} \left((\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}) \ominus \underset{k}{\beta}\right) (-1)^{K_k} \alpha_1 \wedge \cdots \wedge \square_{i_1} \wedge \cdots \wedge \square_{i_k} \wedge \cdots \wedge \alpha_m \qquad 6.29$$

where $K_k == \sum_{\gamma=1}^{k} i_\gamma + \frac{1}{2} k (k+1)$, and $\square_j$ means α_j is missing from the product.

Examples of the Interior Common Factor Theorem

In this section we take the formula just developed and give examples for specific values of *m* and *k*. We do this using the *GrassmannAlgebra* function `InteriorCommonFactorTheorem` which generates the specific case of the theorem when given an argument of the form $\underset{m}{\alpha} \ominus \underset{k}{\beta}$. The grades *m* and *k* must be given as positive integers.

β is a scalar

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus \underset{0}{\beta} == \underset{0}{\beta} (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \qquad 6.30$$

β is a 1-element

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus \beta == \sum_{i=1}^{m} (\alpha_i \ominus \beta) (-1)^{i-1} \alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m \qquad 6.31$$

```
InteriorCommonFactorTheorem[α_2 ⊖ β]
```

$$\underset{2}{\alpha} \ominus \beta == \alpha_1 \wedge \alpha_2 \ominus \beta == -(\beta \ominus \alpha_2)\, \alpha_1 + (\beta \ominus \alpha_1)\, \alpha_2$$

InteriorCommonFactorTheorem$\left[\underset{3}{\alpha}\ominus\beta\right]$

$$\underset{3}{\alpha}\ominus\beta == \alpha_1\wedge\alpha_2\wedge\alpha_3\ominus\beta == (\beta\ominus\alpha_3)\ \alpha_1\wedge\alpha_2 - (\beta\ominus\alpha_2)\ \alpha_1\wedge\alpha_3 + (\beta\ominus\alpha_1)\ \alpha_2\wedge\alpha_3$$

InteriorCommonFactorTheorem$\left[\underset{4}{\alpha}\ominus\beta\right]$

$$\underset{4}{\alpha}\ominus\beta == \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\beta == -(\beta\ominus\alpha_4)\ \alpha_1\wedge\alpha_2\wedge\alpha_3 + (\beta\ominus\alpha_3)\ \alpha_1\wedge\alpha_2\wedge\alpha_4 - (\beta\ominus\alpha_2)\ \alpha_1\wedge\alpha_3\wedge\alpha_4 + (\beta\ominus\alpha_1)\ \alpha_2\wedge\alpha_3\wedge\alpha_4$$

β is a 2-element

$$(\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_m)\ominus\underset{2}{\beta} == \sum_{i,j}\left((\alpha_i\wedge\alpha_j)\ominus\underset{2}{\beta}\right)\ (-1)^{i+j-1}\ \alpha_1\wedge\cdots\wedge\square_i\wedge\cdots\wedge\square_j\wedge\cdots\wedge\alpha_m \qquad 6.32$$

InteriorCommonFactorTheorem$\left[\underset{3}{\alpha}\ominus\underset{2}{\beta}\right]$

$$\underset{3}{\alpha}\ominus\underset{2}{\beta} == \alpha_1\wedge\alpha_2\wedge\alpha_3\ominus\underset{2}{\beta} == \left(\underset{2}{\beta}\ominus\alpha_2\wedge\alpha_3\right)\alpha_1 - \left(\underset{2}{\beta}\ominus\alpha_1\wedge\alpha_3\right)\alpha_2 + \left(\underset{2}{\beta}\ominus\alpha_1\wedge\alpha_2\right)\alpha_3$$

InteriorCommonFactorTheorem$\left[\underset{4}{\alpha}\ominus\underset{2}{\beta}\right]$

$$\underset{4}{\alpha}\ominus\underset{2}{\beta} == \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\underset{2}{\beta} == \left(\underset{2}{\beta}\ominus\alpha_3\wedge\alpha_4\right)\alpha_1\wedge\alpha_2 - \left(\underset{2}{\beta}\ominus\alpha_2\wedge\alpha_4\right)\alpha_1\wedge\alpha_3 + \left(\underset{2}{\beta}\ominus\alpha_2\wedge\alpha_3\right)\alpha_1\wedge\alpha_4 + \left(\underset{2}{\beta}\ominus\alpha_1\wedge\alpha_4\right)\alpha_2\wedge\alpha_3 - \left(\underset{2}{\beta}\ominus\alpha_1\wedge\alpha_3\right)\alpha_2\wedge\alpha_4 + \left(\underset{2}{\beta}\ominus\alpha_1\wedge\alpha_2\right)\alpha_3\wedge\alpha_4$$

β is a 3-element

InteriorCommonFactorTheorem$\left[\underset{4}{\alpha}\ominus\underset{3}{\beta}\right]$

$$\underset{4}{\alpha}\ominus\underset{3}{\beta} == \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\underset{3}{\beta} == -\left(\underset{3}{\beta}\ominus\alpha_2\wedge\alpha_3\wedge\alpha_4\right)\alpha_1 + \left(\underset{3}{\beta}\ominus\alpha_1\wedge\alpha_3\wedge\alpha_4\right)\alpha_2 - \left(\underset{3}{\beta}\ominus\alpha_1\wedge\alpha_2\wedge\alpha_4\right)\alpha_3 + \left(\underset{3}{\beta}\ominus\alpha_1\wedge\alpha_2\wedge\alpha_3\right)\alpha_4$$

The list form of the Interior Common Factor Theorem

We start with the Interior Common Factor Theorem.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \sum_{i=1}^{\nu} \left(\underset{k}{\beta} \ominus \underset{k}{\alpha_i}\right) \underset{m-k}{\alpha_i}$$

$$\underset{m}{\alpha} == \underset{k}{\alpha_1} \wedge \underset{m-k}{\alpha_1} == \underset{k}{\alpha_2} \wedge \underset{m-k}{\alpha_2} == \cdots == \underset{k}{\alpha_\nu} \wedge \underset{m-k}{\alpha_\nu}$$

If we collect together the k-element factors of $\underset{m}{\alpha}$ into a list denoted $\underset{k}{\boldsymbol{\alpha}}$, and the ($m$–$k$)-element factors of $\underset{m}{\alpha}$ into a second list denoted $\underset{m-k}{\boldsymbol{\alpha}}$, we can rewrite the interior product as a list `Dot` product. The `Dot` product is a *Mathematica* function which adds the products of corresponding terms of two lists. Note the larger font size to denote the list corresponding to a given set of elements.

$$\underset{k}{\boldsymbol{\alpha}} == \left\{\underset{k}{\alpha_1},\ \underset{k}{\alpha_2},\ \cdots,\ \underset{k}{\alpha_\nu}\right\}$$

$$\underset{m-k}{\boldsymbol{\alpha}} == \left\{\underset{m-k}{\alpha_1},\ \underset{m-k}{\alpha_2},\ \cdots,\ \underset{m-k}{\alpha_\nu}\right\}$$

Thus:

$$\boxed{\begin{gathered}\underset{m}{\alpha} \ominus \underset{k}{\beta} == \left(\underset{k}{\beta} \ominus \underset{k}{\boldsymbol{\alpha}}\right) . \underset{m-k}{\boldsymbol{\alpha}} \\ \underset{m}{\boldsymbol{\alpha}} == \underset{k}{\boldsymbol{\alpha}} \wedge \underset{m-k}{\boldsymbol{\alpha}}\end{gathered}} \qquad \mathbf{6.33}$$

In list form this can be written:

$$\boxed{\begin{gathered}\underset{m}{\alpha} \ominus \underset{k}{\beta} == \underset{k}{\beta} \ominus \left\{\underset{k}{\alpha_1},\ \underset{k}{\alpha_2},\ \cdots,\ \underset{k}{\alpha_\nu}\right\} . \left\{\underset{m-k}{\alpha_1},\ \underset{m-k}{\alpha_2},\ \cdots,\ \underset{m-k}{\alpha_\nu}\right\} \\ \left\{\underset{m}{\alpha},\ \underset{m}{\alpha},\ \cdots,\ \underset{m}{\alpha}\right\} == \left\{\underset{k}{\alpha_1},\ \underset{k}{\alpha_2},\ \cdots,\ \underset{k}{\alpha_\nu}\right\} \wedge \left\{\underset{m-k}{\alpha_1},\ \underset{m-k}{\alpha_2},\ \cdots,\ \underset{m-k}{\alpha_\nu}\right\}\end{gathered}} \qquad \mathbf{6.34}$$

Further discussion of matrix operations in *GrassmannAlgebra* may be found in Chapter 13: Exploring Grassmann Matrix Algebra.

■ Example: List expansion of an interior product

We wish to expand the interior product of a 3-element and a 2-element $\underset{3}{\alpha} \ominus \underset{2}{\beta}$. For simplicity, we use the *GrassmannAlgebra* functions `ElementList` and `CoelementList` to create the lists, and `MatrixProduct` to take the inner products with $\underset{2}{\beta}$.

$$\underset{2}{\boldsymbol{\alpha}} = \mathbf{ElementList[2]}\left[\underset{3}{\alpha}\right]$$

$$\{\alpha_1 \wedge \alpha_2,\ \alpha_1 \wedge \alpha_3,\ \alpha_2 \wedge \alpha_3\}$$

$$\underset{1}{\boldsymbol{\alpha}} = \mathbf{CoelementList[2]}\left[\underset{3}{\alpha}\right]$$

$$\{\alpha_3,\ -\alpha_2,\ \alpha_1\}$$

Thus $\underset{3}{\alpha} \ominus \underset{2}{\beta}$ can be computed by entering:

$$\text{MatrixProduct}\left[\underset{2}{\beta} \ominus \underset{2}{\alpha}\right] . \underset{1}{\alpha}$$

$$\left(\underset{2}{\beta} \ominus \alpha_2 \wedge \alpha_3\right) \alpha_1 - \left(\underset{2}{\beta} \ominus \alpha_1 \wedge \alpha_3\right) \alpha_2 + \left(\underset{2}{\beta} \ominus \alpha_1 \wedge \alpha_2\right) \alpha_3$$

6.5 The Inner Product

Inner products of general elements

As we have already discussed in Section 6.3 the interior product of two simple elements $\underset{m}{\alpha} == \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$ and $\underset{m}{\beta} == \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m$ of the same grade is called more specifically the *inner product* of $\underset{m}{\alpha}$ and $\underset{m}{\beta}$. In what follows we derive a simple expression for this often-encountered quantity. We begin with the form we wish to re-express.

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} == (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m)$$

From formula 6.21 for extended interior products:

$$\underset{m}{\alpha} \ominus \left(\underset{k1}{\beta} \wedge \underset{k2}{\beta} \wedge \cdots \wedge \underset{kp}{\beta}\right) == \left(\left(\left(\underset{m}{\alpha} \ominus \underset{k1}{\beta}\right) \ominus \underset{k2}{\beta}\right) \ominus \cdots\right) \ominus \underset{kp}{\beta}$$

we can write $(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m)$ as:

$$((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus \beta_1) \ominus (\beta_2 \wedge \cdots \wedge \beta_m)$$

or, by rearranging the β factors to bring β_j to the beginning of the product as:

$$((\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus \beta_j) \ominus ((-1)^{j-1} (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \square_j \wedge \cdots \wedge \beta_m))$$

The Interior Common Factor Theorem gives the first factor of this expression as:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus \beta_j == \sum_{i=1}^{m} (\alpha_i \ominus \beta_j) (-1)^{i-1} \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m$$

Thus, $(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m)$ becomes:

$$\boxed{\begin{gathered}(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) == \\ \sum_{i=1}^{m} (\alpha_i \ominus \beta_j) (-1)^{i+j} (\alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m) \ominus \\ (\beta_1 \wedge \cdots \wedge \square_j \wedge \cdots \wedge \beta_m)\end{gathered}} \qquad 6.35$$

If this process is repeated, we find the strikingly simple formula:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) == \mathrm{Det}[\alpha_i \ominus \beta_j] \qquad 6.36$$

Calculating inner products

The expression for the inner product of two simple elements can be developed in *GrassmannAlgebra* by using the operations `DevelopScalarProductMatrix` and `ToScalarProducts`. `DevelopScalarProductMatrix[α,β]` or `DevelopScalarProductMatrix[α⊖β]` develops the inner product of **α** and **β** into the matrix of the scalar products of their factors. The determinant of the scalar product matrix is the inner product.

We can use `ScalarProductMatrix` purely symbolically. If we apply it to elements of (unspecified) symbolic grade we get:

$$\mathbf{DevelopScalarProductMatrix}\left[\underset{m}{\alpha} \ominus \underset{m}{\beta}\right] \mathbin{//} \mathbf{MatrixForm}$$

$$\begin{pmatrix} \alpha_1 \ominus \beta_1 & \alpha_1 \ominus \beta_2 & \alpha_1 \ominus \cdots & \alpha_1 \ominus \beta_m \\ \alpha_2 \ominus \beta_1 & \alpha_2 \ominus \beta_2 & \alpha_2 \ominus \cdots & \alpha_2 \ominus \beta_m \\ \cdots \ominus \beta_1 & \cdots \ominus \beta_2 & \cdots \ominus \cdots & \cdots \ominus \beta_m \\ \alpha_m \ominus \beta_1 & \alpha_m \ominus \beta_2 & \alpha_m \ominus \cdots & \alpha_m \ominus \beta_m \end{pmatrix}$$

Or we can use the *GrassmannAlgebra* operation `DeterminantForm` to present the result as a determinant.

$$\mathbf{DevelopScalarProductMatrix}\left[\underset{m}{\alpha} \ominus \underset{m}{\beta}\right] \mathbin{//} \mathbf{DeterminantForm}$$

$$\begin{vmatrix} \alpha_1 \ominus \beta_1 & \alpha_1 \ominus \beta_2 & \alpha_1 \ominus \cdots & \alpha_1 \ominus \beta_m \\ \alpha_2 \ominus \beta_1 & \alpha_2 \ominus \beta_2 & \alpha_2 \ominus \cdots & \alpha_2 \ominus \beta_m \\ \cdots \ominus \beta_1 & \cdots \ominus \beta_2 & \cdots \ominus \cdots & \cdots \ominus \beta_m \\ \alpha_m \ominus \beta_1 & \alpha_m \ominus \beta_2 & \alpha_m \ominus \cdots & \alpha_m \ominus \beta_m \end{vmatrix}$$

If the grade of the elements is specified as an integer, we get a more specific result.

$$\mathbf{DevelopScalarProductMatrix}\left[\underset{3}{\xi} \ominus \underset{3}{\psi}\right] \mathbin{//} \mathbf{MatrixForm}$$

$$\begin{pmatrix} \xi_1 \ominus \psi_1 & \xi_1 \ominus \psi_2 & \xi_1 \ominus \psi_3 \\ \xi_2 \ominus \psi_1 & \xi_2 \ominus \psi_2 & \xi_2 \ominus \psi_3 \\ \xi_3 \ominus \psi_1 & \xi_3 \ominus \psi_2 & \xi_3 \ominus \psi_3 \end{pmatrix}$$

Displaying the inner product of two elements as the determinant of the scalar product of their factors shows clearly how the symmetry of the inner product depends on the symmetry of the scalar product (which in turn depends on the symmetry of the metric tensor).

To obtain the actual inner product we take the determinant of this matrix.

Det[DevelopScalarProductMatrix[$\underset{3}{\xi} \ominus \underset{3}{\psi}$]]

$$-(\xi_1 \ominus \psi_3)(\xi_2 \ominus \psi_2)(\xi_3 \ominus \psi_1) + (\xi_1 \ominus \psi_2)(\xi_2 \ominus \psi_3)(\xi_3 \ominus \psi_1) + (\xi_1 \ominus \psi_3)(\xi_2 \ominus \psi_1)(\xi_3 \ominus \psi_2) - (\xi_1 \ominus \psi_1)(\xi_2 \ominus \psi_3)(\xi_3 \ominus \psi_2) - (\xi_1 \ominus \psi_2)(\xi_2 \ominus \psi_1)(\xi_3 \ominus \psi_3) + (\xi_1 \ominus \psi_1)(\xi_2 \ominus \psi_2)(\xi_3 \ominus \psi_3)$$

If the elements of the determinant are given in terms of basis elements, we can obtain a final result. For example, if we are in 3-space and we take the scalar product of the element $\underset{3}{e} == e_1 \wedge e_2 \wedge e_3$ with itself we get:

$\mathbb{V}_3$; DevelopScalarProductMatrix[$\underset{3}{e} \ominus \underset{3}{e}$] // MatrixForm

$$\begin{pmatrix} e_1 \ominus e_1 & e_1 \ominus e_2 & e_1 \ominus e_3 \\ e_2 \ominus e_1 & e_2 \ominus e_2 & e_2 \ominus e_3 \\ e_3 \ominus e_1 & e_3 \ominus e_2 & e_3 \ominus e_3 \end{pmatrix}$$

A = Det[DevelopScalarProductMatrix[$\underset{3}{e} \ominus \underset{3}{e}$]]

$$-(e_1 \ominus e_3)(e_2 \ominus e_2)(e_3 \ominus e_1) + (e_1 \ominus e_2)(e_2 \ominus e_3)(e_3 \ominus e_1) + (e_1 \ominus e_3)(e_2 \ominus e_1)(e_3 \ominus e_2) - (e_1 \ominus e_1)(e_2 \ominus e_3)(e_3 \ominus e_2) - (e_1 \ominus e_2)(e_2 \ominus e_1)(e_3 \ominus e_3) + (e_1 \ominus e_1)(e_2 \ominus e_2)(e_3 \ominus e_3)$$

We can now substitute the metric elements $g_{i,j}$ for the scalar products $e_i \ominus e_j$ by using the *GrassmannAlgebra* function `ToMetricForm`. Declaring a default general metric we get:

DeclareMetric[$\mathbb{g}$]

$$\{\{g_{1,1}, g_{1,2}, g_{1,3}\}, \{g_{1,2}, g_{2,2}, g_{2,3}\}, \{g_{1,3}, g_{2,3}, g_{3,3}\}\}$$

ToMetricForm[A]

$$-g_{1,3}^2 g_{2,2} + 2 g_{1,2} g_{1,3} g_{2,3} - g_{1,1} g_{2,3}^2 - g_{1,2}^2 g_{3,3} + g_{1,1} g_{2,2} g_{3,3}$$

This is **Det[$g_{i,j}$]** as we would expect. Had we declared the default Euclidean metric we would have obtained unity as expected.

To calculate an inner product directly we can use the *GrassmannAlgebra* function `ToScalarProducts`.

ToScalarProducts[(x ∧ y ∧ z) ⊖ (u ∧ v ∧ w)]

$$-(u \ominus z)(v \ominus y)(w \ominus x) + (u \ominus y)(v \ominus z)(w \ominus x) + (u \ominus z)(v \ominus x)(w \ominus y) - (u \ominus x)(v \ominus z)(w \ominus y) - (u \ominus y)(v \ominus x)(w \ominus z) + (u \ominus x)(v \ominus y)(w \ominus z)$$

In fact, `ToScalarProducts` works to convert *any interior* products in a Grassmann expression to scalar products.

X = 1 + x ⊖ a + b (x ∧ y ∧ z) ⊖ w + c ($e_1 \wedge e_2$) ⊖ ($e_2 \wedge e_3$)

$$1 + x \ominus a + c\, e_1 \wedge e_2 \ominus e_2 \wedge e_3 + b\, x \wedge y \wedge z \ominus w$$

```
Y = ToScalarProducts[X]
```

$1 + a\,x - c\,(e_1 \ominus e_3)\,(e_2 \ominus e_2) + c\,(e_1 \ominus e_2)\,(e_2 \ominus e_3) + b\,(w \ominus z)\,x \wedge y - b\,(w \ominus y)\,x \wedge z + b\,(w \ominus x)\,y \wedge z$

```
ToMetricForm[Y]
```

$1 + a\,x - c\,g_{1,3}\,g_{2,2} + c\,g_{1,2}\,g_{2,3} + b\,(w \ominus z)\,x \wedge y - b\,(w \ominus y)\,x \wedge z + b\,(w \ominus x)\,y \wedge z$

Inner products of basis elements

The inner product of two basis *m*-elements has been expressed in various forms previously. Here we collect together the results for reference. Here, $\mathbf{g}_{ij}$ is the metric tensor, $\mathbf{g}$ is its determinant, and $\underset{m}{\alpha}$ and $\underset{m}{\beta}$ refer to any basis elements.

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} == \underset{m}{\beta} \ominus \underset{m}{\alpha} == \overline{\underset{m}{\alpha}} \ominus \overline{\underset{m}{\beta}} == \overline{\underset{m}{\beta}} \ominus \overline{\underset{m}{\alpha}} \qquad 6.37$$

$$\underset{m}{e_i} \ominus \underset{m}{e^j} == \delta_i^j \qquad\qquad \underset{m}{e^i} \ominus \underset{m}{e_j} == \delta_j^i \qquad 6.38$$

$$\underset{m}{e_i} \ominus \underset{m}{e_j} == \overline{\underset{m}{e_i}} \ominus \overline{\underset{m}{e_j}} == \mathbf{g}\, \underline{\underset{m}{e^i}} \ominus \underline{\underset{m}{e^j}} == \overset{m}{\mathbf{g}}_{ij} \qquad 6.39$$

$$\underset{m}{e^i} \ominus \underset{m}{e^j} == \overline{\underset{m}{e^i}} \ominus \overline{\underset{m}{e^j}} == \frac{1}{\mathbf{g}}\, \underline{\underset{m}{e_i}} \ominus \underline{\underset{m}{e_j}} == \overset{m}{\mathbf{g}}^{ij} \qquad 6.40$$

6.6 The Measure of an *m*-element

The definition of measure

The *measure of an element* $\underset{m}{\alpha}$ is denoted $\left|\underset{m}{\alpha}\right|$ and defined as the positive square root of the inner product of $\underset{m}{\alpha}$ with itself.

$$\left|\underset{m}{\alpha}\right| == \sqrt{\underset{m}{\alpha} \ominus \underset{m}{\alpha}} \qquad 6.41$$

If $\underset{m}{\alpha}$ is expressed in terms of components, its measure may be expressed as the positive square root of a quadratic form involving the metric.

$$\underset{m}{\alpha} == \sum_i a^i \underset{m}{e_i} \implies \left|\underset{m}{\alpha}\right| == \sqrt{\sum_{i,j} a^i a^j \overset{m}{g_{ij}}} \qquad 6.42$$

For some metrics this quadratic form may be zero or negative. Thus the measure of an element may be zero or imaginary. An element with zero measure is called *isotropic*. To avoid undue quantities of qualifications we assume in what follows that elements have non-zero measures.

The *measure of a scalar* is its absolute value.

$$|a| == \sqrt{a^2} \qquad 6.43$$

The *measure of unity* (or its negative) is, as would be expected, equal to unity.

$$|1| == 1 \qquad |-1| == 1 \qquad 6.44$$

The *measure of an n-element* $\underset{n}{\alpha} == a\, e_1 \wedge \cdots \wedge e_n$ is the absolute value of its single scalar coefficient multiplied by the determinant g of the metric tensor. Note that *n*-elements cannot be isotropic since we have required that g be non-zero.

$$\underset{n}{\alpha} == a\, e_1 \wedge \cdots \wedge e_n \implies \left|\underset{n}{\alpha}\right| == \sqrt{a^2\, g} == |a| \sqrt{g} \qquad 6.45$$

The *measure of the unit n-element* (or its negative) is, as would be expected, equal to unity.

$$\left|\overline{1}\right| == 1 \qquad \left|-\overline{1}\right| == 1 \qquad 6.46$$

The *measure of an element is equal to the measure of its complement*, since, as we have shown above, $\underset{m}{\alpha} \ominus \underset{m}{\alpha} == \overline{\underset{m}{\alpha}} \ominus \overline{\underset{m}{\alpha}}$.

$$\left|\underset{m}{\alpha}\right| == \left|\overline{\underset{m}{\alpha}}\right| \qquad 6.47$$

The *measure of a simple element* $\underset{m}{\alpha}$ is, as we have shown in the previous section:

$$\left|\underset{m}{\alpha}\right|^2 == (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) == \mathrm{Det}[\alpha_i \ominus \alpha_j] \qquad 6.48$$

In a *Euclidean metric*, the measure of an element is given by the familiar 'root sum of squares' formula yielding a measure that is always real and positive.

$$\underset{m}{\alpha} == \Sigma\, a_i\, \underset{m}{e_i} \quad \Longrightarrow \quad \left|\underset{m}{\alpha}\right| == \sqrt{\Sigma\, {a_i}^2} \qquad 6.49$$

Unit elements

The *unit m-element* associated with the element $\underset{m}{\alpha}$ is denoted $\underset{m}{\hat{\alpha}}$ and may now be defined as:

$$\underset{m}{\hat{\alpha}} == \frac{\underset{m}{\alpha}}{\left|\underset{m}{\alpha}\right|} \qquad 6.50$$

The *unit scalar* $\hat{a}$ corresponding to the scalar a, is equal to $+1$ if a is positive and is equal to -1 if a is negative.

$$\hat{a} == \frac{a}{|a|} == \mathrm{Sign}[a] \qquad 6.51$$

$$\hat{1} == 1 \qquad \widehat{-1} == -1 \qquad 6.52$$

The *unit n-element* $\underset{n}{\hat{\alpha}}$ corresponding to the *n*-element $\underset{n}{\alpha} == a\, e_1 \wedge \cdots \wedge e_n$ is equal to the unit *n*-element $\overline{1}$ if a is positive, and equal to $-\overline{1}$ if a is negative.

$$\underset{n}{\hat{\alpha}} == \frac{\underset{n}{\alpha}}{\left|\underset{n}{\alpha}\right|} == \frac{a\, e_1 \wedge \cdots \wedge e_n}{|a|\,\sqrt{g}} == \mathrm{Sign}[a]\ \overline{1} \qquad 6.53$$

$$\hat{\overline{1}} == \overline{1} \qquad \widehat{-\overline{1}} == -\overline{1} \qquad 6.54$$

The *measure of a unit m-element* $\underset{m}{\hat{\alpha}}$ is therefore always unity.

$$\left|\underset{m}{\hat{\alpha}}\right| == 1 \qquad 6.55$$

Calculating measures

To calculate the measure of any element we can use the *GrassmannAlgebra* function `Measure`.

Measure$[\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m]$

$$\sqrt{\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \ominus \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m}$$

Measure$[\alpha_1 \wedge \alpha_2]$

$$\sqrt{-(\alpha_1 \ominus \alpha_2)^2 + (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_2)}$$

`Measure` will compute the measure of a simple element using the currently declared metric. For example, with the default Euclidean metric:

Measure$[(2\,e_1 - 3\,e_2) \wedge (e_1 + e_2 + e_3) \wedge (5\,e_2 + e_3)]$

5

With a general metric:

$\mathbb{V}_3$**; DeclareMetric[g]**

$\{\{g_{1,1}, g_{1,2}, g_{1,3}\}, \{g_{1,2}, g_{2,2}, g_{2,3}\}, \{g_{1,3}, g_{2,3}, g_{3,3}\}\}$

Measure$[(2\,e_1 - 3\,e_2) \wedge (e_1 + e_2 + e_3) \wedge (5\,e_2 + e_3)]$

$$\sqrt{(50\,g_{1,2}\,g_{1,3}\,g_{2,3} + 25\,g_{1,1}\,g_{2,2}\,g_{3,3} - 25\,(g_{1,3}^2\,g_{2,2} + g_{1,1}\,g_{2,3}^2 + g_{1,2}^2\,g_{3,3}))}$$

The measure of free elements

The measure of an *m*-element has a well-defined geometric significance.

◆ m = 0

$$|a|^2 == a^2$$

The measure of a scalar is its absolute value.

Measure[a]

$$\sqrt{a^2}$$

◆ m = 1

$$|\alpha|^2 == \alpha \ominus \alpha$$

If α is interpreted as a *vector*, then $|\alpha|$ is called the *length* of α.

Measure$[\alpha]$

$$\sqrt{\alpha \ominus \alpha}$$

◆ **m = 2**

$$\left|\underset{2}{\alpha}\right|^2 == (\alpha_1 \wedge \alpha_2) \ominus (\alpha_1 \wedge \alpha_2) == \begin{vmatrix} \alpha_1 \ominus \alpha_1 & \alpha_1 \ominus \alpha_2 \\ \alpha_2 \ominus \alpha_1 & \alpha_2 \ominus \alpha_2 \end{vmatrix}$$

If $\underset{2}{\alpha}$ is interpreted as a *bivector*, then $\left|\underset{2}{\alpha}\right|$ is called the *area* of $\underset{2}{\alpha}$. $\left|\underset{2}{\alpha}\right|$ is in fact the area of the parallelogram formed by the vectors α_1 and α_2.

$$\text{Measure}[\alpha_1 \wedge \alpha_2]$$

$$\sqrt{-(\alpha_1 \ominus \alpha_2)^2 + (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_2)}$$

Graphic showing a parallelogram formed by two vectors α_1 and α_2, and its area.

Because of the nilpotent properties of the exterior product, *the measure of the bivector is independent of the way in which it is expressed in terms of its vector factors*.

$$\text{Measure}[(\alpha_1 + \alpha_2) \wedge \alpha_2]$$

$$\sqrt{-(\alpha_1 \ominus \alpha_2)^2 + (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_2)}$$

Graphic showing a parallelogram formed by vectors $(\alpha_1 + \alpha_2)$ and α_2, and its area.

◆ **m = 3**

$$\left|\underset{3}{\alpha}\right|^2 == (\alpha_1 \wedge \alpha_2 \wedge \alpha_3) \ominus (\alpha_1 \wedge \alpha_2 \wedge \alpha_3) ==$$

$$\begin{vmatrix} \alpha_1 \ominus \alpha_1 & \alpha_1 \ominus \alpha_2 & \alpha_1 \ominus \alpha_3 \\ \alpha_2 \ominus \alpha_1 & \alpha_2 \ominus \alpha_2 & \alpha_2 \ominus \alpha_3 \\ \alpha_3 \ominus \alpha_1 & \alpha_3 \ominus \alpha_2 & \alpha_3 \ominus \alpha_3 \end{vmatrix}$$

If $\underset{3}{\alpha}$ is interpreted as a *trivector*, then $\left|\underset{3}{\alpha}\right|$ is called the *volume* of $\underset{3}{\alpha}$. $\left|\underset{3}{\alpha}\right|$ is in fact the volume of the parallelepiped formed by the vectors α_1, α_2 and α_3.

$$V = \text{Measure}[\alpha_1 \wedge \alpha_2 \wedge \alpha_3]$$

$$\sqrt{}\,(-(\alpha_1 \ominus \alpha_3)^2\ (\alpha_2 \ominus \alpha_2) + 2\ (\alpha_1 \ominus \alpha_2)\ (\alpha_1 \ominus \alpha_3)\ (\alpha_2 \ominus \alpha_3) - (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_3)^2 - (\alpha_1 \ominus \alpha_2)^2\ (\alpha_3 \ominus \alpha_3) + (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_2)\ (\alpha_3 \ominus \alpha_3))$$

Note that this expression has been simplified somewhat as permitted by the symmetry of the scalar product.

Graphic showing a parallelepiped formed by vectors α_1, α_2 and α_3, and its volume.

The measure of bound elements

In this section we explore some relationships for measures of bound elements in an *n*-plane under the hybrid metric [5.33] introduced in Section 5.10. Essentially this metric has the properties that:
1. The scalar product of the origin with itself (and hence its measure) is unity.
2. The scalar product of the origin with any vector or multivector is zero.
3. The scalar product of any two vectors is given by the metric tensor on the *n*-dimensional vector subspace of the *n*-plane.

We summarize these properties as follows:

$$\mathbb{O} \ominus \mathbb{O} == 1 \qquad \underset{m}{\alpha} \ominus \mathbb{O} == 0 \qquad e_i \ominus e_j == g_{ij} \tag{6.56}$$

The measure of a point

Since a point P is defined as the sum of the origin $\mathbb{O}$ and the point's position vector ν relative to the origin, we can write:

$$P \ominus P == (\mathbb{O} + \nu) \ominus (\mathbb{O} + \nu) == \mathbb{O} \ominus \mathbb{O} + 2\, \mathbb{O} \ominus \nu + \nu \ominus \nu == 1 + \nu \ominus \nu$$

$$|P|^2 == 1 + |\nu|^2 \tag{6.57}$$

Although this is at first sight a strange result, it is due entirely to the hybrid metric discussed in Section 5.10. Unlike a vector difference of two points whose measure starts off from zero and increases continuously as the two points move further apart, the measure of a single point starts off from unity as it coincides with the origin and increases continuously as it moves further away from it.

The measure of the difference of two points is, as expected, equal to the measure of the associated vector.

$$\begin{aligned}
&(P_1 - P_2) \ominus (P_1 - P_2) \\
&== P_1 \ominus P_1 - 2\, P_1 \ominus P_2 + P_2 \ominus P_2 \\
&== (1 + \nu_1 \ominus \nu_1) - 2\, (1 + \nu_1 \ominus \nu_2) + (1 + \nu_2 \ominus \nu_2) \\
&== \nu_1 \ominus \nu_1 - 2\, \nu_1 \ominus \nu_2 + \nu_2 \ominus \nu_2 \\
&== (\nu_1 - \nu_2) \ominus (\nu_1 - \nu_2)
\end{aligned}$$

The measure of a bound multivector

Consider now the more general case of a bound *m*-vector $P \wedge \underset{m}{\alpha}$ with $P == \mathbb{O}+\nu$. Applying formula 6.99 derived in Section 6.10, we can write:

$$\left(P \wedge \underset{m}{\alpha}\right) \ominus \left(P \wedge \underset{m}{\alpha}\right) == \left(\underset{m}{\alpha} \ominus \underset{m}{\alpha}\right) (P \ominus P) - \left(\underset{m}{\alpha} \ominus P\right) \ominus \left(\underset{m}{\alpha} \ominus P\right)$$

But $\underset{m}{\alpha} \ominus P == \underset{m}{\alpha} \ominus (\mathbb{O} + \nu) == \underset{m}{\alpha} \ominus \nu$ and $P \ominus P == 1 + \nu \ominus \nu$, so that:

$$\left(P \wedge \underset{m}{\alpha}\right) \ominus \left(P \wedge \underset{m}{\alpha}\right) == \left(\underset{m}{\alpha} \ominus \underset{m}{\alpha}\right) (1 + \nu \ominus \nu) - \left(\underset{m}{\alpha} \ominus \nu\right) \ominus \left(\underset{m}{\alpha} \ominus \nu\right)$$

By applying formula 6.99 again this may be re-expressed as:

$$\left(P \wedge \underset{m}{\alpha}\right) \ominus \left(P \wedge \underset{m}{\alpha}\right) == \underset{m}{\alpha} \ominus \underset{m}{\alpha} + \left(\underset{m}{\alpha} \wedge \nu\right) \ominus \left(\underset{m}{\alpha} \wedge \nu\right) \qquad 6.58$$

$$\left|P \wedge \underset{m}{\alpha}\right|^2 == \left|\underset{m}{\alpha}\right|^2 + \left|\underset{m}{\alpha} \wedge \nu\right|^2 \qquad 6.59$$

$$\left|P \wedge \underset{m}{\hat{\alpha}}\right|^2 == 1 + \left|\underset{m}{\hat{\alpha}} \wedge \nu\right|^2 \qquad 6.60$$

Note that the measure of a multivector bound through the origin ($\nu == 0$) is just the measure of the multivector.

$$\left|\mathbb{O} \wedge \underset{m}{\alpha}\right| == \left|\underset{m}{\alpha}\right| \qquad 6.61$$

$$\left|\mathbb{O} \wedge \underset{m}{\hat{\alpha}}\right| == 1 \qquad 6.62$$

If $\nu^{\perp}$ is the component of ν orthogonal to $\underset{m}{\alpha}$, then by formula 6.99 we have:

$$\left(\underset{m}{\alpha} \wedge \nu\right) \ominus \left(\underset{m}{\alpha} \wedge \nu\right) == \left(\underset{m}{\alpha} \ominus \underset{m}{\alpha}\right) (\nu^{\perp} \ominus \nu^{\perp})$$

We can thus write formulae 6.58 and 6.59 in the alternative forms:

$$\left|P \wedge \underset{m}{\alpha}\right|^2 == \left|\underset{m}{\alpha}\right|^2 (1 + |\nu^{\perp}|^2) \qquad 6.63$$

$$\left|P \wedge \underset{m}{\hat{\alpha}}\right|^2 == 1 + |\nu^{\perp}|^2 \qquad 6.64$$

Thus the measure of a bound unit multivector indicates its minimum distance from the origin.

6.7 The Induced Metric Tensor

Calculating induced metric tensors

In the last chapter we saw how we could use the *GrassmannAlgebra* function `Metric⋀[m]` to calculate the matrix of metric tensor elements induced on $\underset{m}{\Lambda}$ by the metric tensor declared on $\underset{1}{\Lambda}$. For example, we can declare a general metric in a 3-space:

```
𝕍3; DeclareMetric[𝕘]
```

$\{\{g_{1,1}, g_{1,2}, g_{1,3}\}, \{g_{1,2}, g_{2,2}, g_{2,3}\}, \{g_{1,3}, g_{2,3}, g_{3,3}\}\}$

and then ask for the metric induced on $\underset{2}{\Lambda}$.

```
Metric⋀[2]
```

$$\{\{-g_{1,2}^2 + g_{1,1}\, g_{2,2},\ -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3},\ -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3}\},$$
$$\{-g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3},\ -g_{1,3}^2 + g_{1,1}\, g_{3,3},\ -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3}\},$$
$$\{-g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3},\ -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3},\ -g_{2,3}^2 + g_{2,2}\, g_{3,3}\}\}$$

Using scalar products to construct induced metric tensors

An alternative way of viewing and calculating the metric tensor on $\underset{m}{\Lambda}$ is to construct the matrix of inner products of the basis elements of $\underset{m}{\Lambda}$ with themselves, reduce these inner products to scalar products, and then substitute for each scalar product the corresponding metric tensor element from $\underset{1}{\Lambda}$. In what follows we will show an example of how to reproduce the result in the previous section.

First, we calculate the basis of $\underset{2}{\Lambda}$.

```
𝕍3; B = Basis⋀[2]
```

$\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$

Second, we use the *Mathematica* function `Outer` to construct the matrix of all combinations of interior products of these basis elements.

```
M1 = Outer[InteriorProduct, B, B]; MatrixForm[M1]
```

$$\begin{pmatrix} e_1 \wedge e_2 \ominus e_1 \wedge e_2 & e_1 \wedge e_2 \ominus e_1 \wedge e_3 & e_1 \wedge e_2 \ominus e_2 \wedge e_3 \\ e_1 \wedge e_3 \ominus e_1 \wedge e_2 & e_1 \wedge e_3 \ominus e_1 \wedge e_3 & e_1 \wedge e_3 \ominus e_2 \wedge e_3 \\ e_2 \wedge e_3 \ominus e_1 \wedge e_2 & e_2 \wedge e_3 \ominus e_1 \wedge e_3 & e_2 \wedge e_3 \ominus e_2 \wedge e_3 \end{pmatrix}$$

Third, we use the *GrassmannAlgebra* function `DevelopScalarProductMatrix` to convert each of these inner products into matrices of scalar products.

```
M2 = DevelopScalarProductMatrix[M1]; MatrixForm[M2]
```

$$\begin{pmatrix} \begin{pmatrix} e_1 \ominus e_1 & e_1 \ominus e_2 \\ e_2 \ominus e_1 & e_2 \ominus e_2 \end{pmatrix} & \begin{pmatrix} e_1 \ominus e_1 & e_1 \ominus e_3 \\ e_2 \ominus e_1 & e_2 \ominus e_3 \end{pmatrix} & \begin{pmatrix} e_1 \ominus e_2 & e_1 \ominus e_3 \\ e_2 \ominus e_2 & e_2 \ominus e_3 \end{pmatrix} \\ \begin{pmatrix} e_1 \ominus e_1 & e_1 \ominus e_2 \\ e_3 \ominus e_1 & e_3 \ominus e_2 \end{pmatrix} & \begin{pmatrix} e_1 \ominus e_1 & e_1 \ominus e_3 \\ e_3 \ominus e_1 & e_3 \ominus e_3 \end{pmatrix} & \begin{pmatrix} e_1 \ominus e_2 & e_1 \ominus e_3 \\ e_3 \ominus e_2 & e_3 \ominus e_3 \end{pmatrix} \\ \begin{pmatrix} e_2 \ominus e_1 & e_2 \ominus e_2 \\ e_3 \ominus e_1 & e_3 \ominus e_2 \end{pmatrix} & \begin{pmatrix} e_2 \ominus e_1 & e_2 \ominus e_3 \\ e_3 \ominus e_1 & e_3 \ominus e_3 \end{pmatrix} & \begin{pmatrix} e_2 \ominus e_2 & e_2 \ominus e_3 \\ e_3 \ominus e_2 & e_3 \ominus e_3 \end{pmatrix} \end{pmatrix}$$

Fourth, we substitute the metric element $g_{i,j}$ for $e_i \ominus e_j$ in the matrix.

```
DeclareMetric[g]; M3 = ToMetricForm[M2]; MatrixForm[M3]
```

$$\begin{pmatrix} \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{1,2} & g_{2,2} \end{pmatrix} & \begin{pmatrix} g_{1,1} & g_{1,3} \\ g_{1,2} & g_{2,3} \end{pmatrix} & \begin{pmatrix} g_{1,2} & g_{1,3} \\ g_{2,2} & g_{2,3} \end{pmatrix} \\ \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{1,3} & g_{2,3} \end{pmatrix} & \begin{pmatrix} g_{1,1} & g_{1,3} \\ g_{1,3} & g_{3,3} \end{pmatrix} & \begin{pmatrix} g_{1,2} & g_{1,3} \\ g_{2,3} & g_{3,3} \end{pmatrix} \\ \begin{pmatrix} g_{1,2} & g_{2,2} \\ g_{1,3} & g_{2,3} \end{pmatrix} & \begin{pmatrix} g_{1,2} & g_{2,3} \\ g_{1,3} & g_{3,3} \end{pmatrix} & \begin{pmatrix} g_{2,2} & g_{2,3} \\ g_{2,3} & g_{3,3} \end{pmatrix} \end{pmatrix}$$

Fifth, we compute the determinants of each of the elemental matrices.

```
M4 = M3 /. G_?(MetricElementMatrixQ[2]) → Det[G];
MatrixForm[M4]
```

$$\begin{pmatrix} -g_{1,2}^2 + g_{1,1}\, g_{2,2} & -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3} & -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3} \\ -g_{1,2}\, g_{1,3} + g_{1,1}\, g_{2,3} & -g_{1,3}^2 + g_{1,1}\, g_{3,3} & -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3} \\ -g_{1,3}\, g_{2,2} + g_{1,2}\, g_{2,3} & -g_{1,3}\, g_{2,3} + g_{1,2}\, g_{3,3} & -g_{2,3}^2 + g_{2,2}\, g_{3,3} \end{pmatrix}$$

We can have *Mathematica* check that this is the same matrix as obtained from the `MetricΛ` function.

```
M4 == MetricΛ[2]
```

```
True
```

Displaying induced metric tensors as a matrix of matrices

GrassmannAlgebra has an inbuilt function `MetricMatrix` for displaying an induced metric in matrix form. For example, we can display the induced metric on $\underset{3}{\Lambda}$ in a 4-space.

```
𝕍4; DeclareMetric[g]
```

$\{\{g_{1,1}, g_{1,2}, g_{1,3}, g_{1,4}\}, \{g_{1,2}, g_{2,2}, g_{2,3}, g_{2,4}\},$
$\{g_{1,3}, g_{2,3}, g_{3,3}, g_{3,4}\}, \{g_{1,4}, g_{2,4}, g_{3,4}, g_{4,4}\}\}$

```
MetricMatrix[3] // MatrixForm
```

$$
\begin{pmatrix}
\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{1,2} & g_{2,2} & g_{2,3} \\ g_{1,3} & g_{2,3} & g_{3,3} \end{pmatrix} &
\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,4} \\ g_{1,2} & g_{2,2} & g_{2,4} \\ g_{1,3} & g_{2,3} & g_{3,4} \end{pmatrix} &
\begin{pmatrix} g_{1,1} & g_{1,3} & g_{1,4} \\ g_{1,2} & g_{2,3} & g_{2,4} \\ g_{1,3} & g_{3,3} & g_{3,4} \end{pmatrix} &
\begin{pmatrix} g_{1,2} & g_{1,3} & g_{1,4} \\ g_{2,2} & g_{2,3} & g_{2,4} \\ g_{2,3} & g_{3,3} & g_{3,4} \end{pmatrix} \\
\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{1,2} & g_{2,2} & g_{2,3} \\ g_{1,4} & g_{2,4} & g_{3,4} \end{pmatrix} &
\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,4} \\ g_{1,2} & g_{2,2} & g_{2,4} \\ g_{1,4} & g_{2,4} & g_{4,4} \end{pmatrix} &
\begin{pmatrix} g_{1,1} & g_{1,3} & g_{1,4} \\ g_{1,2} & g_{2,3} & g_{2,4} \\ g_{1,4} & g_{3,4} & g_{4,4} \end{pmatrix} &
\begin{pmatrix} g_{1,2} & g_{1,3} & g_{1,4} \\ g_{2,2} & g_{2,3} & g_{2,4} \\ g_{2,4} & g_{3,4} & g_{4,4} \end{pmatrix} \\
\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{1,3} & g_{2,3} & g_{3,3} \\ g_{1,4} & g_{2,4} & g_{3,4} \end{pmatrix} &
\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,4} \\ g_{1,3} & g_{2,3} & g_{3,4} \\ g_{1,4} & g_{2,4} & g_{4,4} \end{pmatrix} &
\begin{pmatrix} g_{1,1} & g_{1,3} & g_{1,4} \\ g_{1,3} & g_{3,3} & g_{3,4} \\ g_{1,4} & g_{3,4} & g_{4,4} \end{pmatrix} &
\begin{pmatrix} g_{1,2} & g_{1,3} & g_{1,4} \\ g_{2,3} & g_{3,3} & g_{3,4} \\ g_{2,4} & g_{3,4} & g_{4,4} \end{pmatrix} \\
\begin{pmatrix} g_{1,2} & g_{2,2} & g_{2,3} \\ g_{1,3} & g_{2,3} & g_{3,3} \\ g_{1,4} & g_{2,4} & g_{3,4} \end{pmatrix} &
\begin{pmatrix} g_{1,2} & g_{2,2} & g_{2,4} \\ g_{1,3} & g_{2,3} & g_{3,4} \\ g_{1,4} & g_{2,4} & g_{4,4} \end{pmatrix} &
\begin{pmatrix} g_{1,2} & g_{2,3} & g_{2,4} \\ g_{1,3} & g_{3,3} & g_{3,4} \\ g_{1,4} & g_{3,4} & g_{4,4} \end{pmatrix} &
\begin{pmatrix} g_{2,2} & g_{2,3} & g_{2,4} \\ g_{2,3} & g_{3,3} & g_{3,4} \\ g_{2,4} & g_{3,4} & g_{4,4} \end{pmatrix}
\end{pmatrix}
$$

Of course, we are only *displaying* the metric tensor in this form. Its elements are correctly the *determinants* of the matrices displayed, not the matrices themselves.

6.8 Product Formulae for Interior Products

Interior product formulae for 1-elements

The fundamental interior product formula

The Interior Common Factor Theorem derived in Section 6.4 may be used to prove an important relationship involving the interior product of a 1-element with the exterior product of two factors, each of which may not be simple. This relationship and the special cases that derive from it find application throughout the explorations in the rest of this book.

Let $\mathbf{x}$ be a 1-element, then:

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus \mathbf{x} == \left(\underset{m}{\alpha} \ominus \mathbf{x}\right) \wedge \underset{k}{\beta} + (-1)^{m}\, \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \ominus \mathbf{x}\right) \qquad 6.65$$

To prove this, suppose initially that $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are simple. Then they can be expressed as $\underset{m}{\alpha} == \alpha_1 \wedge \cdots \wedge \alpha_m$ and $\underset{k}{\beta} == \beta_1 \wedge \cdots \wedge \beta_k$. Thus:

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus \mathbf{x} == ((\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \cdots \wedge \beta_k)) \ominus \mathbf{x}$$

$$== \sum_{i=1}^{m} (-1)^{i-1} (\mathbf{x} \ominus \alpha_i)\, (\alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \cdots \wedge \beta_k)$$

$$+ \sum_{j=1}^{k} (-1)^{m+j-1} (\mathbf{x} \ominus \beta_j)\, (\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge (\beta_1 \wedge \cdots \wedge \square_j \wedge \cdots \wedge \beta_k)$$

$$== \left(\underset{m}{\alpha} \ominus \mathbf{x}\right) \wedge (\beta_1 \wedge \cdots \wedge \beta_k) + (-1)^{m} (\alpha_1 \wedge \cdots \wedge \alpha_m) \wedge \left(\underset{k}{\beta} \ominus \mathbf{x}\right)$$

This formula may be extended in a straightforward manner to the case where $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are not simple: since a non-simple element may be expressed as the sum of simple terms, and the formula is valid for each term, then by addition it can be shown to be valid for the sum.

Extension of the fundamental formula to elements of higher grade

If $\mathbf{x}$ is of grade higher than 1, then similar relations hold, but with extra terms on the right-hand side. For example, if we replace $\mathbf{x}$ by $\mathbf{x_1} \wedge \mathbf{x_2}$ and note that:

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus (\mathbf{x_1} \wedge \mathbf{x_2}) == \left(\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus \mathbf{x_1}\right) \ominus \mathbf{x_2}$$

then the product rule derived above may be applied successively to obtain the following result.

$$\begin{aligned}\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus (\mathbf{x_1} \wedge \mathbf{x_2}) == &\left(\underset{m}{\alpha} \ominus (\mathbf{x_1} \wedge \mathbf{x_2})\right) \wedge \underset{k}{\beta} + \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \ominus (\mathbf{x_1} \wedge \mathbf{x_2})\right) \\ &+ (-1)^{m+1} \left(\left(\underset{m}{\alpha} \ominus \mathbf{x_1}\right) \wedge \left(\underset{k}{\beta} \ominus \mathbf{x_2}\right) - \left(\underset{m}{\alpha} \ominus \mathbf{x_2}\right) \wedge \left(\underset{k}{\beta} \ominus \mathbf{x_1}\right)\right)\end{aligned} \qquad \mathbf{6.66}$$

The complementary form of the fundamental formula

Another form of equation 6.54 which will be found useful is obtained by replacing $\underset{k}{\beta}$ by its complement and then taking the complement of the complete equation.

$$\overline{\left(\underset{m}{\alpha} \wedge \overline{\underset{k}{\beta}}\right) \ominus \mathbf{x}} == \overline{\left(\underset{m}{\alpha} \ominus \mathbf{x}\right) \wedge \overline{\underset{k}{\beta}}} + (-1)^m\, \overline{\underset{m}{\alpha} \wedge \left(\overline{\underset{k}{\beta}} \ominus \mathbf{x}\right)}$$

By converting interior products to regressive products and complements, applying the Complement Axiom [5.3], converting back to interior products, and then interchanging the roles of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ we obtain:

$$\left(\underset{m}{\alpha} \ominus \underset{k}{\beta}\right) \wedge \mathbf{x} == \left(\underset{m}{\alpha} \wedge \mathbf{x}\right) \ominus \underset{k}{\beta} + (-1)^{m-1}\, \underset{m}{\alpha} \ominus \left(\underset{k}{\beta} \ominus \mathbf{x}\right) \qquad \mathbf{6.67}$$

This formula is the basis for the derivation of a set of formulae of geometric interest in the next section: the Triangle Formulae.

The decomposition formula

By putting $\underset{k}{\beta}$ equal to $\mathbf{x}$ and rearranging terms, this formula expresses a 'decomposition' of the element $\underset{m}{\alpha}$ in terms of the 1-element $\mathbf{x}$:

$$\underset{m}{\alpha} == (-1)^m \left(\left(\underset{m}{\alpha} \wedge \hat{\mathbf{x}}\right) \ominus \hat{\mathbf{x}} - \left(\underset{m}{\alpha} \ominus \hat{\mathbf{x}}\right) \wedge \hat{\mathbf{x}}\right) \qquad \mathbf{6.68}$$

where $\hat{\mathbf{x}} == \frac{\mathbf{x}}{\sqrt{\mathbf{x} \ominus \mathbf{x}}}$. Note that $\left(\underset{m}{\alpha} \wedge \hat{\mathbf{x}}\right) \ominus \hat{\mathbf{x}}$ and $\left(\underset{m}{\alpha} \ominus \hat{\mathbf{x}}\right) \wedge \hat{\mathbf{x}}$ are orthogonal.

Interior product formulae for 1-elements from regressive product formulae

Interior Product Formula 1

We start with the regressive product formula [3.41].

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \vee \underset{n-1}{x} == \left(\underset{m}{\alpha} \vee \underset{n-1}{x}\right) \wedge \underset{k}{\beta} + (-1)^{m} \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \vee \underset{n-1}{x}\right)$$

In this formula, put $\underset{n-1}{x}$ equal to $\overline{x}$ (making x a 1-element) to get:

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \vee \overline{x} == \left(\underset{m}{\alpha} \vee \overline{x}\right) \wedge \underset{k}{\beta} + (-1)^{m} \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \vee \overline{x}\right)$$

and then transform to the interior product form.

$$\boxed{\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus x == \left(\underset{m}{\alpha} \ominus x\right) \wedge \underset{k}{\beta} + (-1)^{m} \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \ominus x\right)} \qquad 6.69$$

We have here rederived the fundamental product formula [6.64] of the last section.

Interior Product Formula 2

Now take formula 3.42 (the dual of 3.41).

$$\left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \wedge x == \left(\underset{m}{\alpha} \wedge x\right) \vee \underset{k}{\beta} + (-1)^{n-m} \underset{m}{\alpha} \vee \left(\underset{k}{\beta} \wedge x\right)$$

Here, x is a 1-element. Put $\underset{k}{\beta}$ equal to $\overline{\underset{k}{\beta}}$ and note that:

$$\underset{m}{\alpha} \vee \left(\underset{k}{\beta} \wedge x\right) == (-1)^{n-1} \underset{m}{\alpha} \vee \left(\overline{\underset{k}{\beta} \vee \overline{x}}\right)$$

$$\boxed{\left(\underset{m}{\alpha} \ominus \underset{k}{\beta}\right) \wedge x == \left(\underset{m}{\alpha} \wedge x\right) \ominus \underset{k}{\beta} + (-1)^{m+1} \underset{m}{\alpha} \ominus \left(\underset{k}{\beta} \ominus x\right)} \qquad 6.70$$

We have here rederived the product formula [6.66] of the last section.

Interior Product Formula 3

An interior product of elements can always be expressed as the interior product of their complements in reverse order.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == (-1)^{(m-k)\,(n-m)}\, \overline{\underset{k}{\beta}} \ominus \overline{\underset{m}{\alpha}}$$

When applied to equation 6.69 above, the terms on the right-hand side interchange *forms*.

$$\left(\underset{m}{\alpha} \wedge x\right) \ominus \underset{k}{\beta} == (-1)^{(m-k)\ (n-m)+n+k+1}\ \overline{\underset{k}{\beta}} \ominus \left(\overline{\underset{m}{\alpha}} \ominus x\right)$$

$$\underset{m}{\alpha} \ominus \left(\underset{k}{\beta} \ominus x\right) == (-1)^{(m-k)\ (n-m)+m+1}\ \left(\overline{\underset{k}{\beta}} \wedge x\right) \ominus \overline{\underset{m}{\alpha}}$$

Hence equation 6.69 may be expressed in a variety of ways, for example:

$$\left(\underset{m}{\alpha} \ominus \underset{k}{\beta}\right) \wedge x == \left(\underset{m}{\alpha} \wedge x\right) \ominus \underset{k}{\beta} + (-1)^{(m-k)\ (n-m)}\ \left(\overline{\underset{k}{\beta}} \wedge x\right) \ominus \overline{\underset{m}{\alpha}} \qquad 6.71$$

Interior Product Formula 4

Taking the interior product of equation 6.69 with a second 1-element x_2 gives:

$$\left(\left(\underset{m}{\alpha} \ominus \underset{k}{\beta}\right) \wedge x_1\right) \ominus x_2 ==$$
$$\left(\underset{m}{\alpha} \wedge x_1\right) \ominus \left(\underset{k}{\beta} \wedge x_2\right) + (-1)^{m+k}\ \left(\underset{m}{\alpha} \ominus x_2\right) \ominus \left(\underset{k}{\beta} \ominus x_1\right) \qquad 6.72$$

Interior product formulae for *p*-elements from regressive product formulae

Interior Product Formula 5

Consider equation 3.44.

$$\left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \wedge \underset{p}{x} == \sum_{r=0}^{p} (-1)^{r\ (n-m)} \sum_{i=1}^{\nu} \left(\underset{m}{\alpha} \wedge \underset{p-r}{x_i}\right) \vee \left(\underset{k}{\beta} \wedge \underset{r}{x_i}\right)$$

$$\underset{p}{x} == \underset{r}{x_1} \wedge \underset{p-r}{x_1} == \underset{r}{x_2} \wedge \underset{p-r}{x_2} == \cdots == \underset{r}{x_\nu} \wedge \underset{p-r}{x_\nu}, \quad \nu == \binom{p}{r}$$

Now replace $\underset{k}{\beta}$ with $\overline{\underset{n-k}{\beta}}$, $\overline{\underset{n-k}{\beta} \wedge \underset{r}{x_i}}$ with $(-1)^{r\ (n-r)}\ \overline{\underset{n-k}{\beta} \vee \overline{\underset{r}{x_i}}}$, and *n–k* with *k* to give:

$$\left(\underset{m}{\alpha} \ominus \underset{k}{\beta}\right) \wedge \underset{p}{x} == \sum_{r=0}^{p} (-1)^{r\ (m-r)} \sum_{i=1}^{\nu} \left(\underset{m}{\alpha} \wedge \underset{p-r}{x_i}\right) \ominus \left(\underset{k}{\beta} \ominus \underset{r}{x_i}\right)$$
$$\underset{p}{x} == \underset{r}{x_1} \wedge \underset{p-r}{x_1} == \underset{r}{x_2} \wedge \underset{p-r}{x_2} == \cdots == \underset{r}{x_\nu} \wedge \underset{p-r}{x_\nu}, \quad \nu == \binom{p}{r} \qquad 6.73$$

Interior Product Formula 6

By a similar process equation 3.45 becomes:

$$\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\ominus\underset{p}{x} == \sum_{r=0}^{p}(-1)^{r\,m}\sum_{i=1}^{\nu}\left(\underset{m}{\alpha}\ominus\underset{p-r}{x_i}\right)\wedge\left(\underset{k}{\beta}\ominus\underset{r}{x_i}\right)$$
$$\underset{p}{x} == \underset{r}{x_1}\wedge\underset{p-r}{x_1} == \underset{r}{x_2}\wedge\underset{p-r}{x_2} == \cdots == \underset{r}{x_\nu}\wedge\underset{p-r}{x_\nu},\quad \nu == \binom{p}{r} \qquad 6.74$$

Interior Product Formula 7

In formula 6.72 putting $\underset{k}{\beta}$ equal to $\underset{m}{\alpha}$ expresses a simple *p*-element in terms of products with a unit *m*-element.

$$\underset{p}{x} == \sum_{r=0}^{p}(-1)^{r\,(n-m)}\sum_{i=1}^{\nu}\left(\underset{m}{\hat{\alpha}}\wedge\underset{p-r}{x_i}\right)\ominus\left(\underset{m}{\hat{\alpha}}\ominus\underset{r}{x_i}\right)$$
$$\underset{p}{x} == \underset{r}{x_1}\wedge\underset{p-r}{x_1} == \underset{r}{x_2}\wedge\underset{p-r}{x_2} == \cdots == \underset{r}{x_\nu}\wedge\underset{p-r}{x_\nu},\quad \nu == \binom{p}{r} \qquad 6.75$$

6.9 The Cross Product

Defining a generalized cross product

The cross or vector product of the three-dimensional vector calculus of Gibbs *et al.* [Gibbs 1928] corresponds to two operations in Grassmann's more general calculus. *Taking the cross-product of two vectors in three dimensions corresponds to taking the complement of their exterior product*. However, whilst the usual cross product formulation is valid only for vectors in three dimensions, the exterior product formulation is valid for elements of *any* grade in *any* number of dimensions. Therefore the opportunity exists to generalize the concept.

Because our generalization reduces to the usual definition under the usual circumstances, we take the liberty of continuing to refer to the generalized cross product as, simply, the cross product.

The cross product of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is denoted $\underset{m}{\alpha}\times\underset{k}{\beta}$ and is defined as the complement of their exterior product. The cross product of an *m*-element and a *k*-element is thus an (*n*–(*m*+*k*))-element.

$$\underset{m}{\alpha}\times\underset{k}{\beta} == \overline{\underset{m}{\alpha}\wedge\underset{k}{\beta}} \qquad 6.76$$

This definition preserves the basic property of the cross product: that the cross product of two elements is an element orthogonal to both, and reduces to the usual notion for vectors in a three dimensional metric vector space.

Cross products involving 1-elements

For 1-elements x_i the definition above has the following consequences, independent of the dimension of the space.

The triple cross product

The triple cross product is a 1-element in any number of dimensions.

$$(x_1 \times x_2) \times x_3 == \overline{(\overline{x_1 \wedge x_2}) \wedge x_3} == (x_1 \wedge x_2) \ominus x_3$$

$$\boxed{(x_1 \times x_2) \times x_3 == (x_1 \wedge x_2) \ominus x_3 == (x_3 \ominus x_1)\, x_2 - (x_3 \ominus x_2)\, x_1} \qquad 6.77$$

The box product or triple vector product

The box product, or triple vector product, is an (*n*–3)-element, and therefore a scalar only in three dimensions.

$$(x_1 \times x_2) \ominus x_3 == (\overline{x_1 \wedge x_2}) \ominus x_3 == \overline{x_1 \wedge x_2 \wedge x_3}$$

$$\boxed{(x_1 \times x_2) \ominus x_3 == \overline{x_1 \wedge x_2 \wedge x_3}} \qquad 6.78$$

Because Grassmann identified *n*-elements with their scalar Euclidean complements (see the historical note in the introduction to Chapter 5), he considered $x_1 \wedge x_2 \wedge x_3$ in a 3-space to be a scalar. His notation for the exterior product of elements was to use square brackets $[x_1\ x_2\ x_3]$, thus originating the 'box' product notation for $(x_1 \times x_2) \ominus x_3$ used in the three-dimensional vector calculus in the early decades of the twentieth century.

The scalar product of two cross products

The scalar product of two cross products is a scalar in any number of dimensions.

$$(x_1 \times x_2) \ominus (x_3 \times x_4) == (\overline{x_1 \wedge x_2}) \ominus (\overline{x_3 \wedge x_4}) == (x_1 \wedge x_2) \ominus (x_3 \wedge x_4)$$

$$\boxed{(x_1 \times x_2) \ominus (x_3 \times x_4) == (x_1 \wedge x_2) \ominus (x_3 \wedge x_4)} \qquad 6.79$$

The cross product of two cross products

The cross product of two cross products is a (4–*n*)-element, and therefore a 1-element only in three dimensions. It corresponds to the regressive product of two exterior products.

$$(x_1 \times x_2) \times (x_3 \times x_4) == \overline{(\overline{x_1 \wedge x_2}) \wedge (\overline{x_3 \wedge x_4})} == (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$$

$$\boxed{(x_1 \times x_2) \times (x_3 \times x_4) == (x_1 \wedge x_2) \vee (x_3 \wedge x_4)} \qquad 6.80$$

Implications of the axioms for the cross product

By expressing one or more elements as a complement, axioms for the exterior and regressive products may be rewritten in terms of the cross product, thus yielding some of its more fundamental properties.

◆ ✕ 6: The cross product of an *m*-element and a *k*-element is an (*n*–(*m*+*k*))-element.

The grade of the cross product of an *m*-element and a *k*-element is *n*–(*m*+*k*).

$$\underset{m}{\alpha} \in \underset{m}{\Lambda},\ \underset{k}{\beta} \in \underset{k}{\Lambda} \quad \Longrightarrow \quad \underset{m}{\alpha} \times \underset{k}{\beta} \ \in \underset{n-(m+k)}{\Lambda} \qquad 6.81$$

◆ ✕ 7: The cross product is not associative.

The cross product is not associative. However, it can be expressed in terms of exterior and interior products.

$$\left(\underset{m}{\alpha} \times \underset{k}{\beta}\right) \times \underset{r}{\gamma} == (-1)^{(n-1)\,(m+k)} \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus \underset{r}{\gamma} \qquad 6.82$$

$$\underset{m}{\alpha} \times \left(\underset{k}{\beta} \times \underset{r}{\gamma}\right) == (-1)^{(n-(k+r))\,(m+k+r)} \left(\underset{k}{\beta} \wedge \underset{r}{\gamma}\right) \ominus \underset{m}{\alpha} \qquad 6.83$$

◆ ✕ 8: The cross product with unity yields the complement.

The cross product of an element with the unit scalar **1** yields its complement. Thus the complement operation may be viewed as the cross product with unity.

$$1 \times \underset{m}{\alpha} == \underset{m}{\alpha} \times 1 == \overline{\underset{m}{\alpha}} \qquad 6.84$$

The cross product of an element with a scalar yields that scalar multiple of its complement.

$$a \times \underset{m}{\alpha} == \underset{m}{\alpha} \times a == a\,\overline{\underset{m}{\alpha}} \qquad 6.85$$

◆ ✕ 9: The cross product of unity with itself is the unit *n*-element.

The cross product of **1** with itself is the unit *n*-element. The cross product of a scalar and its reciprocal is the unit *n*-element.

$$1 \times 1 == a \times \frac{1}{a} == \overline{1} \tag{6.86}$$

◆ × 10: **The cross product of two 1-elements anti-commutes.**

The cross product of two elements is equal (apart from a possible sign) to their cross product in reverse order. The cross product of two 1-elements is anti-commutative, just as is the exterior product.

$$\underset{m}{\alpha} \times \underset{k}{\beta} == (-1)^{m k} \underset{k}{\beta} \times \underset{m}{\alpha} \tag{6.87}$$

◆ × 11: **The cross product with zero is zero.**

$$0 \times \underset{m}{\alpha} == 0 == \underset{m}{\alpha} \times 0 \tag{6.88}$$

◆ × 12: **The cross product is both left and right distributive under addition.**

$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \times \underset{r}{\gamma} == \underset{m}{\alpha} \times \underset{r}{\gamma} + \underset{m}{\beta} \times \underset{r}{\gamma} \tag{6.89}$$

$$\underset{m}{\alpha} \times \left(\underset{r}{\beta} + \underset{r}{\gamma}\right) == \underset{m}{\alpha} \times \underset{r}{\beta} + \underset{m}{\alpha} \times \underset{r}{\gamma} \tag{6.90}$$

The cross product as a universal product

We have already shown that all products can be expressed in terms of the exterior product and the complement operation. Additionally, we have shown above that the complement operation may be written as the cross product with unity.

$$\overline{\underset{m}{\alpha}} == 1 \times \underset{m}{\alpha} == \underset{m}{\alpha} \times 1 \tag{6.91}$$

We can therefore write the exterior, regressive, and interior products in terms *only* of the cross product.

$$\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{(m+k)\ (n-(m+k))}\ 1 \times \left(\underset{m}{\alpha} \times \underset{k}{\beta}\right) \tag{6.92}$$

$$\underset{m}{\alpha} \vee \underset{k}{\beta} == (-1)^{m\ (n-m)+k\ (n-k)} \left(1 \times \underset{m}{\alpha}\right) \times \left(1 \times \underset{k}{\beta}\right) \tag{6.93}$$

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == (-1)^{m\ (n-m)} \left(1 \times \underset{m}{\alpha}\right) \times \underset{k}{\beta} \tag{6.94}$$

These formulae show that any result in the Grassmann algebra involving exterior, regressive or interior products, or complements, could be expressed in terms of the generalized cross product alone. This is somewhat reminiscent of the role played by the Scheffer stroke (or Pierce arrow) symbol of Boolean algebra.

Cross product formulae

The complement of a cross product

The complement of a cross product of two elements is, but for a possible sign, the exterior product of the elements.

$$\overline{\underset{m}{\alpha} \times \underset{k}{\beta}} == (-1)^{(m+k)\ (n-(m+k))}\ \underset{m}{\alpha} \wedge \underset{k}{\beta} \tag{6.95}$$

The Common Factor Axiom for cross products

The Common Factor Axiom can be written for $m+k+p = n$.

$$\left(\underset{m}{\alpha} \times \underset{p}{\gamma}\right) \times \left(\underset{k}{\beta} \times \underset{p}{\gamma}\right) == \left(\underset{p}{\gamma} \times \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right)\right) \underset{p}{\gamma} == (-1)^{p\ (n-p)} \left(\underset{p}{\gamma} \ominus \left(\underset{m}{\alpha} \times \underset{k}{\beta}\right)\right) \underset{p}{\gamma} \tag{6.96}$$

Product formulae for cross products

Of course, many of the product formulae derived previously have counterparts involving cross products. For example the complement of formula 3.41 may be written:

$$\left(\underset{m}{\alpha} \times \underset{k}{\beta}\right) \wedge \mathbf{x} == (-1)^{n-1} \left(\left(\underset{m}{\alpha} \ominus \mathbf{x}\right) \times \underset{k}{\beta} + (-1)^{m}\ \underset{m}{\alpha} \times \left(\underset{k}{\beta} \ominus \mathbf{x}\right)\right) \tag{6.97}$$

The measure of a cross product

The measure of the cross product of two elements is equal to the measure of their exterior product.

$$\left|\underset{m}{\alpha} \times \underset{k}{\beta}\right| == \left|\underset{m}{\alpha} \wedge \underset{k}{\beta}\right| \qquad 6.98$$

6.10 The Triangle Formulae

Triangle components

In this section several formulae will be developed which are generalizations of the fundamental Pythagorean relationship for the right-angled triangle and the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$. These formulae will be found useful in determining projections, shortest distances between multiplanes, and a variety of other geometric results. The starting point is the product formula [6.66] with $k = m$.

$$\left(\underset{m}{\alpha} \ominus \underset{m}{\beta}\right) \mathbf{x} == \left(\underset{m}{\alpha} \wedge \mathbf{x}\right) \ominus \underset{m}{\beta} + (-1)^{m-1} \underset{m}{\alpha} \ominus \left(\underset{m}{\beta} \ominus \mathbf{x}\right)$$

By dividing through by the inner product $\underset{m}{\alpha} \ominus \underset{m}{\beta}$, we obtain a decomposition of the 1-element $\mathbf{x}$ into two components.

$$\mathbf{x} == \frac{\left(\underset{m}{\alpha} \wedge \mathbf{x}\right) \ominus \underset{m}{\beta}}{\underset{m}{\alpha} \ominus \underset{m}{\beta}} + (-1)^{m-1} \frac{\underset{m}{\alpha} \ominus \left(\underset{m}{\beta} \ominus \mathbf{x}\right)}{\underset{m}{\alpha} \ominus \underset{m}{\beta}} \qquad 6.99$$

Now take formula 6.71:

$$\left(\left(\underset{m}{\alpha} \ominus \underset{k}{\beta}\right) \wedge \mathbf{x}_1\right) \ominus \mathbf{x}_2 == \left(\underset{m}{\alpha} \wedge \mathbf{x}_1\right) \ominus \left(\underset{k}{\beta} \wedge \mathbf{x}_2\right) + (-1)^{m+k} \left(\underset{m}{\alpha} \ominus \mathbf{x}_2\right) \ominus \left(\underset{k}{\beta} \ominus \mathbf{x}_1\right)$$

and put $k = m$, $\mathbf{x}_1$ equal to $\mathbf{x}$ and $\mathbf{x}_2$ equal to $\mathbf{z}$.

$$\left(\underset{m}{\alpha} \ominus \underset{m}{\beta}\right) (\mathbf{x} \ominus \mathbf{z}) == \left(\underset{m}{\alpha} \wedge \mathbf{x}\right) \ominus \left(\underset{m}{\beta} \wedge \mathbf{z}\right) + \left(\underset{m}{\alpha} \ominus \mathbf{z}\right) \ominus \left(\underset{m}{\beta} \ominus \mathbf{x}\right) \qquad 6.100$$

When $\underset{m}{\alpha}$ is equal to $\underset{m}{\beta}$ and $\mathbf{x}$ is equal to $\mathbf{z}$, this reduces to a relationship similar to the fundamental trigonometric identity.

$$\mathbf{x} \ominus \mathbf{x} == \left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) + \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right) \ominus \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right) \qquad 6.101$$

Or, more succinctly:

$$\left|\underset{m}{\hat{\alpha}} \wedge \hat{\mathbf{x}}\right|^2 + \left|\underset{m}{\hat{\alpha}} \ominus \hat{\mathbf{x}}\right|^2 == 1 \qquad 6.102$$

Similarly, formula 6.98 reduces to a more obvious decomposition for $\mathbf{x}$ in terms of a unit *m*-element.

$$\mathbf{x} == \left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}} + (-1)^{m-1} \underset{m}{\hat{\alpha}} \ominus \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right) \qquad 6.103$$

Because of its geometric significance when $\underset{m}{\alpha}$ is simple, the properties of this equation are worth investigating further. It will be shown that the square (scalar product with itself) of each of its terms is the corresponding term of formula 6.100. That is:

$$\left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right) \ominus \left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right) == \left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \qquad 6.104$$

$$\left(\underset{m}{\hat{\alpha}} \ominus \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right)\right) \ominus \left(\underset{m}{\hat{\alpha}} \ominus \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right)\right) == \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right) \ominus \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right) \qquad 6.105$$

Further, by taking the scalar product of formula 6.102 with itself, and using formulae 6.100, 6.103, and 6.104, yields the result that the terms on the right-hand side of formula 6.102 are orthogonal.

$$\left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right) \ominus \left(\underset{m}{\hat{\alpha}} \ominus \left(\underset{m}{\hat{\alpha}} \ominus \mathbf{x}\right)\right) == 0 \qquad 6.106$$

It is these facts that suggest the name *triangle formulae* for formulae 6.101 and 6.102.

Diagram of a vector $\mathbf{x}$ decomposed into components in and orthogonal to $\underset{m}{\alpha}$.

The measure of the triangle components

Let $\underset{m}{\hat{\alpha}} == \alpha_1 \wedge \cdots \wedge \alpha_m$, then:

$$\left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right) \ominus \left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right) == (-1)^m \left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right)\right) \ominus \underset{m}{\hat{\alpha}}$$

Focussing now on the first factor of the inner product on the right-hand side we get:

$$\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right) == (\alpha_1 \wedge \cdots \wedge \alpha_m \wedge \mathbf{x}) \ominus \left(\left(\underset{m}{\hat{\alpha}} \wedge \mathbf{x}\right) \ominus \underset{m}{\hat{\alpha}}\right)$$

Note that the second factor in the interior product of the right-hand side is of grade 1. We apply the Interior Common Factor Theorem to give:

$$(\alpha_1 \wedge \cdots \wedge \alpha_m \wedge x) \ominus \left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) == (-1)^m \left(\left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) \ominus x\right) \alpha_1 \wedge \cdots \wedge \alpha_m + \sum_i (-1)^{i-1} \left(\left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) \ominus \alpha_i\right) \alpha_1 \wedge \cdots \wedge \square_i \wedge \cdots \wedge \alpha_m \wedge x$$

But the terms in this last sum are zero since:

$$\left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) \ominus \alpha_i == \left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \left(\hat{\underset{m}{\alpha}} \wedge \alpha_i\right) == 0$$

This leaves only the first term in the expansion. Thus:

$$\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) == (-1)^m \left(\left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) \ominus x\right) \hat{\underset{m}{\alpha}} == (-1)^m \left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \left(\hat{\underset{m}{\alpha}} \wedge x\right) \hat{\underset{m}{\alpha}}$$

Since $\hat{\underset{m}{\alpha}} \ominus \hat{\underset{m}{\alpha}} == 1$, we have finally that:

$$\left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) \ominus \left(\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right) == \left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \left(\hat{\underset{m}{\alpha}} \wedge x\right)$$

The measure of $\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}$ is therefore the same as the measure of $\hat{\underset{m}{\alpha}} \wedge x$.

$$\left|\left(\hat{\underset{m}{\alpha}} \wedge x\right) \ominus \hat{\underset{m}{\alpha}}\right| == \left|\hat{\underset{m}{\alpha}} \wedge x\right| \qquad 6.107$$

Equivalent forms for the triangle components

There is an interesting relationship between the two terms $\left(\underset{m}{\alpha} \wedge x\right) \ominus \underset{m}{\alpha}$ and $\underset{m}{\alpha} \ominus \left(\underset{m}{\alpha} \ominus x\right)$ which enables formula 6.104 to be proved in an identical manner to formula 6.103. Using the identity [6.13] we find that each form may be expressed (apart from a possible sign) in the other form provided $\underset{m}{\alpha}$ is replaced by its complement $\overline{\underset{m}{\alpha}}$.

$$\left(\underset{m}{\alpha} \wedge x\right) \ominus \underset{m}{\alpha} == (-1)^{n-m-1}\, \overline{\underset{m}{\alpha}} \ominus (\overline{\underset{m}{\alpha} \wedge x}) == (-1)^{n-m-1}\, \overline{\underset{m}{\alpha}} \ominus \left(\overline{\underset{m}{\alpha}} \ominus x\right)$$

$$\left(\underset{m}{\alpha} \wedge x\right) \ominus \underset{m}{\alpha} == (-1)^{n-m-1}\, \overline{\underset{m}{\alpha}} \ominus \left(\overline{\underset{m}{\alpha}} \ominus x\right) \qquad 6.108$$

In a similar manner it may be shown that:

$$\underset{m}{\alpha} \ominus \left(\underset{m}{\alpha} \ominus x\right) == (-1)^{m-1} \left(\overline{\underset{m}{\alpha}} \wedge x\right) \ominus \overline{\underset{m}{\alpha}} \qquad 6.109$$

Since the proof of formula 6.103 is valid for any simple $\underset{m}{\hat{\alpha}}$ it is also valid for $\overline{\underset{m}{\hat{\alpha}}}$ (which is equal to $\hat{\overline{\underset{m}{\alpha}}}$) since this is also simple. The proof of formula 6.104 is then completed by applying formula 6.107.

The next three sections will look at some applications of these formulae.

6.11 Angle

Defining the angle between elements

The angle between 1-elements

The interior product enables us to define, as is standard practice, the angle between two 1-elements

$$\mathbf{Cos}[\theta]^2 == |\hat{\mathbf{x}}_1 \ominus \hat{\mathbf{x}}_2|^2 \qquad \mathbf{6.110}$$

Diagram of two vectors showing the angle between them.

However, putting $\underset{m}{\alpha}$ equal to $\mathbf{x_1}$ and $\mathbf{x}$ equal to $\mathbf{x_2}$ in formula 6.101 yields:

$$|\hat{\mathbf{x}}_1 \wedge \hat{\mathbf{x}}_2|^2 + |\hat{\mathbf{x}}_1 \ominus \hat{\mathbf{x}}_2|^2 == 1 \qquad \mathbf{6.111}$$

This, together with the definition for angle above implies that:

$$\mathbf{Sin}[\theta]^2 == |\hat{\mathbf{x}}_1 \wedge \hat{\mathbf{x}}_2|^2 \qquad \mathbf{6.112}$$

This is a formula equivalent to the three-dimensional cross product formulation:

$$\mathbf{Sin}[\theta]^2 == |\hat{\mathbf{x}}_1 \times \hat{\mathbf{x}}_2|^2$$

but one which is not restricted to three-dimensional space.

Thus formula 6.110 is an identity equivalent to $\sin^2(\theta) + \cos^2(\theta) = 1$.

The angle between a 1-element and a simple m-element

One may however carry this concept further, and show that it is meaningful to talk about the angle between a 1-element $\mathbf{x}$ and a simple *m*-element $\underset{m}{\alpha}$ for which the general formula [6.102] holds.

$$\left|\underset{m}{\hat{\alpha}} \wedge \hat{x}\right|^2 + \left|\underset{m}{\hat{\alpha}} \ominus \hat{x}\right|^2 == 1$$

$$\boxed{\mathrm{Sin}[\theta]^2 == \left|\underset{m}{\hat{\alpha}} \wedge \hat{x}\right|^2} \qquad 6.113$$

$$\boxed{\mathrm{Cos}[\theta]^2 == \left|\underset{m}{\hat{\alpha}} \ominus \hat{x}\right|^2} \qquad 6.114$$

We will explore this more fully in what follows.

The angle between a vector and a bivector

As a simple example, take the case where $\mathbf{x}$ is rewritten as $\mathbf{x_1}$ and $\underset{m}{\alpha}$ is interpreted as the bivector $\mathbf{x_2} \wedge \mathbf{x_3}$.

Diagram of three vectors showing the angles between them, and the angle between the bivector and the vector.

The angle between any two of the vectors is given by formula 6.109.

$$\mathrm{Cos}[\theta_{ij}]^2 == |\hat{x_i} \ominus \hat{x_j}|^2$$

The cosine of the angle between the vector $\mathbf{x_1}$ and the bivector $\mathbf{x_2} \wedge \mathbf{x_3}$ may be obtained from formula 6.113.

$$\left|\underset{m}{\hat{\alpha}} \ominus \hat{x}\right|^2 == \frac{((x_2 \wedge x_3) \ominus x_1) \ominus ((x_2 \wedge x_3) \ominus x_1)}{((x_2 \wedge x_3) \ominus (x_2 \wedge x_3))\ (x_1 \ominus x_1)}$$

To express the right-hand side in terms of angles, let:

$$A = \frac{((x_2 \wedge x_3) \ominus x_1) \ominus ((x_2 \wedge x_3) \ominus x_1)}{((x_2 \wedge x_3) \ominus (x_2 \wedge x_3))\ (x_1 \ominus x_1)};$$

First, convert the interior products to scalar products and then convert the scalar products to angle form given by formula 6.109. *GrassmannAlgebra* provides the function `ToAngleForm` for doing this in one operation.

`ToAngleForm[A]`

$$(\mathrm{Cos}[\theta_{1,2}]^2 + \mathrm{Cos}[\theta_{1,3}]^2 - 2\,\mathrm{Cos}[\theta_{1,2}]\,\mathrm{Cos}[\theta_{1,3}]\,\mathrm{Cos}[\theta_{2,3}])\ \mathrm{Csc}[\theta_{2,3}]^2$$

This result may be verified by elementary geometry to be $\mathbf{Cos}[\theta]^2$ where θ is defined on the diagram above. Thus we see a verification that formula 6.113 permits the calculation of the angle between a vector and an *m*-vector in terms of the angles between the given vector and any *m* vector factors of the *m*-vector.

The angle between two bivectors

Suppose we have two bivectors $\mathbf{x_1 \wedge x_2}$ and $\mathbf{x_3 \wedge x_4}$.

Diagram of three vectors showing the angles between them, and the angle between the bivector and the vector.

In a 3-space we can find the vector complements of the bivectors, and then find the angle between these vectors. This is equivalent to the cross product formulation. Let **A** be the cosine of the angle between the vectors, then:

$$\mathrm{Cos}[\theta] == \frac{(\overline{x_1 \wedge x_2}) \ominus (\overline{x_3 \wedge x_4})}{\left|\overline{x_1 \wedge x_2}\right| \left|\overline{x_3 \wedge x_4}\right|} == \frac{(x_1 \times x_2) \ominus (x_3 \times x_4)}{\left|x_1 \times x_2\right| \left|x_3 \times x_4\right|}$$

But from formulae 6.25 and 6.46 we can remove the complement operations to get the simpler expression:

$$\mathrm{Cos}[\theta] == \frac{(x_1 \wedge x_2) \ominus (x_3 \wedge x_4)}{\left|x_1 \wedge x_2\right| \left|x_3 \wedge x_4\right|} \qquad 6.115$$

Note that, in contradistinction to the 3-space formulation using cross products, this formulation is valid in a space of any number of dimensions.

An actual calculation is most readably expressed by expanding each of the terms separately. We can either represent the $\mathbf{x_i}$ in terms of basis elements or deal with them directly. A direct expansion, valid for any metric is:

ToScalarProducts[($x_1 \wedge x_2$) ⊖ ($x_3 \wedge x_4$)]

$$-(x_1 \ominus x_4)\ (x_2 \ominus x_3) + (x_1 \ominus x_3)\ (x_2 \ominus x_4)$$

Measure[$x_1 \wedge x_2$] Measure[$x_3 \wedge x_4$]

$$\sqrt{-(x_1 \ominus x_2)^2 + (x_1 \ominus x_1)\ (x_2 \ominus x_2)}\ \sqrt{-(x_3 \ominus x_4)^2 + (x_3 \ominus x_3)\ (x_4 \ominus x_4)}$$

The volume of a parallelepiped

We can calculate the volume of a parallelepiped as the measure of the trivector whose vectors make up the sides of the parallelepiped. *GrassmannAlgebra* provides the function `Measure` for expressing the volume in terms of scalar products:

V = Measure[$\alpha_1 \wedge \alpha_2 \wedge \alpha_3$]

$$\sqrt{}\,(-(\alpha_1 \ominus \alpha_3)^2\ (\alpha_2 \ominus \alpha_2) + 2\ (\alpha_1 \ominus \alpha_2)\ (\alpha_1 \ominus \alpha_3)\ (\alpha_2 \ominus \alpha_3) - (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_3)^2 - (\alpha_1 \ominus \alpha_2)^2\ (\alpha_3 \ominus \alpha_3) + (\alpha_1 \ominus \alpha_1)\ (\alpha_2 \ominus \alpha_2)\ (\alpha_3 \ominus \alpha_3))$$

Note that this has been simplified somewhat as permitted by the symmetry of the scalar product.

By putting this in its angle form we get the usual expression for the volume of a parallelepiped:

```
ToAngleForm[V]
```

$$\sqrt{(-|\alpha_1|^2\,|\alpha_2|^2\,|\alpha_3|^2\,(-1+\mathrm{Cos}[\theta_{1,2}]^2+\mathrm{Cos}[\theta_{1,3}]^2-2\,\mathrm{Cos}[\theta_{1,2}]\,\mathrm{Cos}[\theta_{1,3}]\,\mathrm{Cos}[\theta_{2,3}]+\mathrm{Cos}[\theta_{2,3}]^2))}$$

A slight rearrangement gives the volume of the parallelepiped as:

$$\boxed{\begin{gathered}V == |\alpha_1|\,|\alpha_2|\,|\alpha_3| \\ \sqrt{(1+2\,\mathrm{Cos}[\theta_{1,2}]\,\mathrm{Cos}[\theta_{1,3}]\,\mathrm{Cos}[\theta_{2,3}]-\mathrm{Cos}[\theta_{1,2}]^2-\mathrm{Cos}[\theta_{1,3}]^2-\mathrm{Cos}[\theta_{2,3}]^2)}\end{gathered}} \qquad 6.116$$

We can of course use the same approach in any number of dimensions. For example, the 'volume' of a 4-dimensional parallelepiped in terms of the lengths of its sides and the angles between them is:

```
ToAngleForm[Measure[α1 ∧ α2 ∧ α3 ∧ α4]]
```

$$\begin{aligned}&\sqrt{}(|\alpha_1|^2\,|\alpha_2|^2\,|\alpha_3|^2\,|\alpha_4|^2\,(1-\mathrm{Cos}[\theta_{2,3}]^2-\mathrm{Cos}[\theta_{2,4}]^2+\\&\quad 2\,\mathrm{Cos}[\theta_{1,3}]\,\mathrm{Cos}[\theta_{1,4}]\,\mathrm{Cos}[\theta_{3,4}]-\mathrm{Cos}[\theta_{3,4}]^2+\\&\quad 2\,\mathrm{Cos}[\theta_{2,3}]\,\mathrm{Cos}[\theta_{2,4}]\,(-\mathrm{Cos}[\theta_{1,3}]\,\mathrm{Cos}[\theta_{1,4}]+\mathrm{Cos}[\theta_{3,4}])+\\&\quad 2\,\mathrm{Cos}[\theta_{1,2}]\,(\mathrm{Cos}[\theta_{1,4}]\,(\mathrm{Cos}[\theta_{2,4}]-\mathrm{Cos}[\theta_{2,3}]\,\mathrm{Cos}[\theta_{3,4}])+\\&\qquad \mathrm{Cos}[\theta_{1,3}]\,(\mathrm{Cos}[\theta_{2,3}]-\mathrm{Cos}[\theta_{2,4}]\,\mathrm{Cos}[\theta_{3,4}]))-\\&\quad \mathrm{Cos}[\theta_{1,4}]^2\,\mathrm{Sin}[\theta_{2,3}]^2-\mathrm{Cos}[\theta_{1,3}]^2\,\mathrm{Sin}[\theta_{2,4}]^2-\\&\quad \mathrm{Cos}[\theta_{1,2}]^2\,\mathrm{Sin}[\theta_{3,4}]^2))\end{aligned}$$

6.12 Projection

To be completed.

6.13 Interior Products of Interpreted Elements

To be completed.

6.14 The Closest Approach of Multiplanes

To be completed.

6.15 Historical Notes

Grassmann's definition of the interior product

Grassmann and workers in the Grassmannian tradition define the interior product of two elements as the product of one with the complement of the other, the product being either exterior or regressive depending on which interpretation produces a non-zero result. Furthermore, when the grades of the elements are equal, it is defined either way. This definition involves the confusion between scalars and n-elements discussed in Chapter 5, Section 5.1 (equivalent to assuming a Euclidean metric and identifying scalars with pseudo-scalars). It is to obviate this inconsistency and restriction on generality that the approach adopted here bases its definition of the interior product explicitly on the *regressive* exterior product.

7 Exploring Screw Algebra

7.1 Introduction

In Chapter 8: Exploring Mechanics, we will see that systems of forces and momenta, and the velocity and infinitesimal displacement of a rigid body may be represented by the sum of a bound vector and a bivector. We have already noted in Chapter 1 that a single force is better represented by a bound vector than by a (free) vector. Systems of forces are then better represented by sums of bound vectors; and a sum of bound vectors may always be reduced to the sum of a single bound vector and a single bivector.

We call the sum of a bound vector and a bivector a 2-*entity*. These geometric entities are therefore worth exploring for their ability to represent the principal physical entities of mechanics.

In this chapter we begin by establishing some properties of 2-entities in an *n*-plane, and then show how, in a 3-plane, they take on a particularly symmetrical and potent form. This form, a 2-entity in a 3-plane, is called a *screw*. Since it is in the 3-plane that we wish to explore 3-dimensional mechanics we then explore the properties of screws in more detail.

The objective of this chapter then, is to lay the algebraic and geometric foundations for the chapter on mechanics to follow.

Historical Note

The classic text on screws and their application to mechanics is by Sir Robert Stawell Ball: *A Treatise on the Theory of Screws* (1900). Ball was aware of Grassmann's work as he explains in the Biographical Notes to this text in a comment on the *Ausdehnungslehre* of 1862.

> This remarkable work, a development of an earlier volume (1844), by the same author, contains much that is of instruction and interest in connection with the present theory.
>
> ... Here we have a very general theory, which includes screw coordinates as a particular case.

The principal proponent of screw theory from a Grassmannian viewpoint was Edward Wyllys Hyde. In 1888 he wrote a paper entitled 'The Directional Theory of Screws' on which Ball comments, again in the Biographical Notes to *A Treatise on the Theory of Screws*.

> The author writes: "I shall define a screw to be the sum of a point-vector and a plane-vector perpendicular to it, the former being a directed and posited line, the latter the product of two vectors, hence a directed but not posited plane." Prof. Hyde proves by his [sic] calculus many of the fundamental theorems in the present theory in a very concise manner.

7.2 A Canonical Form for a 2-Entity

The canonical form

The most general 2-entity in a bound vector space may always be written as the sum of a bound vector and a bivector.

$$\mathbf{S} == \mathbf{P} \wedge \alpha + \beta$$

Here, $\mathbf{P}$ is a point, α a vector and β a bivector. Remember that only in vector 1, 2 and 3-spaces are bivectors necessarily simple. In what follows we will show that in a metric space the point $\mathbf{P}$ may always be chosen in such a way that *the bivector β is orthogonal to the vector α*. This property, when specialised to three-dimensional space, is important for the theory of screws to be developed in the rest of the chapter. We show it as follows:

To the above equation, add and subtract the bound vector $\mathbf{P}^* \wedge \alpha$ such that $(\mathbf{P} - \mathbf{P}^*) \ominus \alpha == 0$, giving:

$$\mathbf{S} == \mathbf{P}^* \wedge \alpha + \beta^*$$

$$\beta^* == \beta + (P - P^*) \wedge \alpha$$

We want to choose P^* such that β^* is orthogonal to α, that is:

$$(\beta + (P - P^*) \wedge \alpha) \ominus \alpha == 0$$

Expanding the left-hand side of this equation gives:

$$\beta \ominus \alpha + (\alpha \ominus (P - P^*))\, \alpha - (\alpha \ominus \alpha)\, (P - P^*) == 0$$

But from our first condition on the choice of P^* (that is, $(P - P^*) \ominus \alpha == 0$) we see that the second term is zero giving:

$$P^* == P - \frac{\beta \ominus \alpha}{\alpha \ominus \alpha}$$

whence β^* may be expressed as:

$$\beta^* == \beta + \left(\frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha$$

Finally, substituting for P^* and β^* in the expression for S above gives the *canonical form* for the 2-entity S, in which the new bivector component is now orthogonal to the vector α. The bound vector component defines a line called the *central axis* of S.

$$\boxed{S == \left(P - \frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha + \left(\beta + \left(\frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha\right)} \quad 7.1$$

Canonical forms in an *n*-plane

Canonical forms in $\mathbb{P}_1$

In a bound vector space of one dimension, that is a 1-plane, there are points, vectors, and bound vectors. There are no bivectors. Hence every 2-entity is of the form $S == P \wedge \alpha$ and is therefore in some sense already in its canonical form.

Canonical forms in $\mathbb{P}_2$

In a bound vector space of two dimensions (the plane), every bound element can be expressed as a bound vector.

To see this we note that any bivector is simple and can be expressed using the vector of the bound vector as one of its factors. For example, if P is a point and α, β_1, β_2, are vectors, any bound element in the plane can be expressed in the form:

$$P \wedge \alpha + \beta_1 \wedge \beta_2$$

But because any bivector in the plane can be expressed as a scalar factor times any other bivector, we can also write:

$$\beta_1 \wedge \beta_2 == \kappa \wedge \alpha$$

so that the bound element may now be written as the bound vector:

$$\mathbf{P} \wedge \alpha + \beta_1 \wedge \beta_2 == \mathbf{P} \wedge \alpha + \kappa \wedge \alpha == (\mathbf{P} + \kappa) \wedge \alpha == \mathbf{P}^* \wedge \alpha$$

We can use *GrassmannAlgebra* to verify that formula 7.1 gives this result in a 2-space. First we declare the 2-space, and write α and β in terms of basis elements:

$$\mathbb{P}_2;\ \alpha = a\, e_1 + b\, e_2;\ \beta = c\, e_1 \wedge e_2;$$

For the canonical expression [7.1] above we wish to show that the bivector term is zero. We do this by converting the interior products to scalar products using `ToScalarProducts`.

$$\mathtt{Simplify}\left[\mathtt{ToScalarProducts}\left[\beta + \left(\frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha\right]\right]$$

0

In sum: A bound 2-element in the plane, $\mathbf{P} \wedge \alpha + \beta$, may always be expressed as a bound vector:

$$\mathbf{S} == \mathbf{P} \wedge \alpha + \beta == \left(\mathbf{P} - \frac{\beta \ominus \alpha}{\alpha \ominus \alpha}\right) \wedge \alpha \qquad 7.2$$

Canonical forms in $\mathbb{P}_3$

In a bound vector space of three dimensions, that is a 3-plane, every 2-element can be expressed as the sum of a bound vector and a bivector orthogonal to the vector of the bound vector. (The bivector is necessarily simple, because all bivectors in a 3-space are simple.)

Such canonical forms are called *screws*, and will be discussed in more detail in the sections to follow.

Creating 2-entities

A 2-entity may be created by applying the *GrassmannAlgebra* function `CreateElement`. For example in 3-dimensional space we create a 2-entity based on the symbol **s** by entering:

$$\mathbb{P}_3;\ \mathbf{S} = \mathtt{CreateElement}\left[\underset{2}{\mathbf{s}}\right]$$

$$s_1\, \mathbb{O} \wedge e_1 + s_2\, \mathbb{O} \wedge e_2 + s_3\, \mathbb{O} \wedge e_3 + s_4\, e_1 \wedge e_2 + s_5\, e_1 \wedge e_3 + s_6\, e_2 \wedge e_3$$

Note that `CreateElement` automatically declares the generated symbols s_i as scalars by adding the pattern **s_**. We can confirm this by entering `Scalars`.

```
Scalars
```

$$\{a, b, c, d, e, f, g, h, \Bbbk, (\_\ominus\_)\ ?\,\mathtt{InnerProductQ}, s\_, \underset{0}{\_}\}$$

To explicitly factor out the origin and express the 2-element in the form $\mathbf{S} == \mathbf{P} \wedge \alpha + \beta$, we can use the *GrassmannAlgebra* function `GrassmannSimplify`.

$\mathcal{G}[\mathbf{S}]$

$\mathbb{O} \wedge (e_1\, s_1 + e_2\, s_2 + e_3\, s_3) + s_4\, e_1 \wedge e_2 + s_5\, e_1 \wedge e_3 + s_6\, e_2 \wedge e_3$

7.3 The Complement of 2-Entity

Complements in an *n*-plane

In this section an expression will be developed for the complement of a 2-entity in an *n*-plane with a metric. In an *n*-plane, the complement of the sum of a bound vector and a bivector is the sum of a bound (*n*-2)-vector and an (*n*-1)-vector. It is of pivotal consequence for the theory of mechanics that such a complement in a 3-plane (the usual three-dimensional space) is itself the sum of a bound vector and a bivector. Geometrically, a quantity and its complement have the same measure but are orthogonal. In addition, for a 2-element (such as the sum of a bound vector and a bivector) in a 3-plane, the complement of the complement of an entity is the entity itself. These results will find application throughout Chapter 8: Exploring Mechanics.

The metric we choose to explore is the hybrid metric $\mathbf{G_{ij}}$ defined in [5.33], in which the origin is orthogonal to all vectors, but otherwise the vector space metric is arbitrary.

The complement referred to the origin

Consider the general sum of a bound vector and a bivector expressed in its simplest form referred to the origin. "Referring" an element to the origin means expressing its bound component as bound through the origin, rather than through some other more general point.

$$\mathbf{X} == \mathbb{O} \wedge \alpha + \beta$$

The complement of $\mathbf{X}$ is then:

$$\overline{\mathbf{X}} == \overline{\mathbb{O} \wedge \alpha} + \overline{\beta}$$

which, by formulae 5.41 and 5.43 derived in Chapter 5, gives:

$$\overline{\mathbf{X}} == \mathbb{O} \wedge \vec{\beta} + \vec{\alpha}$$

$$\boxed{\mathbf{X} == \mathbb{O} \wedge \alpha + \beta \quad \Longleftrightarrow \quad \overline{\mathbf{X}} == \mathbb{O} \wedge \vec{\beta} + \vec{\alpha}} \qquad 7.3$$

It may be seen that the effect of taking the complement of $\mathbf{X}$ when it is in a form referred to the origin is equivalent to interchanging the vector α and the bivector β whilst taking their vector space complements. This means that the vector that was bound through the origin becomes a free (*n*–1)-vector, whilst the bivector that was free becomes an (*n*–2)-vector bound through the origin.

The complement referred to a general point

Let $\mathbf{X} == \mathbb{O}\wedge\alpha + \beta$ and $\mathbf{P} == \mathbb{O} + \nu$. We can refer $\mathbf{X}$ to the point $\mathbf{P}$ by adding and subtracting $\nu\wedge\alpha$ to and from the above expression for $\mathbf{X}$.

$$\mathbf{X} == \mathbb{O}\wedge\alpha + \nu\wedge\alpha + \beta - \nu\wedge\alpha == (\mathbb{O} + \nu)\wedge\alpha + (\beta - \nu\wedge\alpha)$$

Or, equivalently:

$$\mathbf{X} == \mathbf{P}\wedge\alpha + \beta_{\mathbf{p}} \qquad\qquad \beta_{\mathbf{p}} == \beta - \nu\wedge\alpha$$

By manipulating the complement $\overline{\mathbf{P}\wedge\alpha}$, we can write it in the alternative form $-\overline{\alpha}\ominus\mathbf{P}$.

$$\overline{\mathbf{P}\wedge\alpha} == -\overline{\alpha\wedge\mathbf{P}} == -\overline{\alpha}\vee\overline{\mathbf{P}} == -\overline{\alpha}\ominus\mathbf{P}$$

Further, from formula 5.41, we have that:

$$-\overline{\alpha}\ominus\mathbf{P} == (\mathbb{O}\wedge\vec{\alpha})\ominus\mathbf{P}$$

$$\overline{\beta_{\mathbf{p}}} = \mathbb{O}\wedge\overrightarrow{\beta_{\mathbf{p}}}$$

So that the relationship between $\mathbf{X}$ and its complement $\overline{\mathbf{X}}$, can finally be written:

$$\boxed{\mathbf{X} == \mathbf{P}\wedge\alpha + \beta_{\mathbf{p}} \quad\Longleftrightarrow\quad \overline{\mathbf{X}} == \mathbb{O}\wedge\overrightarrow{\beta_{\mathbf{p}}} + (\mathbb{O}\wedge\vec{\alpha})\ominus\mathbf{P}} \qquad \mathbf{7.4}$$

Remember, this formula is valid for the hybrid metric [5.33] in an *n*-plane of arbitrary dimension. We explore its application to 3-planes in the section below.

7.4 The Screw

The definition of a screw

A screw is the *canonical form of a 2-entity in a three-plane*, and may always be written in the form:

$$\boxed{\mathbf{S} == \mathbf{P}\wedge\alpha + \mathbf{s}\,\vec{\alpha}} \qquad \mathbf{7.5}$$

where:

α	is the *vector* of the screw
$\mathbf{P}\wedge\alpha$	is the *central axis* of the screw
$\mathbf{s}$	is the *pitch* of the screw
$\vec{\alpha}$	is the *bivector* of the screw

Remember that $\vec{\alpha}$ is the (free) complement of the vector α in the three-plane, and hence is a simple bivector.

The unit screw

Let the scalar $\mathbf{a}$ denote the magnitude of the vector α of the screw, so that $\alpha == \mathbf{a}\,\hat{\alpha}$. A unit screw $\hat{\mathbf{S}}$ may then be defined by $\mathbf{S} == \mathbf{a}\,\hat{\mathbf{S}}$, and written as:

$$\hat{\mathbf{S}} = \mathbf{P} \wedge \hat{\alpha} + \mathbf{s}\,\vec{\hat{\alpha}} \qquad \mathbf{7.6}$$

Note that the unit screw does not have unit magnitude. The magnitude of a screw will be discussed in Section 7.5.

The pitch of a screw

An explicit formula for the pitch $\mathbf{s}$ of a screw is obtained by taking the exterior product of the screw expression with its vector α.

$$\mathbf{S} \wedge \alpha == (\mathbf{P} \wedge \alpha + \mathbf{s}\,\vec{\alpha}) \wedge \alpha$$

The first term in the expansion of the right hand side is zero, leaving:

$$\mathbf{S} \wedge \alpha == \mathbf{s}\,\vec{\alpha} \wedge \alpha$$

Further, by taking the free complement of this expression and invoking the definition of the interior product we get:

$$\overrightarrow{\mathbf{S} \wedge \alpha} == \mathbf{s}\,\overrightarrow{\vec{\alpha} \wedge \alpha} == \mathbf{s}\,\alpha \vee \vec{\alpha} == \mathbf{s}\,\alpha \ominus \alpha$$

Dividing through by the square of the magnitude of α gives:

$$\overrightarrow{\hat{\mathbf{S}} \wedge \hat{\alpha}} == \mathbf{s}\,\hat{\alpha} \ominus \hat{\alpha} == \mathbf{s}$$

In sum: We can obtain the pitch of a screw from the any of the following formulae:

$$\mathbf{s} == \frac{\mathbf{S} \wedge \alpha}{\vec{\alpha} \wedge \alpha} == \frac{\overrightarrow{\mathbf{S} \wedge \alpha}}{\alpha \ominus \alpha} == \overrightarrow{\hat{\mathbf{S}} \wedge \hat{\alpha}} \qquad \mathbf{7.7}$$

The central axis of a screw

An explicit formula for the central axis $\mathbf{P} \wedge \alpha$ of a screw is obtained by taking the interior product of the screw expression with its vector α.

$$\mathbf{S} \ominus \alpha == (\mathbf{P} \wedge \alpha + \mathbf{s}\,\vec{\alpha}) \ominus \alpha$$

The second term in the expansion of the right-hand side is zero (since an element is always orthogonal to its complement), leaving:

$$\mathbf{S}\ominus\alpha == (\mathbf{P}\wedge\alpha)\ominus\alpha$$

The right-hand side of this equation may be expanded using the Interior Common Factor Theorem to give:

$$\mathbf{S}\ominus\alpha == (\alpha\ominus\mathbf{P})\ \alpha - (\alpha\ominus\alpha)\ \mathbf{P}$$

By taking the exterior product of this expression with $\boldsymbol{\alpha}$, we eliminate the first term on the right-hand side to get:

$$(\mathbf{S}\ominus\alpha)\wedge\alpha == -(\alpha\ominus\alpha)\ (\mathbf{P}\wedge\alpha)$$

By dividing through by the square of the magnitude of $\boldsymbol{\alpha}$, we can express this in terms of unit elements.

$$(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} == -(\mathbf{P}\wedge\alpha)$$

In sum: We can obtain the central axis $\mathbf{P}\wedge\boldsymbol{\alpha}$ of a screw from any of the following formulae:

$$\boxed{\mathbf{P}\wedge\alpha == -\frac{(\mathbf{S}\ominus\alpha)\wedge\alpha}{\alpha\ominus\alpha} == -(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} == \hat{\alpha}\wedge(\mathbf{S}\ominus\hat{\alpha})} \qquad \mathbf{7.8}$$

Orthogonal decomposition of a screw

By taking the expression for a screw and substituting the expressions derived in [7.7] for the pitch and [7.8] for the central axis we obtain:

$$\mathbf{S} == -(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} + \frac{\overrightarrow{\mathbf{S}\wedge\alpha}}{\alpha\ominus\alpha}\ \vec{\alpha}$$

In order to transform the second term into the form we want, we note first that $\overrightarrow{\mathbf{S}\wedge\alpha}$ is a scalar. So that we can write:

$$\overrightarrow{\mathbf{S}\wedge\alpha}\ \vec{\alpha} == \overrightarrow{\mathbf{S}\wedge\alpha}\wedge\vec{\alpha} == \overrightarrow{\overrightarrow{\mathbf{S}\wedge\alpha}\wedge\alpha} == (\mathbf{S}\wedge\alpha)\ominus\alpha$$

Hence $\mathbf{S}$ can be written as:

$$\boxed{\mathbf{S} == -(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha} + (\mathbf{S}\wedge\hat{\alpha})\ominus\hat{\alpha}} \qquad \mathbf{7.9}$$

This type of decomposition is in fact valid for any *m*-element $\mathbf{S}$ and 1-element $\boldsymbol{\alpha}$, as we have shown in Chapter 6, formula 6.68. The central axis is the term $-(\mathbf{S}\ominus\hat{\alpha})\wedge\hat{\alpha}$ which is the component of $\mathbf{S}$ parallel to $\boldsymbol{\alpha}$, and the term $(\mathbf{S}\wedge\hat{\alpha})\ominus\hat{\alpha}$ is $\mathbf{s}\ \vec{\alpha}$, the component of $\mathbf{S}$ orthogonal to $\boldsymbol{\alpha}$.

7.5 The Algebra of Screws

To be completed

7.6 Computing with Screws

To be completed

8 Exploring Mechanics

8.1 Introduction

Grassmann algebra applied to the field of mechanics performs an interesting synthesis between what now seem to be regarded as disparate concepts. In particular we will explore the synthesis it can effect between the concepts of force and moment; velocity and angular velocity; and linear momentum and angular momentum.

This synthesis has an important concomitant, namely that the form in which the mechanical entities are represented, is for many results of interest, *independent of the dimension of the space involved*. It may be argued therefore, that such a representation is more fundamental than one specifically requiring a three-dimensional context (as indeed does that which uses the Gibbs-Heaviside vector algebra).

This is a more concrete result than may be apparent at first sight since the form, as well as being valid for spaces of dimension three or greater, is also valid for spaces of dimension zero, one or two. Of most interest, however, is the fact that the complementary form of a mechanical entity takes on a different form depending on the number of dimensions concerned. For example, the velocity of a rigid body is the sum of a bound vector and a bivector in a space of any dimensions. Its complement is dependent on the dimension of the space, but in each case it may be viewed as representing the points of the body (if they exist) which have *zero velocity*. In three dimensions the complementary velocity can specify the *axis of rotation* of the rigid body, while in two dimensions it can specify the *centre* (*point*) of rotation.

Furthermore, some results in mechanics (for example, Newton's second law or the conditions of equilibrium of a rigid body) will be shown *not to require the space to be a metric space*. On the other hand, use of the Gibbs-Heaviside 'cross product' to express angular momentum or the moment condition immediately supposes the space to possess a metric.

Mechanics as it is known today is in the strange situation of being a field of mathematical physics in which *location* is very important, and of which the calculus traditionally used (being vectorial) can take no proper account. As already discussed in Chapter 1, one may take the example of the concept of *force*. A (physical) force is not satisfactorily represented by a vector, yet contemporary practice is still to use a vector calculus for this task. To patch up this inadequacy the concept of moment is introduced and the conditions of equilibrium of a rigid body augmented by a condition on the sum of the moments. The justification for this condition is often not well treated in contemporary texts.

Many of these texts will of course rightly state that forces are not (represented by) free vectors and yet will proceed to use the Gibbs-Heaviside calculus to denote and manipulate them. Although confusing, this inaptitude is usually offset by various comments attached to the symbolic descriptions and calculations. For example, a position is represented by a 'position vector'. A position vector is described as a (free?) vector with its tail (fixed?) to the origin (point?) of the coordinate system. The coordinate system itself is supposed to consist of an origin *point* and a number of basis vectors. But whereas the vector calculus can cope with the vectors, it cannot cope with the origin. This confusion between vectors, free vectors, bound vectors, sliding vectors and position vectors would not occur if the calculus used to describe them were a calculus of *position* (points) as well as *direction* (vectors).

In order to describe the phenomena of mechanics in purely vectorial terms it has been necessary therefore to devise the notions of couple and moment as notions almost distinct from that of force, thus effectively splitting in two all the results of mechanics. In traditional mechanics, to every 'linear' quantity: force, linear momentum, translational velocity, etc., corresponds an 'angular' quantity: moment, angular momentum, angular velocity.

In this chapter we will show that by representing mechanical quantities correctly in terms of elements of a Grassmann algebra this dichotomy disappears and mechanics takes on a more unified form. In particular we will show that there exists a screw $\mathbb{F}$ representing (*including* moments) a system of forces (remember that a system of forces in three dimensions is not necessarily replaceable by a single force); a screw $\mathbb{L}$ representing the momentum of a system of bodies (linear *and* angular), and a screw $\mathbb{V}$ representing the velocity of a rigid body (linear and angular): all invariant combinations of the linear and angular components. For example, the velocity $\mathbb{V}$ is a complete characterization of the kinematics of the rigid body independent of any particular point on the body used to specify its motion. Expressions for work, power and kinetic energy of a system of forces and a rigid body will be shown to be determinable by an interior product between the relevant screws. For example, the interior product of $\mathbb{F}$ with $\mathbb{V}$ will give the power of the system of forces acting on the rigid body, that is, the *sum* of the translational and rotational powers.

Historical Note

The application of the *Ausdehnungslehre* to mechanics has obtained far fewer proponents than might be expected. This may be due to the fact that Grassmann himself was late going into the subject. His '*Die Mechanik nach den principien der Ausdehnungslehre*' (1877) was written just a few weeks before his death. Furthermore, in the period before his ideas became completely submerged by the popularity of the Gibbs-Heaviside system (around the turn of the century) there were few people with sufficient understanding of the *Ausdehnungslehre* to break new ground using its methods.

There are only three people who have written substantially in English using the original concepts of the *Ausdehnungslehre*: Edward Wyllys Hyde (1888), Alfred North Whitehead (1898), and Henry James Forder (1941). Each of them has discussed the theory of screws in more or less detail, but none has addressed the complete field of mechanics. The principal works in other languages are in German, and apart from Grassmann's paper in 1877 mentioned above, are the book by Jahnke (1905) which includes applications to mechanics, and the short monograph by Lotze (1922) which lays particular emphasis on rigid body mechanics.

8.2 Force

Representing force

The notion of force as used in mechanics involves the concepts of magnitude, sense, and line of action. It will be readily seen in what follows that such a physical entity may be faithfully represented by a *bound vector*. Similarly, all the usual properties of systems of forces are faithfully represented by the analogous properties of sums of bound vectors. For the moment it is not important whether the space has a metric, or what its dimension is.

Let $\mathbb{F}$ denote a force. Then:

$$\mathbb{F} == \mathbf{P} \wedge \mathbf{f} \tag{8.1}$$

The vector $\mathbf{f}$ is called the *force vector* and expresses the sense and direction of the force. In a metric space it would also express its magnitude.

The point $\mathbf{P}$ is any point on the line of action of the force. It can be expressed as the sum of the origin point $\mathbb{O}$ and the position vector $\mathbf{v}$ of the point.

This simple representation delineates clearly the role of the (free) vector $\mathbf{f}$. The vector $\mathbf{f}$ represents all the properties of the force *except* the position of its line of action. The operation $\mathbf{P}\wedge$ has the effect of 'binding' the force vector to the line through $\mathbf{P}$.

On the other hand the expression of a bound vector as a product of points is also useful when expressing certain types of forces, for example gravitational forces. Newton's law of gravitation for the force exerted on a *point* mass $\mathbf{m_1}$ (weighted point) by a *point* mass $\mathbf{m_2}$ may be written:

$$\mathbb{F}_{12} == \frac{G\, m_1 \wedge m_2}{R^2} \tag{8.2}$$

Note that this expression correctly changes sign if the masses are interchanged.

Systems of forces

A *system of forces* may be represented by a *sum of bound vectors*.

$$\mathbb{F} == \sum \mathbb{F}_i == \sum \mathbf{P}_i \wedge \mathbf{f}_i \tag{8.3}$$

A sum of bound vectors is not necessarily reducible to a bound vector: a system of forces is not necessarily reducible to a single force. However, a sum of bound vectors is in general reducible to the sum of a bound vector and a bivector. This is done simply by adding and subtracting the term $\mathbf{P} \wedge (\sum \mathbf{f}_i)$ to and from the sum [8.3].

$$\mathbb{F} == \mathbf{P} \wedge \left(\sum \mathbf{f_i}\right) + \sum (\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i} \qquad \mathbf{8.4}$$

This 'adding and subtracting' operation is a common one in our treatment of mechanics. We call it *referring the sum to the point* $\mathbf{P}$.

Note that although the expression for $\mathbb{F}$ now involves the point $\mathbf{P}$, it is completely independent of $\mathbf{P}$ since the terms involving $\mathbf{P}$ cancel. The sum may be said to be *invariant* with respect to any point used to express it.

Examining the terms in the expression 8.4 above we can see that since $\sum \mathbf{f_i}$ is a vector ($\mathbf{f}$, say) the first term reduces to the bound vector $\mathbf{P}\wedge\mathbf{f}$ representing a force through the chosen point $\mathbf{P}$. The vector $\mathbf{f}$ is called the *resultant force vector*.

$$\mathbf{f} == \sum \mathbf{f_i}$$

The second term is a sum of bivectors of the form $(\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i}$. Such a bivector may be seen to faithfully represent a *moment*: specifically, *the moment of the force represented by* $\mathbf{P_i} \wedge \mathbf{f_i}$ *about the point* $\mathbf{P}$. Let this moment be denoted $\mathbb{G}_{\mathbf{iP}}$.

$$\mathbb{G}_{\mathbf{iP}} == (\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i}$$

To see that it is more reasonable that a moment be represented as a bivector rather than as a vector, one has only to consider that the physical dimensions of a moment are the *product* of a length and a force unit.

The expression for moment above clarifies distinctly how it can arise from two located entities: the bound vector $\mathbf{P_i} \wedge \mathbf{f_i}$ and the point $\mathbf{P}$, and yet itself be a 'free' entity. The bivector, it will be remembered, has no concept of location associated with it. This has certainly been a source of confusion among students of mechanics using the usual Gibbs-Heaviside three-dimensional vector calculus. It is well known that the moment of a force about a point does not possess the property of location, *and yet it still depends on that point*. While notions of 'free' and 'bound' are not properly mathematically characterized, this type confusion is likely to persist.

The second term in [8.4] representing the sum of the moments of the forces about the point $\mathbf{P}$ will be denoted:

$$\mathbb{G}_{\mathbf{P}} == \sum (\mathbf{P_i} - \mathbf{P}) \wedge \mathbf{f_i}$$

Then, *any system of forces may be represented by the sum of a bound vector and a bivector*.

$$\mathbb{F} == \mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} \qquad \mathbf{8.5}$$

The bound vector $\mathbf{P}\wedge\mathbf{f}$ represents a force through an arbitrary point $\mathbf{P}$ with force vector equal to the sum of the force vectors of the system.

The bivector $\mathbb{G}_{\mathbf{P}}$ represent the sum of the moments of the forces about the same point $\mathbf{P}$.

Suppose the system of forces be referred to some other point $\mathbf{P}^*$. The system of forces may then be written in either of the two forms:

$$\mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} == \mathbf{P}^* \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}^*}$$

Solving for $\mathbb{G}_{P^*}$ gives us a formula relating the moment sum about different points.

$$\mathbb{G}_{P^*} == \mathbb{G}_P + (P - P^*) \wedge f$$

If the bound vector term $P \wedge f$ in formula 8.5 is zero then the system of forces is called a *couple*. If the bivector term $\mathbb{G}_P$ is zero then the system of forces *reduces to a single force*.

Equilibrium

If a sum of forces is zero, that is:

$$\mathbb{F} == P \wedge f + \mathbb{G}_P == 0$$

then it is straightforward to show that each of $P \wedge f$ and $\mathbb{G}_P$ must be zero. Indeed if such were not the case then the bound vector would be equal to the negative of the bivector, a possibility excluded by the implicit presence of the origin in a bound vector, and its absence in a bivector. Furthermore $P \wedge f$ being zero implies f is zero.

These considerations lead us directly to the basic theorem of statics. *For a body to be in equilibrium, the sum of the forces must be zero*.

$$\boxed{\mathbb{F} == \sum \mathbb{F}_i == 0} \qquad 8.6$$

This one expression encapsulates *both* the usual conditions for equilibrium of a body, that is: that the sum of the force vectors be zero; and that the sum of the moments of the forces about an arbitrary point P be zero.

Force in a metric 3-plane

In a 3-plane the bivector $\mathbb{G}_P$ is necessarily simple. In a metric 3-plane, the vector-space complement of the simple bivector $\mathbb{G}_P$ is just the usual moment *vector* of the three-dimensional Gibbs-Heaviside vector calculus.

In three dimensions, the complement of an exterior product is equivalent to the usual cross product.

$$\overrightarrow{\mathbb{G}_P} == \sum (P_i - P) \times f_i$$

Thus, a system of forces in a metric 3-plane may be reduced to a single force through an arbitrarily chosen point P *plus* the *vector-space complement* of the usual moment vector about P.

$$\boxed{\mathbb{F} == P \wedge f + \overline{\overrightarrow{\sum (P_i - P) \times f_i}}} \qquad 8.7$$

8.3 Momentum

The velocity of a particle

Suppose a point $\mathbf{P}$ with position vector ν, that is $\mathbf{P} == \mathbb{O} + \nu$.

Since the origin is fixed, the velocity $\overset{\circ}{\mathbf{P}}$ of the point $\mathbf{P}$ is clearly a vector given by the time-derivative of the position vector of $\mathbf{P}$.

$$\overset{\circ}{\mathbf{P}} == \frac{d\,\mathbf{P}}{d\,t} == \frac{d\,\nu}{d\,t} == \overset{\circ}{\nu} \qquad 8.8$$

Representing momentum

As may be suspected from Newton's second law, momentum is of the same tensorial nature as force. The momentum of a particle is represented by a bound vector as is a force. The momentum of a system of particles (or of a rigid body, or of a system of rigid bodies) is representable by a sum of bound vectors as is a system of forces.

The momentum of a particle comprises three factors: the *mass*, the *position*, and the *velocity* of the particle. The mass is represented by a scalar, the position by a point, and the velocity by a vector.

$$\mathbb{L} == \mathbf{P} \wedge \left(m\,\overset{\circ}{\mathbf{P}}\right) \qquad 8.9$$

Here $\mathbb{L}$ is the particle momentum.
m is the particle mass.
$\mathbf{P}$ is the particle position point.
$\dot{\mathbf{P}}$ is the particle velocity.
$m\,\mathbf{P}$ is the particle point mass.
$m\,\overset{\circ}{\mathbf{P}}$ is the particle momentum vector.

The momentum of a particle may be viewed either as a momentum vector bound to a line through the position of the particle, or as a velocity vector bound to a line through the point mass. The simple description [8.9] delineates clearly the role of the (free) vector $m\,\overset{\circ}{\mathbf{P}}$. It represents all the properties of the momentum of a particle *except* its position.

The momentum of a system of particles

The momentum of a system of particles is denoted by $\mathbb{L}$, and may be represented by the sum of bound vectors:

$$\mathbb{L} == \sum \mathbb{L}_i == \sum \mathbf{P_i} \wedge \left(m_i \overset{\circ}{\mathbf{P_i}}\right) \qquad \mathbf{8.10}$$

A sum of bound vectors is not necessarily reducible to a bound vector: the momentum of a system of particles is not necessarily reducible to the momentum of a single particle. However, a sum of bound vectors is in general reducible to the sum of a bound vector and a bivector. This is done simply by adding and subtracting the terms $\mathbf{P} \wedge \sum m_i \overset{\circ}{\mathbf{P_i}}$ to and from the sum [8.10].

$$\mathbb{L} == \mathbf{P} \wedge \sum m_i \overset{\circ}{\mathbf{P_i}} + \sum (\mathbf{P_i} - \mathbf{P}) \wedge \left(m_i \overset{\circ}{\mathbf{P_i}}\right) \qquad \mathbf{8.11}$$

It cannot be too strongly emphasized that although the momentum of the system has now been referred to the point $\mathbf{P}$, it is completely independent of $\mathbf{P}$.

Examining the terms in formula 8.11 we see that since $\sum m_i \overset{\circ}{\mathbf{P_i}}$ is a vector ($\mathbf{l}$, say) the first term reduces to the bound vector $\mathbf{P} \wedge \mathbf{l}$ representing the (linear) momentum of a 'particle' situated at the point $\mathbf{P}$. The vector $\mathbf{l}$ is called the *linear momentum of the system*. The 'particle' may be viewed as a particle with mass equal to the total mass of the system and with velocity equal to the velocity of the centre of mass.

$$\mathbf{l} == \sum m_i \overset{\circ}{\mathbf{P_i}}$$

The second term is a sum of bivectors of the form $\sum (\mathbf{P_i} - \mathbf{P}) \wedge \left(m_i \overset{\circ}{\mathbf{P_i}}\right)$. Such a bivector may be seen to faithfully represent the *moment of momentum* or *angular momentum* of the system of particles about the point $\mathbf{P}$. Let this moment of momentum for particle *i* be denoted $\mathbf{H_{iP}}$.

$$\mathbf{H_{iP}} == (\mathbf{P_i} - \mathbf{P}) \wedge \left(m_i \overset{\circ}{\mathbf{P_i}}\right)$$

The expression for angular momentum $\mathbf{H_{ip}}$ above clarifies distinctly how it can arise from two *located* entities: the bound vector $\mathbf{P_i} \wedge \left(m_i \overset{\circ}{\mathbf{P_i}}\right)$ and the point $\mathbf{P}$, and yet itself be a 'free' entity. Similarly to the notion of moment, the notion of angular momentum treated using the three-dimensional Gibbs-Heaviside vector calculus has caused some confusion amongst students of mechanics: the angular momentum of a particle about a point depends on the positions of the particle and of the point and yet itself has no property of location.

The second term in formula 8.11 representing the sum of the moments of momenta about the point $\mathbf{P}$ will be denoted $\mathbb{H}_\mathbf{P}$.

$$\mathbb{H}_\mathbf{P} == \sum (\mathbf{P_i} - \mathbf{P}) \wedge \left(m_i \overset{\circ}{\mathbf{P_i}}\right)$$

Then *the momentum* $\mathbb{L}$ *of any system of particles may be represented by the sum of a bound vector and a bivector*.

$$\mathbb{L} == \mathbf{P} \wedge \mathbf{l} + \mathbb{H}_\mathbf{P} \qquad \mathbf{8.12}$$

The bound vector $\mathbf{P} \wedge \mathbf{l}$ represents the linear momentum of the system referred to an arbitrary point $\mathbf{P}$.

The bivector $\mathbb{H}_{\mathbf{P}}$ represents the angular momentum of the system about the point $\mathbf{P}$.

The momentum of a system of bodies

The momentum of a system of bodies (rigid or non-rigid) is of the same form as that for a system of particles [8.12], since to calculate momentum it is not necessary to consider any constraints between the particles.

Suppose a number of bodies with momenta $\mathbb{L}_{\mathbf{i}}$, then the total momentum $\sum \mathbb{L}_{\mathbf{i}}$ may be written:

$$\sum \mathbb{L}_{\mathbf{i}} == \sum \mathbf{P}_{\mathbf{i}} \wedge \mathbf{l}_{\mathbf{i}} + \sum \mathbb{H}_{\mathbf{P}_{\mathbf{i}}}$$

Referring this momentum sum to a point $\mathbf{P}$ by adding and subtracting the term $\mathbf{P} \wedge \sum \mathbf{l}_{\mathbf{i}}$ we obtain:

$$\sum \mathbb{L}_{\mathbf{i}} == \mathbf{P} \wedge \sum \mathbf{l}_{\mathbf{i}} + \sum (\mathbf{P}_{\mathbf{i}} - \mathbf{P}) \wedge \mathbf{l}_{\mathbf{i}} + \sum \mathbb{H}_{\mathbf{P}_{\mathbf{i}}}$$

It is thus natural to represent the *linear momentum vector of the system of bodies* $\mathbf{l}$ by:

$$\mathbf{l} == \sum \mathbf{l}_{\mathbf{i}}$$

and the *angular momentum of the system of bodies* $\mathbb{H}_{\mathbf{P}}$ by

$$\mathbb{H}_{\mathbf{P}} == \sum (\mathbf{P}_{\mathbf{i}} - \mathbf{P}) \wedge \mathbf{l}_{\mathbf{i}} + \sum \mathbb{H}_{\mathbf{P}_{\mathbf{i}}}$$

That is, the momentum of a system of bodies is of the same form as the momentum of a system of particles. The linear momentum vector of the system is the sum of the linear momentum vectors of the bodies. The angular momentum of the system is made up of two parts: the sum of the angular momenta of the bodies about their respective chosen points; and the sum of the moments of the linear momenta of the bodies (referred to their respective chosen points) about the point $\mathbf{P}$.

Again it may be worth emphasizing that the total momentum $\mathbb{L}$ is independent of the point $\mathbf{P}$, whilst its component terms are dependent on it. However, we can easily convert the formulae to refer to some other point say $\mathbf{P}^*$.

$$\mathbb{L} == \mathbf{P} \wedge \mathbf{l} + \mathbb{H}_{\mathbf{P}} == \mathbf{P}^* \wedge \mathbf{l} + \mathbb{H}_{\mathbf{P}^*}$$

Hence the angular momentum referred to the new point is given by:

$$\mathbb{H}_{\mathbf{P}^*} == \mathbb{H}_{\mathbf{P}} + (\mathbf{P} - \mathbf{P}^*) \wedge \mathbf{l}$$

Linear momentum and the mass centre

There is a simple relationship between the linear momentum of the system, and the momentum of a 'particle' at the mass centre. The centre of mass $\mathbf{P}_{\mathbf{G}}$ of a system may be defined by:

$$\mathbf{M\,P_G} == \sum \mathbf{m_i\;P_i}$$

Here $\mathbf{M}$ is the total mass of the system.
$\mathbf{P_i}$ is the position of the *i*th particle or centre of mass of the *i*th body.
$\mathbf{P_G}$ is the position of the centre of mass of the system.

Differentiating this equation yields:

$$\mathbf{M\,\overset{\circ}{P}_G} == \sum \mathbf{m_i\;\overset{\circ}{P}_i} == \mathbf{l}$$

Formula 8.12 then may then be written:

$$\mathbb{L} == \mathbf{P} \wedge \left(\mathbf{M\,\overset{\circ}{P}_G}\right) + \mathbb{H}_{\mathbf{P}}$$

If now we refer the momentum to the centre of mass, that is, we choose $\mathbf{P}$ to be $\mathbf{P_G}$, we can write the momentum of the system as:

$$\boxed{\mathbb{L} == \mathbf{P_G} \wedge \left(\mathbf{M\,\overset{\circ}{P}_G}\right) + \mathbb{H}_{\mathbf{P_G}}} \qquad \mathbf{8.13}$$

Thus the momentum $\mathbb{L}$ of a system is equivalent to the momentum of a particle of mass equal to the total mass $\mathbf{M}$ of the system situated at the centre of mass $\mathbf{P_G}$ plus the angular momentum about the centre of mass.

Momentum in a metric 3-plane

In a 3-plane the bivector $\mathbb{H}_{\mathbf{P}}$ is necessarily simple. In a metric 3-plane, the vector-space complement of the simple bivector $\mathbb{H}_{\mathbf{P}}$ is just the usual momentum *vector* of the three-dimensional Gibbs-Heaviside vector calculus.

In three dimensions, the complement of the exterior product of two vectors is equivalent to the usual cross product.

$$\overrightarrow{\mathbb{H}_{\mathbf{P}}} == \sum \mathbf{(P_i - P) \times l_i}$$

Thus, the momentum of a system in a metric 3-plane may be reduced to the momentum of a single particle through an arbitrarily chosen point $\mathbf{P}$ *plus* the *vector-space complement* of the usual moment of moment vector about $\mathbf{P}$.

$$\boxed{\mathbb{L} == \mathbf{P} \wedge \mathbf{l} + \overrightarrow{\sum \mathbf{(P_i - P) \times l_i}}} \qquad \mathbf{8.14}$$

8.4 Newton's Law

Rate of change of momentum

The momentum of a system of particles has been given by [8.10] as:

$$\mathbb{L} == \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \mathring{\mathbf{P}}_\mathbf{i}\right)$$

The rate of change of this momentum with respect to time is:

$$\mathring{\mathbb{L}} == \sum \mathring{\mathbf{P}}_\mathbf{i} \wedge \left(\mathbf{m_i}\, \mathring{\mathbf{P}}_\mathbf{i}\right) + \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \mathring{\mathring{\mathbf{P}}}_\mathbf{i}\right) + \sum \mathbf{P_i} \wedge \left(\mathring{\mathbf{m}}_\mathbf{i}\, \mathring{\mathbf{P}}_\mathbf{i}\right)$$

In what follows, it will be supposed for simplicity that the masses are not time varying, and since the first term is zero, this expression becomes:

$$\boxed{\mathring{\mathbb{L}} == \sum \mathbf{P_i} \wedge \left(\mathbf{m_i}\, \mathring{\mathring{\mathbf{P}}}_\mathbf{i}\right)} \quad \mathbf{8.15}$$

Consider now the system with momentum:

$$\mathbb{L} == \mathbf{P} \wedge \mathbf{l} + \mathbb{H}_\mathbf{P}$$

The rate of change of momentum with respect to time is:

$$\boxed{\mathring{\mathbb{L}} == \mathring{\mathbf{P}} \wedge \mathbf{l} + \mathbf{P} \wedge \mathring{\mathbf{l}} + \mathring{\mathbb{H}}_\mathbf{P}} \quad \mathbf{8.16}$$

The term $\mathring{\mathbf{P}} \wedge \mathbf{l}$, or what is equivalent, the term $\mathring{\mathbf{P}} \wedge \left(\mathbf{M}\, \mathring{\mathbf{P}}_\mathbf{G}\right)$, is zero under the following conditions:

- $\mathbf{P}$ is a fixed point.
- $\mathbf{l}$ is zero.
- $\mathring{\mathbf{P}}$ is parallel to $\mathring{\mathbf{P}}_\mathbf{G}$.
- $\mathbf{P}$ is equal to $\mathbf{P_G}$.

In particular then, when the point to which the momentum is referred is the centre of mass, the rate of change of momentum [4.2] becomes:

$$\boxed{\mathring{\mathbb{L}} == \mathbf{P_G} \wedge \mathring{\mathbf{l}} + \mathring{\mathbb{H}}_{\mathbf{P_G}}} \quad \mathbf{8.17}$$

Newton's second law

For a system of bodies of momentum $\mathbb{L}$ acted upon by a system of forces $\mathbb{F}$, Newton's second law may be expressed as:

$$\mathbb{F} == \overset{\circ}{\mathbb{L}} \tag{8.18}$$

This equation captures Newton's law in its most complete form, encapsulating both linear and angular terms. Substituting for $\mathbb{F}$ and $\overset{\circ}{\mathbb{L}}$ from equations 8.5 and 8.16 we have:

$$\mathbf{P} \wedge \mathbf{f} + \mathbb{G}_{\mathbf{P}} == \overset{\circ}{\mathbf{P}} \wedge \mathbf{l} + \mathbf{P} \wedge \overset{\circ}{\mathbf{l}} + \overset{\circ}{\mathbb{H}}_{\mathbf{P}} \tag{8.19}$$

By equating the bound vector terms of [8.19] we obtain the vector equation:

$$\mathbf{f} == \overset{\circ}{\mathbf{l}}$$

By equating the bivector terms of [8.19] we obtain the bivector equation:

$$\mathbb{G}_{\mathbf{P}} == \overset{\circ}{\mathbf{P}} \wedge \mathbf{l} + \overset{\circ}{\mathbb{H}}_{\mathbf{P}}$$

In a metric 3-plane, it is the vector complement of this bivector equation which is usually given as the moment condition.

$$\overrightarrow{\mathbb{G}_{\mathbf{P}}} == \overset{\circ}{\mathbf{P}} \times \mathbf{l} + \overrightarrow{\overset{\circ}{\mathbb{H}}_{\mathbf{P}}}$$

If $\mathbf{P}$ *is a fixed point* so that $\overset{\circ}{\mathbf{P}}$ is zero, Newton's law [8.18] is equivalent to the pair of equations:

$$\begin{gathered} \mathbf{f} == \overset{\circ}{\mathbf{l}} \\ \mathbb{G}_{\mathbf{P}} == \overset{\circ}{\mathbb{H}}_{\mathbf{P}} \end{gathered} \tag{8.20}$$

8.5 The Angular Velocity of a Rigid Body

To be completed.

8.6 The Momentum of a Rigid Body

To be completed.

8.7 The Velocity of a Rigid Body

`To be completed.`

8.8 The Complementary Velocity of a Rigid Body

`To be completed.`

8.9 The Infinitesimal Displacement of a Rigid Body

`To be completed.`

8.10 Work, Power and Kinetic Energy

`To be completed.`

9 Grassmann Algebra

9.1 Introduction

The Grassmann algebra is an algebra of "numbers" composed of linear sums of elements from any of the exterior linear spaces generated from a given linear space. These are numbers in the same sense that, for example, a complex number or a matrix is a number. The essential property that an algebra has, in addition to being a linear space is, broadly speaking, one of closure under a product operation. That is, the algebra has a product operation such that the product of any elements of the algebra is also an element of the algebra.

Thus the exterior product spaces $\underset{m}{\Lambda}$ discussed up to this point, are not algebras, since products (exterior, regressive or interior) of their elements are not generally elements of the same space. However, there are two important exceptions. $\underset{0}{\Lambda}$ is not only a linear space, but an algebra (and a field) as well under the exterior and interior products. $\underset{n}{\Lambda}$ is an algebra and a field under the regressive product.

Many of our examples will be using *GrassmannAlgebra* functions, and because we will often be changing the dimension of the space depending on the example under consideration, we will indicate a change without comment simply by entering the appropriate `DeclareBasis` symbol from the palette. For example the following entry:

```
𝕍3;
```

effects the change to a 3-dimensional linear or vector space.

9.2 Grassmann Numbers

Creating Grassmann numbers

A *Grassmann number* is a sum of elements from any of the exterior product spaces $\underset{m}{\Lambda}$. They thus form a linear space in their own right, which we call Λ.

A basis for Λ is obtained from the current declared basis of $\underset{1}{\Lambda}$ by collecting together all the basis elements of the various $\underset{m}{\Lambda}$. This may be generated by entering `BasisΛ[]`.

```
𝕍3; BasisΛ[]
```

$\{1, e_1, e_2, e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3\}$

If we wish to generate a general symbolic Grassmann number in the currently declared basis, we can use the function *GrassmannAlgebra* function `CreateGrassmannNumber`. `CreateGrassmannNumber[symbol]` creates a Grassmann number with scalar coefficients formed by subscripting the symbol given. For example, if we enter `GrassmannNumber[ξ]`, we obtain:

```
CreateGrassmannNumber[ξ]
```

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1 \wedge e_2 + \xi_5\,e_1 \wedge e_3 + \xi_6\,e_2 \wedge e_3 + \xi_7\,e_1 \wedge e_2 \wedge e_3$$

The positional ordering of the ξ_i with respect to the basis elements in any given term is due to *Mathematica*'s internal ordering routines, and does not affect the meaning of the result.

The `CreateGrassmannNumber` operation automatically adds the pattern for the generated coefficients ξ_i to the list of declared scalars, as we see by entering `Scalars`.

```
Scalars
```

$\{a, b, c, d, e, f, g, h, \Bbbk, (\_\ominus\_)\,?\text{InnerProductQ}, \xi\_, \underset{0}{\_}\}$

If we wish to enter our own coefficients, we can use `CreateGrassmannNumber` with a placeholder (□, ESCplESC) as argument, and then tab through the placeholders generated, replacing them with the desired coefficients:

```
CreateGrassmannNumber[□]
```

$$\square + \square\,e_1 + \square\,e_2 + \square\,e_3 + \square\,e_1 \wedge e_2 + \square\,e_1 \wedge e_3 + \square\,e_2 \wedge e_3 + \square\,e_1 \wedge e_2 \wedge e_3$$

Note that the coefficients entered here will *not* be automatically put on the list of declared scalars.

If several Grassmann numbers are required, we can apply the operation to a list of symbols. That is, `CreateGrassmannNumber` is `Listable`. Below we generate three numbers in 2-space.

```
𝕍2; CreateGrassmannNumber[{ξ, ψ, ζ}] // MatrixForm
```

$$\begin{pmatrix} \xi_1 + e_1\,\xi_3 + e_2\,\xi_4 + \xi_2\,e_1 \wedge e_2 \\ \psi_1 + e_1\,\psi_3 + e_2\,\psi_4 + \psi_2\,e_1 \wedge e_2 \\ \zeta_1 + e_1\,\zeta_3 + e_2\,\zeta_4 + \zeta_2\,e_1 \wedge e_2 \end{pmatrix}$$

Grassmann numbers can of course be entered in general symbolic form, not just using basis elements. For example a Grassmann number might take the form:

$$\mathbf{2 + x - 3\,y + 5\,x \wedge y + x \wedge y \wedge z}$$

In some cases it might be faster to change basis temporarily in order to generate a template with all but the required scalars formed.

```
DeclareBasis[{x, y, z}];
T = Table[CreateGrassmannNumber[□], {3}]; 𝕍3; MatrixForm[T]
```

$$\begin{pmatrix} \square + x\,\square + y\,\square + z\,\square + \square\, x\wedge y + \square\, x\wedge z + \square\, y\wedge z + \square\, x\wedge y\wedge z \\ \square + x\,\square + y\,\square + z\,\square + \square\, x\wedge y + \square\, x\wedge z + \square\, y\wedge z + \square\, x\wedge y\wedge z \\ \square + x\,\square + y\,\square + z\,\square + \square\, x\wedge y + \square\, x\wedge z + \square\, y\wedge z + \square\, x\wedge y\wedge z \end{pmatrix}$$

Body and soul

For certain operations (particularly the inverse) it is critically important whether or not a Grassmann number has a (non-zero) scalar component.

The scalar component of a Grassmann number is called its *body*. The body may be obtained from a Grassmann number by using the `Body` function. Suppose we have a general Grassmann number **X** in a 3-space.

```
𝕍3; X = CreateGrassmannNumber[ξ]
```

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\, e_1\wedge e_2 + \xi_5\, e_1\wedge e_3 + \xi_6\, e_2\wedge e_3 + \xi_7\, e_1\wedge e_2\wedge e_3$$

The body of **X** is then:

```
Body[X]
```

$$\xi_0$$

The rest of a Grassmann number is called its *soul*. The soul may be obtained from a Grassmann number by using the `Soul` function. The soul of the number **X** is:

```
Soul[X]
```

$$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\, e_1\wedge e_2 + \xi_5\, e_1\wedge e_3 + \xi_6\, e_2\wedge e_3 + \xi_7\, e_1\wedge e_2\wedge e_3$$

`Body` and `Soul` apply to Grassmann numbers whose components are given in any form.

```
Z = (x⊖y) + ((x∧(y + 2))⊖(z - 2)) + (z∨(y∧x));
```

```
{Body[Z], Soul[Z]}
```

$$\{x\ominus y + 2\,(x\ominus z) + z\vee y\wedge x,\ -4\,x + x\wedge y\ominus z - 2\,x\wedge y\}$$

The first two terms of the body are scalar because they are interior products of 1-elements (scalar products). The third term of the body is a scalar in a space of 3 dimensions.

`Body` and `Soul` are both `Listable`. Taking the `Soul` of the list of Grassmann numbers generated in 2-space above gives

$$\text{Soul}\left[\begin{pmatrix} \xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1\wedge e_2 \\ \psi_0 + e_1\,\psi_1 + e_2\,\psi_2 + \psi_3\,e_1\wedge e_2 \\ \zeta_0 + e_1\,\zeta_1 + e_2\,\zeta_2 + \zeta_3\,e_1\wedge e_2 \end{pmatrix}\right] \text{ // MatrixForm}$$

$$\begin{pmatrix} e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1\wedge e_2 \\ e_1\,\psi_1 + e_2\,\psi_2 + \psi_3\,e_1\wedge e_2 \\ e_1\,\zeta_1 + e_2\,\zeta_2 + \zeta_3\,e_1\wedge e_2 \end{pmatrix}$$

Even and odd components

Changing the order of the factors in an exterior product changes its sign according to the axiom:

$$\underset{m}{\alpha}\wedge\underset{k}{\beta} = (-1)^{m\,k}\,\underset{k}{\beta}\wedge\underset{m}{\alpha}$$

Thus factors of odd grade anti-commute; whereas, if one of the factors is of even grade, they commute. This means that, if **X** is a Grassmann number whose components are all of odd grade, then it is nilpotent.

The even components of a Grassmann number can be extracted with the function `EvenGrade` and the odd components with `OddGrade`.

X

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge e_3$$

EvenGrade[X]

$$\xi_0 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3$$

OddGrade[X]

$$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_7\,e_1\wedge e_2\wedge e_3$$

It should be noted that the result returned by `EvenGrade` or `OddGrade` is dependent on the dimension of the current space. For example, the calculations above for the even and odd components of the number **X** were carried out in a 3-space. If now we change to a 2-space and repeat the calculations, we will get a different result for `OddGrade[X]` because the term of grade three is necessarily zero in this 2-space.

$\mathbb{V}_2$**; OddGrade[X]**

$$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3$$

For `EvenGrade` and `OddGrade` to apply, it is not necessary that the Grassmann number be in terms of basis elements or simple exterior products. Applying `EvenGrade` and `OddGrade` to the number **Z** below still separates the number into its even and odd components:

$$Z = (x \ominus y) + ((x\wedge(y+2))\ominus(z-2)) + (z\vee(y\wedge x));$$

```
𝕍3; {EvenGrade[Z], OddGrade[Z]}
```

$\{x \ominus y + 2\ (x \ominus z) + z \vee y \wedge x - 2\ x \wedge y,\ -4\ x + x \wedge y \ominus z\}$

`EvenGrade` and `OddGrade` are `Listable`.

To *test* whether a number is even or odd we use `EvenGradeQ` and `OddGradeQ`. Consider again the general Grassmann number **X** in 3-space.

```
{EvenGradeQ[X], OddGradeQ[X],
 EvenGradeQ[EvenGrade[X]], OddGradeQ[EvenGrade[X]]}
```

```
{False, False, True, False}
```

`EvenGradeQ` and `OddGradeQ` are *not* `Listable`. When applied to a list of elements, they require *all* the elements in the list to conform in order to return `True`. They can of course still be mapped over a list of elements to question the evenness or oddness of the individual components.

```
EvenGradeQ /@ {1, x, x ∧ y, x ∧ y ∧ z}
```

```
{True, False, True, False}
```

Finally, there is the question as to whether the Grassmann number **0** is even or odd. It may be recalled from Chapter 2 that the single symbol **0** is actually a shorthand for the zero element of *any* of the exterior linear spaces, and hence is of indeterminate grade. Entering `Grade[0]` will return a flag `Grade0` which can be manipulated as appropriate. Therefore, both `EvenGradeQ[0]` and `OddGradeQ[0]` will return `False`.

```
{Grade[0], EvenGradeQ[0], OddGradeQ[0]}
```

```
{Grade0, False, False}
```

The grades of a Grassmann number

The *GrassmannAlgebra* function `Grade` will determine the grades of each of the components in a Grassmann number, and return a list of the grades. For example, applying `Grade` to a general number **X** in 3-space shows that it contains components of all grades up to 3.

```
X
```

$$\xi_0 + e_1\ \xi_1 + e_2\ \xi_2 + e_3\ \xi_3 + \xi_4\ e_1 \wedge e_2 + \xi_5\ e_1 \wedge e_3 + \xi_6\ e_2 \wedge e_3 + \xi_7\ e_1 \wedge e_2 \wedge e_3$$

```
Grade[X]
```

```
{0, 1, 2, 3}
```

`Grade` will also work with more general Grassmann number expressions. For example we can apply it to the number **Z** which we previously defined above.

```
Z
```

$x \ominus y + x \wedge (2 + y) \ominus (-2 + z) + z \vee y \wedge x$

```
Grade[Z]
```

```
{0, 1, 2}
```

Because it may be necessary to expand out a Grassmann number into a sum of terms (each of which has a definite grade) before the grades can be calculated, and because *Mathematica* will reorganize the ordering of those terms in the sum according to its own internal algorithms, direct correspondence with the original form is often lost. However, if a list of the grades of each term in an expansion of a Grassmann number is required, we should:

- Expand and simplify the number into a sum of terms.
- Create a list of the terms.
- Map `Grade` over the list.

For example:

```
A = 𝒢[Z]
```

$-4\,x + x \ominus y + 2\,(x \ominus z) + x \wedge y \ominus z - x \wedge y \wedge z \vee 1 - 2\,x \wedge y$

```
A = List @@ A
```

$\{-4\,x,\ x \ominus y,\ 2\,(x \ominus z),\ x \wedge y \ominus z,\ -(x \wedge y \wedge z \vee 1),\ -2\,x \wedge y\}$

```
Grade /@ A
```

```
{1, 0, 0, 1, 0, 2}
```

To extract the components of a Grassmann number of a given grade or grades, we can use the *GrassmannAlgebra* `ExtractGrade[m][X]` function which takes a Grassmann number and extracts the components of grade *m*.

Extracting the components of grade 1 from the number **Z** defined above gives:

```
ExtractGrade[1][Z]
```

$-4\,x + x \wedge y \ominus z$

`ExtractGrade` also works on lists or tensors of GrassmannNumbers.

```
DeclareExtraScalars[{ψ_, ζ_}]
```

$\{a, b, c, d, e, f, g, h, \Bbbk, (\_ \ominus \_)\,?\,\mathrm{InnerProductQ}, \zeta\_, \xi\_, \psi\_, \underset{0}{\_}\}$

$$\mathrm{ExtractGrade}[1]\left[\begin{pmatrix} \xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1 \wedge e_2 \\ \psi_0 + e_1\,\psi_1 + e_2\,\psi_2 + \psi_3\,e_1 \wedge e_2 \\ \zeta_0 + e_1\,\zeta_1 + e_2\,\zeta_2 + \zeta_3\,e_1 \wedge e_2 \end{pmatrix}\right] \ // \ \mathrm{MatrixForm}$$

$$\begin{pmatrix} e_1\,\xi_1 + e_2\,\xi_2 \\ e_1\,\psi_1 + e_2\,\psi_2 \\ e_1\,\zeta_1 + e_2\,\zeta_2 \end{pmatrix}$$

Working with complex scalars

The imaginary unit i is treated by *Mathematica* and the *GrassmannAlgebra* package just as if it were a numeric quantity. This means that just like other numeric quantities, it will not appear in the list of declared scalars, even if explicitly entered.

```
DeclareScalars[{a, b, c, d, 2, ⅈ, ⅇ}]
```

```
{a, b, c, d}
```

However, i or any other numeric quantity is treated as a scalar.

```
ScalarQ[{a, b, c, 2, ⅈ, ⅇ, 2 - 3 ⅈ}]
```

```
{True, True, True, True, True, True, True}
```

This feature allows the *GrassmannAlgebra* package to deal with complex numbers just as it would any other scalars.

```
𝒢[((a + ⅈ b) x) ∧ (ⅈ y)]
```

$(i\, a - b)\, x \wedge y$

9.3 Operations with Grassmann Numbers

The exterior product of Grassmann numbers

We have already shown how to create a general Grassmann number in 3-space:

```
X = CreateGrassmannNumber[ξ]
```

$$\xi_0 + e_1\, \xi_1 + e_2\, \xi_2 + e_3\, \xi_3 + \xi_4\, e_1 \wedge e_2 + \xi_5\, e_1 \wedge e_3 + \xi_6\, e_2 \wedge e_3 + \xi_7\, e_1 \wedge e_2 \wedge e_3$$

We now create a second Grassmann number **Y** so that we can look at various operations applied to two Grassmann numbers in 3-space.

```
Y = CreateGrassmannNumber[ψ]
```

$$\psi_0 + e_1\, \psi_1 + e_2\, \psi_2 + e_3\, \psi_3 + \psi_4\, e_1 \wedge e_2 + \psi_5\, e_1 \wedge e_3 + \psi_6\, e_2 \wedge e_3 + \psi_7\, e_1 \wedge e_2 \wedge e_3$$

The exterior product of two general Grassmann numbers in 3-space is obtained by multiplying out the numbers termwise and simplifying the result.

$\mathcal{G}$[X ∧ Y]

$$\xi_0\,\psi_0 + e_1\,(\xi_1\,\psi_0 + \xi_0\,\psi_1) + e_2\,(\xi_2\,\psi_0 + \xi_0\,\psi_2) + e_3\,(\xi_3\,\psi_0 + \xi_0\,\psi_3) + (\xi_4\,\psi_0 - \xi_2\,\psi_1 + \xi_1\,\psi_2 + \xi_0\,\psi_4)\,e_1 \wedge e_2 + (\xi_5\,\psi_0 - \xi_3\,\psi_1 + \xi_1\,\psi_3 + \xi_0\,\psi_5)\,e_1 \wedge e_3 + (\xi_6\,\psi_0 - \xi_3\,\psi_2 + \xi_2\,\psi_3 + \xi_0\,\psi_6)\,e_2 \wedge e_3 + (\xi_7\,\psi_0 + \xi_6\,\psi_1 - \xi_5\,\psi_2 + \xi_4\,\psi_3 + \xi_3\,\psi_4 - \xi_2\,\psi_5 + \xi_1\,\psi_6 + \xi_0\,\psi_7)\,e_1 \wedge e_2 \wedge e_3$$

When the bodies of the numbers are zero we get only components of grades two and three.

$\mathcal{G}$[Soul[X] ∧ Soul[Y]]

$$(-\xi_2\,\psi_1 + \xi_1\,\psi_2)\,e_1 \wedge e_2 + (-\xi_3\,\psi_1 + \xi_1\,\psi_3)\,e_1 \wedge e_3 + (-\xi_3\,\psi_2 + \xi_2\,\psi_3)\,e_2 \wedge e_3 + (\xi_6\,\psi_1 - \xi_5\,\psi_2 + \xi_4\,\psi_3 + \xi_3\,\psi_4 - \xi_2\,\psi_5 + \xi_1\,\psi_6)\,e_1 \wedge e_2 \wedge e_3$$

The regressive product of Grassmann numbers

The regressive product of X and Y is again obtained by multiplying out the numbers termwise and simplifying.

Z = $\mathcal{G}$[X ∨ Y]

$$e_1 \wedge e_2 \wedge e_3 \vee (\xi_7\,\psi_0 + \xi_6\,\psi_1 - \xi_5\,\psi_2 + \xi_4\,\psi_3 + \xi_3\,\psi_4 - \xi_2\,\psi_5 + \xi_1\,\psi_6 + \xi_0\,\psi_7 + e_1\,(\xi_7\,\psi_1 - \xi_5\,\psi_4 + \xi_4\,\psi_5 + \xi_1\,\psi_7) + e_2\,(\xi_7\,\psi_2 - \xi_6\,\psi_4 + \xi_4\,\psi_6 + \xi_2\,\psi_7) + e_3\,(\xi_7\,\psi_3 - \xi_6\,\psi_5 + \xi_5\,\psi_6 + \xi_3\,\psi_7) + (\xi_7\,\psi_4 + \xi_4\,\psi_7)\,e_1 \wedge e_2 + (\xi_7\,\psi_5 + \xi_5\,\psi_7)\,e_1 \wedge e_3 + (\xi_7\,\psi_6 + \xi_6\,\psi_7)\,e_2 \wedge e_3 + \xi_7\,\psi_7\,e_1 \wedge e_2 \wedge e_3)$$

We cannot obtain the result in the form of a pure exterior product without relating the unit *n*-element to the basis *n*-element. In general these two elements may be related by a scalar multiple which we have denoted $\mathbb{k}$. In 3-space this relationship is given by $\underset{3}{\mathbf{1}} == \mathbb{k}\, \mathbf{e_1 \wedge e_2 \wedge e_3}$. To obtain the result as an exterior product, but one that will also involve the scalar $\mathbb{k}$, we use the *GrassmannAlgebra* function `ToCongruenceForm`.

Z1 = ToCongruenceForm[Z]

$$\frac{1}{\mathbb{k}}\,(\xi_7\,\psi_0 + \xi_6\,\psi_1 - \xi_5\,\psi_2 + \xi_4\,\psi_3 + \xi_3\,\psi_4 - \xi_2\,\psi_5 + \xi_1\,\psi_6 + \xi_0\,\psi_7 + e_1\,(\xi_7\,\psi_1 - \xi_5\,\psi_4 + \xi_4\,\psi_5 + \xi_1\,\psi_7) + e_2\,(\xi_7\,\psi_2 - \xi_6\,\psi_4 + \xi_4\,\psi_6 + \xi_2\,\psi_7) + e_3\,(\xi_7\,\psi_3 - \xi_6\,\psi_5 + \xi_5\,\psi_6 + \xi_3\,\psi_7) + (\xi_7\,\psi_4 + \xi_4\,\psi_7)\,e_1 \wedge e_2 + (\xi_7\,\psi_5 + \xi_5\,\psi_7)\,e_1 \wedge e_3 + (\xi_7\,\psi_6 + \xi_6\,\psi_7)\,e_2 \wedge e_3 + \xi_7\,\psi_7\,e_1 \wedge e_2 \wedge e_3)$$

In a space with a Euclidean metric $\mathbb{k}$ is unity. `GrassmannSimplify` looks at the currently declared metric and substitutes the value of $\mathbb{k}$. Since the currently declared metric is by default Euclidean, applying `GrassmannSimplify` puts $\mathbb{k}$ to unity.

```
𝒢[Z1]
```

$\xi_7\,\psi_0 + \xi_6\,\psi_1 - \xi_5\,\psi_2 + \xi_4\,\psi_3 + \xi_3\,\psi_4 - \xi_2\,\psi_5 + \xi_1\,\psi_6 + \xi_0\,\psi_7 + e_1\,(\xi_7\,\psi_1 - \xi_5\,\psi_4 + \xi_4\,\psi_5 + \xi_1\,\psi_7) + e_2\,(\xi_7\,\psi_2 - \xi_6\,\psi_4 + \xi_4\,\psi_6 + \xi_2\,\psi_7) + e_3\,(\xi_7\,\psi_3 - \xi_6\,\psi_5 + \xi_5\,\psi_6 + \xi_3\,\psi_7) + (\xi_7\,\psi_4 + \xi_4\,\psi_7)\,e_1\wedge e_2 + (\xi_7\,\psi_5 + \xi_5\,\psi_7)\,e_1\wedge e_3 + (\xi_7\,\psi_6 + \xi_6\,\psi_7)\,e_2\wedge e_3 + \xi_7\,\psi_7\,e_1\wedge e_2\wedge e_3$

The complement of a Grassmann number

Consider again a general Grassmann number **X** in 3-space.

```
X
```

$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge e_3$

The complement of **X** is denoted $\overline{\mathbf{X}}$. Entering $\overline{\mathbf{X}}$ applies the complement operation, but does not simplify it in any way.

$\overline{\mathbf{X}}$

$\overline{\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge}$

Further simplification to an explicit Grassmann number depends on the metric of the space concerned and may be accomplished by applying `GrassmannSimplify`. `GrassmannSimplify` will look at the metric and make the necessary transformations. In the Euclidean metric assumed by *GrassmannAlgebra* as the default we have:

$\mathcal{G}[\overline{\mathbf{X}}]$

$e_3\,\xi_4 - e_2\,\xi_5 + e_1\,\xi_6 + \xi_7 + \xi_3\,e_1\wedge e_2 - \xi_2\,e_1\wedge e_3 + \xi_1\,e_2\wedge e_3 + \xi_0\,e_1\wedge e_2\wedge e_3$

For metrics other than Euclidean, the expression for the complement will be more complex. We can explore the complement of a general Grassmann number in a general metric by entering `DeclareMetric[𝕘]` and then applying `GrassmannSimplify` to $\overline{\mathbf{X}}$. As an example, we take the complement of a general Grassmann number in a 2-space with a general metric. We choose a 2-space rather than a 3-space, because the result is less complex to display.

```
𝕍2; DeclareMetric[𝕘]
```

$\{\{g_{1,1}, g_{1,2}\}, \{g_{1,2}, g_{2,2}\}\}$

```
U = CreateGrassmannNumber[υ]
```

$\upsilon_0 + e_1\,\upsilon_1 + e_2\,\upsilon_2 + \upsilon_3\,e_1\wedge e_2$

$\mathcal{G}[\overline{\mathbf{U}}]$ **// Simplify**

$$\left(-\upsilon_3\,g_{1,2}^2 + e_2\,(\upsilon_1\,g_{1,1} + \upsilon_2\,g_{1,2}) + \upsilon_3\,g_{1,1}\,g_{2,2} - e_1\,(\upsilon_1\,g_{1,2} + \upsilon_2\,g_{2,2}) + \upsilon_0\,e_1\wedge e_2\right) \Big/ \left(\sqrt{-g_{1,2}^2 + g_{1,1}\,g_{2,2}}\right)$$

The interior product of Grassmann numbers

The interior product of two elements $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ where *m* is less than *k*, is always zero. Thus the interior product $\mathbf{X} \ominus \mathbf{Y}$ is zero if the components of $\mathbf{X}$ are of lesser grade than those of $\mathbf{Y}$. For two general Grassmann numbers we will therefore have some of the products of the components being zero.

By way of example we take two general Grassmann numbers **U** and **V** in 2-space:

```
𝕍2; {U = CreateGrassmannNumber[υ],
 V = CreateGrassmannNumber[ω]}
```

$\{\upsilon_0 + e_1\,\upsilon_1 + e_2\,\upsilon_2 + \upsilon_3\,e_1 \wedge e_2,\ \omega_0 + e_1\,\omega_1 + e_2\,\omega_2 + \omega_3\,e_1 \wedge e_2\}$

The interior product of **U** with **V** is:

```
W1 = 𝒢[U ⊖ V]
```

$$\upsilon_0\,\omega_0 + e_1\,\upsilon_1\,\omega_0 + e_2\,\upsilon_2\,\omega_0 + ((e_1 \ominus e_1)\,\upsilon_1 + (e_1 \ominus e_2)\,\upsilon_2)\,\omega_1 + (e_1 \wedge e_2 \ominus e_1)\,\upsilon_3\,\omega_1 + ((e_1 \ominus e_2)\,\upsilon_1 + (e_2 \ominus e_2)\,\upsilon_2)\,\omega_2 + (e_1 \wedge e_2 \ominus e_2)\,\upsilon_3\,\omega_2 + (e_1 \wedge e_2 \ominus e_1 \wedge e_2)\,\upsilon_3\,\omega_3 + \upsilon_3\,\omega_0\,e_1 \wedge e_2$$

Note that there are no products of the form $\mathbf{e_i} \ominus (\mathbf{e_j} \wedge \mathbf{e_k})$ as these have been put to zero by `GrassmannSimplify`. If we wish, we can convert the remaining interior products to scalar products using the *GrassmannAlgebra* function `ToScalarProducts`.

```
W2 = ToScalarProducts[W1]
```

$$\upsilon_0\,\omega_0 + e_1\,\upsilon_1\,\omega_0 + e_2\,\upsilon_2\,\omega_0 + (e_1 \ominus e_1)\,\upsilon_1\,\omega_1 + (e_1 \ominus e_2)\,\upsilon_2\,\omega_1 - (e_1 \ominus e_2)\,e_1\,\upsilon_3\,\omega_1 + (e_1 \ominus e_1)\,e_2\,\upsilon_3\,\omega_1 + (e_1 \ominus e_2)\,\upsilon_1\,\omega_2 + (e_2 \ominus e_2)\,\upsilon_2\,\omega_2 - (e_2 \ominus e_2)\,e_1\,\upsilon_3\,\omega_2 + (e_1 \ominus e_2)\,e_2\,\upsilon_3\,\omega_2 - (e_1 \ominus e_2)^2\,\upsilon_3\,\omega_3 + (e_1 \ominus e_1)\,(e_2 \ominus e_2)\,\upsilon_3\,\omega_3 + \upsilon_3\,\omega_0\,e_1 \wedge e_2$$

Finally, we can substitute the values from the currently declared metric tensor for the scalar products using the *GrassmannAlgebra* function `ToMetricForm`. For example, if we first declare a general metric:

```
DeclareMetric[𝕘]
```

$\{\{g_{1,1}, g_{1,2}\}, \{g_{1,2}, g_{2,2}\}\}$

```
ToMetricForm[W2]
```

$$\upsilon_0\,\omega_0 + e_1\,\upsilon_1\,\omega_0 + e_2\,\upsilon_2\,\omega_0 + \upsilon_1\,\omega_1\,g_{1,1} + e_2\,\upsilon_3\,\omega_1\,g_{1,1} + \upsilon_2\,\omega_1\,g_{1,2} - e_1\,\upsilon_3\,\omega_1\,g_{1,2} + \upsilon_1\,\omega_2\,g_{1,2} + e_2\,\upsilon_3\,\omega_2\,g_{1,2} - \upsilon_3\,\omega_3\,g_{1,2}^2 + \upsilon_2\,\omega_2\,g_{2,2} - e_1\,\upsilon_3\,\omega_2\,g_{2,2} + \upsilon_3\,\omega_3\,g_{1,1}\,g_{2,2} + \upsilon_3\,\omega_0\,e_1 \wedge e_2$$

The interior product of two general Grassmann numbers in 3-space is

$$\xi_0\,\psi_0+\xi_1\,\psi_1+\xi_2\,\psi_2+\xi_3\,\psi_3+\xi_4\,\psi_4+e_3\,(\xi_3\,\psi_0+\xi_5\,\psi_1+\xi_6\,\psi_2+\xi_7\,\psi_4)+\xi_5\,\psi_5+e_2\,(\xi_2\,\psi_0+\xi_4\,\psi_1-\xi_6\,\psi_3-\xi_7\,\psi_5)+\xi_6\,\psi_6+e_1\,(\xi_1\,\psi_0-\xi_4\,\psi_2-\xi_5\,\psi_3+\xi_7\,\psi_6)+\xi_7\,\psi_7+(\xi_4\,\psi_0+\xi_7\,\psi_3)\,e_1\wedge e_2+(\xi_5\,\psi_0-\xi_7\,\psi_2)\,e_1\wedge e_3+(\xi_6\,\psi_0+\xi_7\,\psi_1)\,e_2\wedge e_3+\xi_7\,\psi_0\,e_1\wedge e_2\wedge e_3$$

Now consider the case of two general Grassmann number in 3-space.

$\mathbb{V}_3$; X = CreateGrassmannNumber[ξ]

$$\xi_0+e_1\,\xi_1+e_2\,\xi_2+e_3\,\xi_3+\xi_4\,e_1\wedge e_2+\xi_5\,e_1\wedge e_3+\xi_6\,e_2\wedge e_3+\xi_7\,e_1\wedge e_2\wedge e_3$$

Y = CreateGrassmannNumber[ψ]

$$\psi_0+e_1\,\psi_1+e_2\,\psi_2+e_3\,\psi_3+\psi_4\,e_1\wedge e_2+\psi_5\,e_1\wedge e_3+\psi_6\,e_2\wedge e_3+\psi_7\,e_1\wedge e_2\wedge e_3$$

The interior product of **X** with **Y** is:

Z = $\mathcal{G}$[X $\ominus$ Y]

$$\xi_0\,\psi_0+e_1\,\xi_1\,\psi_0+e_2\,\xi_2\,\psi_0+e_3\,\xi_3\,\psi_0+((e_1\ominus e_1)\,\xi_1+(e_1\ominus e_2)\,\xi_2+(e_1\ominus e_3)\,\xi_3)\,\psi_1+(e_1\wedge e_2\ominus e_1)\,\xi_4\,\psi_1+(e_1\wedge e_3\ominus e_1)\,\xi_5\,\psi_1+(e_2\wedge e_3\ominus e_1)\,\xi_6\,\psi_1+(e_1\wedge e_2\wedge e_3\ominus e_1)\,\xi_7\,\psi_1+((e_1\ominus e_2)\,\xi_1+(e_2\ominus e_2)\,\xi_2+(e_2\ominus e_3)\,\xi_3)\,\psi_2+(e_1\wedge e_2\ominus e_2)\,\xi_4\,\psi_2+(e_1\wedge e_3\ominus e_2)\,\xi_5\,\psi_2+(e_2\wedge e_3\ominus e_2)\,\xi_6\,\psi_2+(e_1\wedge e_2\wedge e_3\ominus e_2)\,\xi_7\,\psi_2+((e_1\ominus e_3)\,\xi_1+(e_2\ominus e_3)\,\xi_2+(e_3\ominus e_3)\,\xi_3)\,\psi_3+(e_1\wedge e_2\ominus e_3)\,\xi_4\,\psi_3+(e_1\wedge e_3\ominus e_3)\,\xi_5\,\psi_3+(e_2\wedge e_3\ominus e_3)\,\xi_6\,\psi_3+(e_1\wedge e_2\wedge e_3\ominus e_3)\,\xi_7\,\psi_3+((e_1\wedge e_2\ominus e_1\wedge e_2)\,\xi_4+(e_1\wedge e_2\ominus e_1\wedge e_3)\,\xi_5+(e_1\wedge e_2\ominus e_2\wedge e_3)\,\xi_6)\,\psi_4+(e_1\wedge e_2\wedge e_3\ominus e_1\wedge e_2)\,\xi_7\,\psi_4+((e_1\wedge e_2\ominus e_1\wedge e_3)\,\xi_4+(e_1\wedge e_3\ominus e_1\wedge e_3)\,\xi_5+(e_1\wedge e_3\ominus e_2\wedge e_3)\,\xi_6)\,\psi_5+(e_1\wedge e_2\wedge e_3\ominus e_1\wedge e_3)\,\xi_7\,\psi_5+((e_1\wedge e_2\ominus e_2\wedge e_3)\,\xi_4+(e_1\wedge e_3\ominus e_2\wedge e_3)\,\xi_5+(e_2\wedge e_3\ominus e_2\wedge e_3)\,\xi_6)\,\psi_6+(e_1\wedge e_2\wedge e_3\ominus e_2\wedge e_3)\,\xi_7\,\psi_6+(e_1\wedge e_2\wedge e_3\ominus e_1\wedge e_2\wedge e_3)\,\xi_7\,\psi_7+\xi_4\,\psi_0\,e_1\wedge e_2+\xi_5\,\psi_0\,e_1\wedge e_3+\xi_6\,\psi_0\,e_2\wedge e_3+\xi_7\,\psi_0\,e_1\wedge e_2\wedge e_3$$

We can expand these interior products to inner products and replace them by elements of the metric tensor by using the *GrassmannAlgebra* function `ToMetricForm`. For example in the case of a Euclidean metric we have:

$\mathbb{E}$; ToMetricForm[Z]

$$\xi_0\,\psi_0+e_1\,\xi_1\,\psi_0+e_2\,\xi_2\,\psi_0+e_3\,\xi_3\,\psi_0+\xi_1\,\psi_1+e_2\,\xi_4\,\psi_1+e_3\,\xi_5\,\psi_1+\xi_2\,\psi_2-e_1\,\xi_4\,\psi_2+e_3\,\xi_6\,\psi_2+\xi_3\,\psi_3-e_1\,\xi_5\,\psi_3-e_2\,\xi_6\,\psi_3+\xi_4\,\psi_4+e_3\,\xi_7\,\psi_4+\xi_5\,\psi_5-e_2\,\xi_7\,\psi_5+\xi_6\,\psi_6+e_1\,\xi_7\,\psi_6+\xi_7\,\psi_7+\xi_4\,\psi_0\,e_1\wedge e_2+\xi_7\,\psi_3\,e_1\wedge e_2+\xi_5\,\psi_0\,e_1\wedge e_3-\xi_7\,\psi_2\,e_1\wedge e_3+\xi_6\,\psi_0\,e_2\wedge e_3+\xi_7\,\psi_1\,e_2\wedge e_3+\xi_7\,\psi_0\,e_1\wedge e_2\wedge e_3$$

9.4 Simplifying Grassmann Numbers

Elementary simplifying operations

All of the elementary simplifying operations that work with elements of a single exterior linear space also work with Grassmann numbers. In this section we take a Grassmann number **A** in 2-space involving both exterior and interior products, and explore the application of those component operations of `GrassmannSimplify` which apply to exterior and interior products.

```
A = ((1 + x) ∧ (2 + y + y ∧ x)) ⊖ (3 + z)
```

$(1+x)\wedge(2+y+y\wedge x)\ominus(3+z)$

Expanding products

`ExpandProducts` performs the first level of the simplification operation by simply expanding out any products. It treats scalars as elements of the Grassmann algebra, just like those of higher grade.Taking the expression **A** and expanding out both the exterior and interior products gives:

$$\mathbf{A_1 = ExpandProducts[A]}$$

$$1\wedge 2\ominus 3+1\wedge 2\ominus z+1\wedge y\ominus 3+1\wedge y\ominus z+x\wedge 2\ominus 3+x\wedge 2\ominus z+x\wedge y\ominus 3+x\wedge y\ominus z+1\wedge y\wedge x\ominus 3+1\wedge y\wedge x\ominus z+x\wedge y\wedge x\ominus 3+x\wedge y\wedge x\ominus z$$

Factoring scalars

`FactorScalars` collects all the scalars in each term, which figure either by way of ordinary multiplication or by way of exterior multiplication, and multiplies them as a scalar factor at the beginning of each term. Remember that both the exterior and interior product of scalars is a scalar. *Mathematica* will also automatically rearrange the terms in this sum according to its internal ordering algorithm.

$$\mathbf{A_2 = FactorScalars[A_1]}$$

$$6+6\,x+3\,y+2\,(1\ominus z)+2\,(x\ominus z)+y\ominus z+x\wedge y\ominus z+y\wedge x\ominus z+x\wedge y\wedge x\ominus z+3\,x\wedge y+3\,y\wedge x+3\,x\wedge y\wedge x$$

Checking for zero terms

`ZeroSimplify` looks at each term and decides if it is zero, either from nilpotency or from expotency. A term is *nilpotent* if it contains repeated 1-element factors. A term is *expotent* if the number of 1-element factors in it is greater than the dimension of the space. (We have coined this term 'expotent' for convenience.) An interior product is zero if the grade of the first factor is less than the grade of the second factor. `ZeroSimplify` then puts those terms it finds to be zero, equal to zero.

$$\mathbf{A_3 = ZeroSimplify[A_2]}$$

$$6 + 6\,x + 3\,y + 2\,(x \ominus z) + y \ominus z + x \wedge y \ominus z + y \wedge x \ominus z + 3\,x \wedge y + 3\,y \wedge x$$

Reordering factors

In order to simplify further it is necessary to put the factors of each term into a canonical order so that terms of opposite sign in an exterior product will cancel and any inner product will be transformed into just one of its two symmetric forms. This reordering may be achieved by using the *GrassmannAlgebra* `ToStandardOrdering` function. (The rules for *GrassmannAlgebra* standard ordering are given in the package documentation.)

$$\mathbf{A_4 = ToStandardOrdering[A_3]}$$

$$6 + 6\,x + 3\,y + 2\,(x \ominus z) + y \ominus z$$

Simplifying expressions

We can perform all the simplifying operations above with `GrassmannSimplify`. Applying `GrassmannSimplify` to our original expression **A** we get the expression $\mathbf{A_4}$.

$$\mathcal{G}[\mathbf{A_4}]$$

$$6 + 6\,x + 3\,y + 2\,(x \ominus z) + y \ominus z$$

9.5 Powers of Grassmann Numbers

Direct computation of powers

The zeroth power of a Grassmann number is defined to be the unit scalar **1**. The first power is defined to be the Grassmann number itself. Higher powers can be obtained by simply taking the exterior product of the number with itself the requisite number of times, and then applying `GrassmannSimplify` to simplify the result.

To *refer* to an exterior power *n* of a Grassmann number **X** (but not to compute it), we will use the usual notation $\mathbf{X^n}$.

As an example we take the general element of 3-space, **X**, introduced at the beginning of this chapter, and calculate some powers:

```
X = CreateGrassmannNumber[ξ]
```

$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge e_3$

```
𝒢[X ∧ X]
```

$\xi_0^2 + 2\,e_1\,\xi_0\,\xi_1 + 2\,e_2\,\xi_0\,\xi_2 + 2\,e_3\,\xi_0\,\xi_3 + 2\,\xi_0\,\xi_4\,e_1\wedge e_2 + 2\,\xi_0\,\xi_5\,e_1\wedge e_3 + 2\,\xi_0\,\xi_6\,e_2\wedge e_3 + (-2\,\xi_2\,\xi_5 + 2\,(\xi_3\,\xi_4 + \xi_1\,\xi_6 + \xi_0\,\xi_7))\,e_1\wedge e_2\wedge e_3$

```
𝒢[X ∧ X ∧ X]
```

$\xi_0^3 + 3\,e_1\,\xi_0^2\,\xi_1 + 3\,e_2\,\xi_0^2\,\xi_2 + 3\,e_3\,\xi_0^2\,\xi_3 + 3\,\xi_0^2\,\xi_4\,e_1\wedge e_2 + 3\,\xi_0^2\,\xi_5\,e_1\wedge e_3 + 3\,\xi_0^2\,\xi_6\,e_2\wedge e_3 + (6\,\xi_0\,\xi_3\,\xi_4 - 6\,\xi_0\,\xi_2\,\xi_5 + 6\,\xi_0\,\xi_1\,\xi_6 + 3\,\xi_0^2\,\xi_7)\,e_1\wedge e_2\wedge e_3$

Direct computation is generally an inefficient way to calculate a power, as the terms are all expanded out before checking to see which are zero. It will be seen below that we can generally calculate powers more efficiently by using knowledge of which terms in the product would turn out to be zero.

Powers of even Grassmann numbers

We take the even component of **X**, and construct a table of its first, second, third and fourth powers.

```
Xe = EvenGrade[X]
```

$\xi_0 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3$

```
𝒢[{Xe, Xe ∧ Xe, Xe ∧ Xe ∧ Xe, Xe ∧ Xe ∧ Xe ∧ Xe}] // Simplify //
 TableForm
```

$\xi_0 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3$

$\xi_0\,(\xi_0 + 2\,(\xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3))$

$\xi_0^2\,(\xi_0 + 3\,(\xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3))$

$\xi_0^3\,(\xi_0 + 4\,(\xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3))$

The soul of the even part of a general Grassmann number in 3-space just happens to be nilpotent due to its being simple. (See Section 2.10.)

```
Xes = EvenSoul[X]
```

$\xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3$

```
𝒢[Xes ∧ Xes]
```

0

However, in the general case an even soul will not be nilpotent, but its powers will become zero when the grades of the terms in the power expansion exceed the dimension of the space. We can show this by calculating the powers of an even soul in spaces of successively higher dimensions. Let

```
Zes = m∧p + q∧r∧s∧t + u∧v∧w∧x∧y∧z;
```

```
Table[{𝕍n; Dimension, 𝒢[Zes ∧ Zes]}, {n, 3, 8}] // TableForm
```

3	0
4	0
5	0
6	$2\, m \wedge p \wedge q \wedge r \wedge s \wedge t$
7	$2\, m \wedge p \wedge q \wedge r \wedge s \wedge t$
8	$2\, m \wedge p \wedge q \wedge r \wedge s \wedge t + 2\, m \wedge p \wedge u \wedge v \wedge w \wedge x \wedge y \wedge z$

Powers of odd Grassmann numbers

Odd Grassmann numbers are nilpotent. Hence all positive integer powers except the first are zero.

```
Xo = OddGrade[X]
```

$e_1\, \xi_1 + e_2\, \xi_2 + e_3\, \xi_3 + \xi_7\, e_1 \wedge e_2 \wedge e_3$

```
𝒢[Xo ∧ Xo]
```

0

Computing positive powers of Grassmann numbers

We can make use of the nilpotence of the odd component to compute powers of Grassmann numbers more efficiently than the direct approach described at the beginning of this section by noting that it will cause a binomial expansion of the sum of the even and odd components to terminate after the first two terms. Let $X == X_e + X_o$, then:

$$X^p == (X_e + X_o)^p == X_e{}^p + p\, X_e{}^{p-1} \wedge X_o + \cdots$$

$$\boxed{X^p == X_e{}^{p-1} \wedge (X_e + p\, X_o)} \qquad 9.1$$

where the powers of p and $p-1$ refer to the exterior powers of the Grassmann numbers. For example, the cube of a general Grassmann number X in 3-space may then be calculated as

```
𝕍3; Xe = EvenGrade[X]; Xo = OddGrade[X]; 𝒢[Xe ∧ Xe ∧ (Xe + 3 Xo)]
```

$\xi_0^3 + 3\, e_1\, \xi_0^2\, \xi_1 + 3\, e_2\, \xi_0^2\, \xi_2 + 3\, e_3\, \xi_0^2\, \xi_3 + 3\, \xi_0^2\, \xi_4\, e_1 \wedge e_2 + 3\, \xi_0^2\, \xi_5\, e_1 \wedge e_3 +$
$3\, \xi_0^2\, \xi_6\, e_2 \wedge e_3 + (\xi_0\, (-6\, \xi_2\, \xi_5 + 6\, (\xi_3\, \xi_4 + \xi_1\, \xi_6)) + 3\, \xi_0^2\, \xi_7)\, e_1 \wedge e_2 \wedge e_3$

This algorithm is used in the general power function `GrassmannPower` described below.

Powers of Grassmann numbers with no body

A further simplification in the computation of powers of Grassmann numbers arises when they have no body. Powers of numbers with no body will eventually become zero. In this section we determine formulae for the highest non-zero power of such numbers.

The formula developed above for positive powers of Grassmann number still applies to the specific case of numbers without a body. If we can determine an expression for the highest non-zero power of an even number with no body, we can substitute it into the formula to obtain the required result.

Let $\mathbf{S}$ be an *even* Grassmann number with no body. Express $\mathbf{S}$ as a sum of components $\underset{i}{\sigma}$, where *i* is the grade of the component. (Components may be a sum of several terms. Components with no underscript are 1-elements by default.)

As a first example we take a general element in 3-space.

$$\mathbf{S} = \sigma + \underset{2}{\sigma} + \underset{3}{\sigma};$$

The (exterior) square of $\mathbf{S}$ may be obtained by expanding and simplifying $\mathbf{S} \wedge \mathbf{S}$.

$$\mathcal{G}[\mathbf{S} \wedge \mathbf{S}]$$

$$\sigma \wedge \underset{2}{\sigma} + \underset{2}{\sigma} \wedge \sigma$$

which simplifies to:

$$\mathbf{S} \wedge \mathbf{S} == 2\, \sigma \wedge \underset{2}{\sigma}$$

It is clear from this that because this expression is of grade three, further multiplication by $\mathbf{S}$ would give zero. It thus represents an expression for the highest non-zero power (the square) of a general bodyless element in 3-space.

To generalize this result, we proceed as follows:
1) Determine an expression for the highest non-zero power of an even Grassmann number with no body in an even-dimensional space.
2) Substitute this expression into the general expression for the positive power of a Grassmann number in the last section.
3) Repeat these steps for the case of an odd-dimensional space.

The highest non-zero power p_{max} of an even Grassmann number (with no body) can be seen to be that which enables the smallest term (of grade 2 and hence commutative) to be multiplied by itself the largest number of times, without the result being zero. For a space of *even* dimension *n* this is obviously $p_{max} = \frac{n}{2}$.

For example, let $\underset{2}{\mathbf{X}}$ be a 2-element in a 4-space.

$$\mathbb{V}_4;\ \underset{2}{X} = \mathbf{CreateElement}\left[\underset{2}{\xi}\right]$$

$$\xi_1\, e_1 \wedge e_2 + \xi_2\, e_1 \wedge e_3 + \xi_3\, e_1 \wedge e_4 + \xi_4\, e_2 \wedge e_3 + \xi_5\, e_2 \wedge e_4 + \xi_6\, e_3 \wedge e_4$$

The square of this number is a 4-element. Any further multiplication by a bodyless Grassmann number will give zero.

$$\mathcal{G}\left[\underset{2}{X} \wedge \underset{2}{X}\right]$$

$$(-2\, \xi_2\, \xi_5 + 2\, (\xi_3\, \xi_4 + \xi_1\, \xi_6))\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

Substituting $\underset{2}{\sigma}$ for X_e and $\frac{n}{2}$ for *p* into formula 9.1 for the *p*th power gives:

$$X^p = \underset{2}{\sigma}^{\frac{n}{2}-1} \wedge \left(\underset{2}{\sigma} + \frac{n}{2}\, X_o\right)$$

The only odd component X_o capable of generating a non-zero term is the one of grade 1. Hence X_o is equal to σ and we have finally that:

The maximum non-zero power of a bodyless Grassmann number in a space of ***even*** *dimension n is equal to* $\frac{n}{2}$ *and may be computed from the formula*

$$X^{p_{max}} = X^{\frac{n}{2}} = \underset{2}{\sigma}^{\frac{n}{2}-1} \wedge \left(\underset{2}{\sigma} + \frac{n}{2}\, \sigma\right) \qquad n \text{ even} \qquad 9.2$$

A similar argument may be offered for odd-dimensional spaces. The highest non-zero power of an even Grassmann number (with no body) can be seen to be that which enables the smallest term (of grade 2 and hence commutative) to be multiplied by itself the largest number of times, without the result being zero. For a space of *odd* dimension *n* this is obviously $\frac{n-1}{2}$. However the highest power of a general bodyless number will be one greater than this due to the possibility of multiplying this even product by the element of grade 1. Hence $p_{max} = \frac{n+1}{2}$.

Substituting $\underset{2}{\sigma}$ for X_e, σ for X_o, and $\frac{n+1}{2}$ for *p* into the formula for the *p*th power and noting that the term involving the power of $\underset{2}{\sigma}$ only is zero, leads to the following:

The maximum non-zero power of a bodyless Grassmann number in a space of ***odd*** *dimension n is equal to* $\frac{n+1}{2}$ *and may be computed from the formula*

$$X^{p_{max}} = X^{\frac{n+1}{2}} = \frac{(n+1)}{2}\, \underset{2}{\sigma}^{\frac{n-1}{2}} \wedge \sigma \qquad n \text{ odd} \qquad 9.3$$

A formula which gives the highest power for both even and odd spaces is

$$p_{max} = \frac{1}{4}\, (2\, n + 1 - (-1)^n) \qquad 9.4$$

The following table gives the maximum non-zero power of a bodiless Grassmann number and the formula for it for spaces up to dimension 8.

n	p_{max}	$X^{p_{max}}$
1	1	σ
2	1	$\sigma + \underset{2}{\sigma}$
3	2	$2\,\sigma \wedge \underset{2}{\sigma}$
4	2	$\left(2\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma}$
5	3	$3\,\sigma \wedge \underset{2}{\sigma} \wedge \underset{2}{\sigma}$
6	3	$\left(3\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma} \wedge \underset{2}{\sigma}$
7	4	$4\,\sigma \wedge \underset{2}{\sigma} \wedge \underset{2}{\sigma} \wedge \underset{2}{\sigma}$
8	4	$\left(4\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma} \wedge \underset{2}{\sigma} \wedge \underset{2}{\sigma}$

Verifying the formulae in a 3-space

Declare a 3-space and create a Grassmann number $\mathbf{X_3}$ with no body.

$\mathbb{V}_3$**; X₃ = Soul[CreateGrassmannNumber[ξ]]**

$$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1 \wedge e_2 + \xi_5\,e_1 \wedge e_3 + \xi_6\,e_2 \wedge e_3 + \xi_7\,e_1 \wedge e_2 \wedge e_3$$

$$\left\{\sigma = \mathbf{ExtractGrade[1][X_3]},\ \underset{2}{\sigma} = \mathbf{ExtractGrade[2][X_3]}\right\}$$

$$\{e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3,\ \xi_4\,e_1 \wedge e_2 + \xi_5\,e_1 \wedge e_3 + \xi_6\,e_2 \wedge e_3\}$$

$$\mathcal{G}\left[X_3 \wedge X_3 == 2\,\sigma \wedge \underset{2}{\sigma}\right]$$

```
True
```

Verifying the formulae in a 4-space

Declare a 4-space and create a Grassmann number $\mathbf{X_4}$ with no body.

$\mathbb{V}_4$**; X₄ = Soul[CreateGrassmannNumber[ξ]]**

$$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + e_4\,\xi_4 + \xi_5\,e_1 \wedge e_2 + \xi_6\,e_1 \wedge e_3 + \xi_7\,e_1 \wedge e_4 + \xi_8\,e_2 \wedge e_3 + \xi_9\,e_2 \wedge e_4 + \xi_{10}\,e_3 \wedge e_4 + \xi_{11}\,e_1 \wedge e_2 \wedge e_3 + \xi_{12}\,e_1 \wedge e_2 \wedge e_4 + \xi_{13}\,e_1 \wedge e_3 \wedge e_4 + \xi_{14}\,e_2 \wedge e_3 \wedge e_4 + \xi_{15}\,e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

$$\left\{\sigma = \mathbf{ExtractGrade[1][X_4]},\ \underset{2}{\sigma} = \mathbf{ExtractGrade[2][X_4]}\right\}$$

$$\{e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + e_4\,\xi_4,\ \xi_5\,e_1 \wedge e_2 + \xi_6\,e_1 \wedge e_3 + \xi_7\,e_1 \wedge e_4 + \xi_8\,e_2 \wedge e_3 + \xi_9\,e_2 \wedge e_4 + \xi_{10}\,e_3 \wedge e_4\}$$

$$\mathcal{G}\left[X_4 \wedge X_4 == \left(2\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma}\right]$$

```
True
```

The inverse of a Grassmann number

Let **X** be a general Grassmann number in 3-space.

```
𝕍3; X = CreateGrassmannNumber[ξ]
```

$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge e_3$

Finding an inverse of **X** is equivalent to finding the Grassmann number **A** such that **X∧A==1** or **A∧X==1**. In what follows we shall show that **A** commutes with **X** and therefore is a unique inverse which we may call *the inverse* of **X**. To find **A** we need to solve the equation **X∧A-1==0**. First we create a general Grassmann number for **A**.

```
A = CreateGrassmannNumber[α]
```

$\alpha_0 + e_1\,\alpha_1 + e_2\,\alpha_2 + e_3\,\alpha_3 + \alpha_4\,e_1\wedge e_2 + \alpha_5\,e_1\wedge e_3 + \alpha_6\,e_2\wedge e_3 + \alpha_7\,e_1\wedge e_2\wedge e_3$

The calculate and simplify the expression **X∧A-1**.

```
𝒢[X ∧ A - 1]
```

$-1 + \alpha_0\,\xi_0 + e_1\,(\alpha_1\,\xi_0 + \alpha_0\,\xi_1) + e_2\,(\alpha_2\,\xi_0 + \alpha_0\,\xi_2) + e_3\,(\alpha_3\,\xi_0 + \alpha_0\,\xi_3) + (\alpha_4\,\xi_0 + \alpha_2\,\xi_1 - \alpha_1\,\xi_2 + \alpha_0\,\xi_4)\,e_1\wedge e_2 + (\alpha_5\,\xi_0 + \alpha_3\,\xi_1 - \alpha_1\,\xi_3 + \alpha_0\,\xi_5)\,e_1\wedge e_3 + (\alpha_6\,\xi_0 + \alpha_3\,\xi_2 - \alpha_2\,\xi_3 + \alpha_0\,\xi_6)\,e_2\wedge e_3 + (\alpha_7\,\xi_0 + \alpha_6\,\xi_1 - \alpha_5\,\xi_2 + \alpha_4\,\xi_3 + \alpha_3\,\xi_4 - \alpha_2\,\xi_5 + \alpha_1\,\xi_6 + \alpha_0\,\xi_7)\,e_1\wedge e_2\wedge e_3$

To make this expression zero we need to solve for the α_i in terms of the ξ_i which make the coefficients zero. This is most easily done with the `GrassmannSolve` function to be introduced later. Here, to see more clearly the process, we do it directly with *Mathematica*'s `Solve` function.

```
S = Solve[{-1 + α0 ξ0 == 0, (α1 ξ0 + α0 ξ1) == 0, (α2 ξ0 + α0 ξ2) == 0,
     (α3 ξ0 + α0 ξ3) == 0, (α4 ξ0 + α2 ξ1 - α1 ξ2 + α0 ξ4) == 0,
     (α5 ξ0 + α3 ξ1 - α1 ξ3 + α0 ξ5)  == 0,
     (α6 ξ0 + α3 ξ2 - α2 ξ3 + α0 ξ6)  == 0,
     (α7 ξ0 + α6 ξ1 - α5 ξ2 + α4 ξ3 + α3 ξ4 - α2 ξ5 + α1 ξ6 + α0 ξ7)  ==
      0}] // Flatten
```

$$\left\{\alpha_7 \to \frac{2\,\xi_3\,\xi_4 - 2\,\xi_2\,\xi_5 + 2\,\xi_1\,\xi_6 - \xi_0\,\xi_7}{\xi_0^3},\ \alpha_4 \to -\frac{\xi_4}{\xi_0^2},\ \alpha_5 \to -\frac{\xi_5}{\xi_0^2},\ \alpha_6 \to -\frac{\xi_6}{\xi_0^2},\ \alpha_1 \to -\frac{\xi_1}{\xi_0^2},\ \alpha_2 \to -\frac{\xi_2}{\xi_0^2},\ \alpha_3 \to -\frac{\xi_3}{\xi_0^2},\ \alpha_0 \to \frac{1}{\xi_0}\right\}$$

We denote the inverse of **X** by $\mathbf{X_r}$. We obtain an explicit expression for $\mathbf{X_r}$ by substituting the values obtained above in the formula for **A**.

X_r = A /. S

$$\frac{1}{\xi_0} - \frac{e_1\,\xi_1}{\xi_0^2} - \frac{e_2\,\xi_2}{\xi_0^2} - \frac{e_3\,\xi_3}{\xi_0^2} - \frac{\xi_4\,e_1 \wedge e_2}{\xi_0^2} - \frac{\xi_5\,e_1 \wedge e_3}{\xi_0^2} - \frac{\xi_6\,e_2 \wedge e_3}{\xi_0^2} + \frac{(2\,\xi_3\,\xi_4 - 2\,\xi_2\,\xi_5 + 2\,\xi_1\,\xi_6 - \xi_0\,\xi_7)\,e_1 \wedge e_2 \wedge e_3}{\xi_0^3}$$

To verify that this is indeed the inverse and that the inverse commutes, we calculate the products of **X** and **X_r**

$\mathcal{G}$[{X ∧ X_r, X_r ∧ X}] // Simplify

{1, 1}

To avoid having to rewrite the coefficients for the `Solve` equations, we can use the *GrassmannAlgebra* function `GrassmannSolve` (discussed in Section 9.6) to get the same results. To use `GrassmannSolve` we only need to enter a single undefined symbol (here we have used **Y**)for the Grassmann number we are looking to solve for.

GrassmannSolve[X ∧ Y == 1, Y]

$$\left\{\left\{Y \to \frac{1}{\xi_0} - \frac{e_1\,\xi_1}{\xi_0^2} - \frac{e_2\,\xi_2}{\xi_0^2} - \frac{e_3\,\xi_3}{\xi_0^2} - \frac{\xi_4\,e_1 \wedge e_2}{\xi_0^2} - \frac{\xi_5\,e_1 \wedge e_3}{\xi_0^2} - \frac{\xi_6\,e_2 \wedge e_3}{\xi_0^2} - \frac{(-2\,\xi_3\,\xi_4 + 2\,\xi_2\,\xi_5 - 2\,\xi_1\,\xi_6 + \xi_0\,\xi_7)\,e_1 \wedge e_2 \wedge e_3}{\xi_0^3}\right\}\right\}$$

It is evident from this example that (at least in 3-space), for a Grassmann number to have an inverse, it must have a non-zero body.

To calculate an inverse directly, we can use the function `GrassmannInverse`. To see the pattern for inverses in general, we calculate the inverses of a general Grassmann number in 1, 2, 3, and 4-spaces:

Inverse in a 1-space

$\mathbb{V}_1$; X_1 = CreateGrassmannNumber[ξ]

$$\xi_0 + e_1\,\xi_1$$

$X_{1\,r}$ = GrassmannInverse[X_1]

$$\frac{1}{\xi_0} - \frac{e_1\,\xi_1}{\xi_0^2}$$

Inverse in a 2-space

$\mathbb{V}_2$; X_2 = CreateGrassmannNumber[ξ]

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1 \wedge e_2$$

$X_{2\,r}$ = GrassmannInverse[X_2]

$$\frac{1}{\xi_0} - \frac{e_1\,\xi_1}{\xi_0^2} - \frac{e_2\,\xi_2}{\xi_0^2} - \frac{\xi_3\,e_1 \wedge e_2}{\xi_0^2}$$

Inverse in a 3-space

```
𝕍3; X3 = CreateGrassmannNumber[ξ]
```

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge e_3$$

```
X3r = GrassmannInverse[X3]
```

$$\frac{1}{\xi_0} - \frac{e_1\,\xi_1}{\xi_0^2} - \frac{e_2\,\xi_2}{\xi_0^2} - \frac{e_3\,\xi_3}{\xi_0^2} - \frac{\xi_4\,e_1\wedge e_2}{\xi_0^2} - \frac{\xi_5\,e_1\wedge e_3}{\xi_0^2} - \frac{\xi_6\,e_2\wedge e_3}{\xi_0^2} + \left(\frac{-2\,\xi_2\,\xi_5 + 2\,(\xi_3\,\xi_4 + \xi_1\,\xi_6)}{\xi_0^3} - \frac{\xi_7}{\xi_0^2}\right) e_1\wedge e_2\wedge e_3$$

Inverse in a 4-space

```
𝕍4; X4 = CreateGrassmannNumber[ξ]
```

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + e_4\,\xi_4 + \xi_5\,e_1\wedge e_2 + \xi_6\,e_1\wedge e_3 + \xi_7\,e_1\wedge e_4 + \xi_8\,e_2\wedge e_3 + \xi_9\,e_2\wedge e_4 + \xi_{10}\,e_3\wedge e_4 + \xi_{11}\,e_1\wedge e_2\wedge e_3 + \xi_{12}\,e_1\wedge e_2\wedge e_4 + \xi_{13}\,e_1\wedge e_3\wedge e_4 + \xi_{14}\,e_2\wedge e_3\wedge e_4 + \xi_{15}\,e_1\wedge e_2\wedge e_3\wedge e_4$$

```
X4r = GrassmannInverse[X4]
```

$$\frac{1}{\xi_0} - \frac{e_1\,\xi_1}{\xi_0^2} - \frac{e_2\,\xi_2}{\xi_0^2} - \frac{e_3\,\xi_3}{\xi_0^2} - \frac{e_4\,\xi_4}{\xi_0^2} - \frac{\xi_5\,e_1\wedge e_2}{\xi_0^2} - \frac{\xi_6\,e_1\wedge e_3}{\xi_0^2} - \frac{\xi_7\,e_1\wedge e_4}{\xi_0^2} - \frac{\xi_8\,e_2\wedge e_3}{\xi_0^2} - \frac{\xi_9\,e_2\wedge e_4}{\xi_0^2} - \frac{\xi_{10}\,e_3\wedge e_4}{\xi_0^2} + \left(\frac{-2\,\xi_2\,\xi_6 + 2\,(\xi_3\,\xi_5 + \xi_1\,\xi_8)}{\xi_0^3} - \frac{\xi_{11}}{\xi_0^2}\right) e_1\wedge e_2\wedge e_3 + \left(\frac{-2\,\xi_2\,\xi_7 + 2\,(\xi_4\,\xi_5 + \xi_1\,\xi_9)}{\xi_0^3} - \frac{\xi_{12}}{\xi_0^2}\right) e_1\wedge e_2\wedge e_4 + \left(\frac{-2\,\xi_3\,\xi_7 + 2\,(\xi_4\,\xi_6 + \xi_1\,\xi_{10})}{\xi_0^3} - \frac{\xi_{13}}{\xi_0^2}\right) e_1\wedge e_3\wedge e_4 + \left(\frac{-2\,\xi_3\,\xi_9 + 2\,(\xi_4\,\xi_8 + \xi_2\,\xi_{10})}{\xi_0^3} - \frac{\xi_{14}}{\xi_0^2}\right) e_2\wedge e_3\wedge e_4 + \left(\frac{-2\,\xi_6\,\xi_9 + 2\,(\xi_7\,\xi_8 + \xi_5\,\xi_{10})}{\xi_0^3} - \frac{\xi_{15}}{\xi_0^2}\right) e_1\wedge e_2\wedge e_3\wedge e_4$$

The form of the inverse of a Grassmann number

To understand the form of the inverse of a Grassmann number it is instructive to generate it by yet another method. Let $\boldsymbol{\beta}$ be a bodiless Grassmann number and $\boldsymbol{\beta^q}$ its qth exterior power. Then the following identity will hold.

$$(1 + \beta) \wedge (1 - \beta + \beta^2 - \beta^3 + \beta^4 - \ ...\ \pm \beta^q) == 1 \pm \beta^{q+1}$$

We can see this most easily by writing the expansion in the form below and noting that all the intermediate terms cancel.

$$1 - \beta + \beta^2 - \beta^3 + \beta^4 - \;\ldots \pm \beta^q$$
$$+ \beta - \beta^2 + \beta^3 - \beta^4 + \;\ldots \mp \beta^q \pm \beta^{q+1}$$

Alternatively, consider the product in the reverse order $(1 - \beta + \beta^2 - \beta^3 + \beta^4 - \;\ldots \pm \beta^q) \wedge (1 + \beta)$. Expanding this gives precisely the same result. *Hence a Grassmann number and its inverse commute.*

Furthermore, it has been shown in the previous section that for a bodiless Grassmann number in a space of *n* dimensions, the greatest non-zero power is $p_{max} = \frac{1}{4}\,(2\,n + 1 + (-1)^n)$. Thus if *q* is equal to p_{max}, β^{q+1} is equal to zero, and the identity becomes:

$$(1 + \beta) \wedge (1 - \beta + \beta^2 - \beta^3 + \beta^4 - \;\ldots \pm \beta^{p_{max}}) == 1 \qquad 9.5$$

We have thus shown that $1 - \beta + \beta^2 - \beta^3 + \beta^4 \;\ldots \pm \beta^{p_{max}}$ is the inverse of $1+\beta$.

If now we have a general Grassmann number X, say, we can write X as $X == \xi_0\;(1 + \beta)$, so that if X_s is the soul of X, then

$$X == \xi_0\;(1 + \beta) == \xi_0 + X_s \qquad \beta == \frac{X_s}{\xi_0}$$

The inverse of X then becomes:

$$X^{-1} = \frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \left(\frac{X_s}{\xi_0}\right)^2 - \left(\frac{X_s}{\xi_0}\right)^3 + \;\ldots \pm \left(\frac{X_s}{\xi_0}\right)^{p_{max}}\right) \qquad 9.6$$

We tabulate some examples for $X == \xi_0 + X_s$, expressing the inverse X^{-1} both in terms of X_s and ξ_0 alone, and in terms of the even and odd components σ and $\underset{2}{\sigma}$ of X_s.

n	p_{max}	X^{-1}	X^{-1}
1	1	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0}\right)$	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0}\right)$
2	1	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0}\right)$	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0}\right)$
3	2	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \left(\frac{X_s}{\xi_0}\right)^2\right)$	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \frac{2}{\xi_0{}^2}\,\sigma \wedge \underset{2}{\sigma}\right)$
4	2	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \left(\frac{X_s}{\xi_0}\right)^2\right)$	$\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \frac{1}{\xi_0{}^2}\left(2\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma}\right)$

It is easy to verify the formulae of the table for dimensions 1 and 2 from the results above. The formulae for spaces of dimension 3 and 4 are given below. But for a slight rearrangement of terms they are identical to the results above.

Verifying the formula in a 3-space

```
𝕍3; Xs = Soul[CreateGrassmannNumber[ξ]]
```

$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1 \wedge e_2 + \xi_5\,e_1 \wedge e_3 + \xi_6\,e_2 \wedge e_3 + \xi_7\,e_1 \wedge e_2 \wedge e_3$

$\{\sigma = \text{ExtractGrade}[1][X_s],\ \underset{2}{\sigma} = \text{ExtractGrade}[2][X_s]\}$

$\{e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3,\ \xi_4\,e_1 \wedge e_2 + \xi_5\,e_1 \wedge e_3 + \xi_6\,e_2 \wedge e_3\}$

$$\mathcal{G}\left[\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \frac{2}{\xi_0^{\,2}}\,\sigma \wedge \underset{2}{\sigma}\right) \wedge (\xi_0 + X_s)\right]$$

1

Verifying the formula in a 4-space

```
𝕍4; Xs = Soul[CreateGrassmannNumber[ξ]]
```

$e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + e_4\,\xi_4 + \xi_5\,e_1 \wedge e_2 + \xi_6\,e_1 \wedge e_3 + \xi_7\,e_1 \wedge e_4 + \xi_8\,e_2 \wedge e_3 + \xi_9\,e_2 \wedge e_4 + \xi_{10}\,e_3 \wedge e_4 + \xi_{11}\,e_1 \wedge e_2 \wedge e_3 + \xi_{12}\,e_1 \wedge e_2 \wedge e_4 + \xi_{13}\,e_1 \wedge e_3 \wedge e_4 + \xi_{14}\,e_2 \wedge e_3 \wedge e_4 + \xi_{15}\,e_1 \wedge e_2 \wedge e_3 \wedge e_4$

$\{\sigma = \text{ExtractGrade}[1][X_s],\ \underset{2}{\sigma} = \text{ExtractGrade}[2][X_s]\}$

$\{e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + e_4\,\xi_4,\ \xi_5\,e_1 \wedge e_2 + \xi_6\,e_1 \wedge e_3 + \xi_7\,e_1 \wedge e_4 + \xi_8\,e_2 \wedge e_3 + \xi_9\,e_2 \wedge e_4 + \xi_{10}\,e_3 \wedge e_4\}$

$$\mathcal{G}\left[\frac{1}{\xi_0}\left(1 - \frac{X_s}{\xi_0} + \frac{1}{\xi_0^{\,2}}\left(2\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma}\right) \wedge (\xi_0 + X_s)\right]$$

1

Integer powers of a Grassmann number

In the *GrassmannAlgebra* function `GrassmannPower` we collect together the algorithm introduced above for positive integer powers of Grassmann numbers with that for calculating inverses, so that `GrassmannPower` works with any integer power, positive or negative.

```
? GrassmannPower
```

```
GrassmannPower[X,n] computes the nth power of a Grassmann
  number X in a space of the currently declared number of
  dimensions. n may be any numeric or symbolic quantity.
```

Powers in terms of basis elements

```
𝕍2; X = CreateGrassmannNumber[ξ]
```

$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1 \wedge e_2$

```
Y = GrassmannPower[X, 3]
```

$\xi_0^3 + 3\,e_1\,\xi_0^2\,\xi_1 + 3\,e_2\,\xi_0^2\,\xi_2 + 3\,\xi_0^2\,\xi_3\,e_1 \wedge e_2$

```
Z = GrassmannPower[X, -3]
```

$\frac{1}{\xi_0^3} - \frac{3\,e_1\,\xi_1}{\xi_0^4} - \frac{3\,e_2\,\xi_2}{\xi_0^4} - \frac{3\,\xi_3\,e_1 \wedge e_2}{\xi_0^4}$

As usual, we verify with `GrassmannSimplify`.

```
𝒢[{Y ∧ Z, Z ∧ Y}]
```

{1, 1}

General powers

`GrassmannPower` will also work with general elements that are not expressed in terms of basis elements.

```
X = 1 + x + x ∧ y + x ∧ y ∧ z;
```

```
{Y = GrassmannPower[X, 3], Z = GrassmannPower[X, -3]}
```

$\{1 + 3\,x + 3\,x \wedge y,\ 1 - 3\,x - 3\,x \wedge y\}$

```
𝒢[{Y ∧ Z, Z ∧ Y}]
```

{1, 1}

Symbolic powers

We take a general Grassmann number in 2-space.

```
𝕍2; X = CreateGrassmannNumber[ξ]
```

$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1 \wedge e_2$

```
Y = GrassmannPower[X, n]
```

$\xi_0^n + n\,e_1\,\xi_0^{-1+n}\,\xi_1 + n\,e_2\,\xi_0^{-1+n}\,\xi_2 + n\,\xi_0^{-1+n}\,\xi_3\,e_1 \wedge e_2$

```
Z = GrassmannPower[X, -n]
```

$\xi_0^{-n} - n\,e_1\,\xi_0^{-1-n}\,\xi_1 - n\,e_2\,\xi_0^{-1-n}\,\xi_2 - n\,\xi_0^{-1-n}\,\xi_3\,e_1 \wedge e_2$

Before we verify that these are indeed inverses of each other, we need to declare the power *n* to be a scalar.

```
DeclareExtraScalars[n];
```

```
𝒢[{Y ∧ Z, Z ∧ Y}]
```

{1, 1}

The *n*th power of a general Grassmann number in 3-space is:

```
𝕍3; X = CreateGrassmannNumber[ξ]; Y = GrassmannPower[X, n]
```

$$\xi_0^n + n\, e_1\, \xi_0^{-1+n}\, \xi_1 + n\, e_2\, \xi_0^{-1+n}\, \xi_2 + n\, e_3\, \xi_0^{-1+n}\, \xi_3 + n\, \xi_0^{-1+n}\, \xi_4\, e_1 \wedge e_2 + n\, \xi_0^{-1+n}\, \xi_5\, e_1 \wedge e_3 + n\, \xi_0^{-1+n}\, \xi_6\, e_2 \wedge e_3 + (\xi_0^{-2+n}\, ((n - n^2)\, \xi_2\, \xi_5 + (-n + n^2)\, (\xi_3\, \xi_4 + \xi_1\, \xi_6)) + n\, \xi_0^{-1+n}\, \xi_7)\, e_1 \wedge e_2 \wedge e_3$$

```
Z = GrassmannPower[X, -n]
```

$$\xi_0^{-n} - n\, e_1\, \xi_0^{-1-n}\, \xi_1 - n\, e_2\, \xi_0^{-1-n}\, \xi_2 - n\, e_3\, \xi_0^{-1-n}\, \xi_3 - n\, \xi_0^{-1-n}\, \xi_4\, e_1 \wedge e_2 - n\, \xi_0^{-1-n}\, \xi_5\, e_1 \wedge e_3 - n\, \xi_0^{-1-n}\, \xi_6\, e_2 \wedge e_3 + (\xi_0^{-2-n}\, (\xi_3\, (n\, \xi_4 + n^2\, \xi_4) - n\, \xi_2\, \xi_5 - n^2\, \xi_2\, \xi_5 + n\, \xi_1\, \xi_6 + n^2\, \xi_1\, \xi_6) - n\, \xi_0^{-1-n}\, \xi_7)\, e_1 \wedge e_2 \wedge e_3$$

```
𝒢[{Y ∧ Z, Z ∧ Y}]
```

```
{1, 1}
```

General powers of a Grassmann number

As well as integer powers, `GrassmannPower` is able to compute non-integer powers.

The square root of a Grassmann number

The square root of a general Grassmann number **X** in 3-space is given by:

```
𝕍3; X = CreateGrassmannNumber[ξ]; Y = GrassmannPower[X, 1/2]
```

$$\sqrt{\xi_0} + \frac{e_1\, \xi_1}{2\sqrt{\xi_0}} + \frac{e_2\, \xi_2}{2\sqrt{\xi_0}} + \frac{e_3\, \xi_3}{2\sqrt{\xi_0}} + \frac{\xi_4\, e_1 \wedge e_2}{2\sqrt{\xi_0}} + \frac{\xi_5\, e_1 \wedge e_3}{2\sqrt{\xi_0}} + \frac{\xi_6\, e_2 \wedge e_3}{2\sqrt{\xi_0}} + \left(\frac{-\xi_3\, \xi_4 + \xi_2\, \xi_5 - \xi_1\, \xi_6}{4\, \xi_0^{3/2}} + \frac{\xi_7}{2\sqrt{\xi_0}}\right) e_1 \wedge e_2 \wedge e_3$$

We verify that this is indeed the square root.

```
Simplify[𝒢[X == Y ∧ Y]]
```

```
True
```

The πth power of a Grassmann number

Consider a general Grassmann number **X** in 2-space raised to the power of π and $-\pi$.

```
𝕍2; X = CreateGrassmannNumber[ξ]; Y = GrassmannPower[X, π]
```

$$\xi_0^{\pi} + \pi\, e_1\, \xi_0^{-1+\pi}\, \xi_1 + \pi\, e_2\, \xi_0^{-1+\pi}\, \xi_2 + \pi\, \xi_0^{-1+\pi}\, \xi_3\, e_1 \wedge e_2$$

```
Z = GrassmannPower[X, -π]
```

$\xi_0^{-\pi} - \pi\, e_1\, \xi_0^{-1-\pi}\, \xi_1 - \pi\, e_2\, \xi_0^{-1-\pi}\, \xi_2 - \pi\, \xi_0^{-1-\pi}\, \xi_3\, e_1 \wedge e_2$

We verify that these are indeed inverses.

```
𝒢[{Y ∧ Z, Z ∧ Y}]
```

{1, 1}

9.6 Solving Equations

Solving for unknown coefficients

Suppose we have a Grassmann number whose coefficients are functions of a number of parameters, and we wish to find the values of the parameters which will make the number zero. For example:

```
Z = a - b - 1 + (b + 3 - 5 a) e1 + (d - 2 c + 6) e3 + (e + 4 a) e1 ∧ e2 +
    (c + 2 + 2 g) e2 ∧ e3 + (-1 + e - 7 g + b) e1 ∧ e2 ∧ e3;
```

We can calculate the values of the parameters by using the *GrassmannAlgebra* function `GrassmannScalarSolve`.

```
GrassmannScalarSolve[Z == 0]
```

$\left\{\left\{a \to \frac{1}{2},\ b \to -\frac{1}{2},\ c \to -1,\ d \to -8,\ e \to -2,\ g \to -\frac{1}{2}\right\}\right\}$

`GrassmannScalarSolve` will also work for several equations and for general symbols (other than basis elements).

```
Z1 = {a (-x) + b y ∧ (-2 x) + f == c x + d x ∧ y,
    (7 - a) y == (13 - d) y ∧ x - f};
```

```
GrassmannScalarSolve[Z1]
```

$\left\{\left\{a \to 7,\ c \to -7,\ d \to 13,\ b \to \frac{13}{2},\ f \to 0\right\}\right\}$

`GrassmannScalarSolve` can be used in several other ways. As with other *Mathematica* functions its usage statement can be obtained by entering **`?GrassmannScalarSolve`**.

? GrassmannScalarSolve

```
GrassmannScalarSolve[eqns] attempts to find the values of those (
  declared) scalar variables which make the equations True.
  GrassmannScalarSolve[eqns,scalars] attempts to find the
  values of the scalars which make the equations True. If
  not already in the DeclaredScalars list the scalars will
  be added to it. GrassmannScalarSolve[eqns,scalars,elims]
  attempts to find the values of the scalars which make the
  equations True while eliminating the scalars elims. If
  the equations are not fully solvable, GrassmannScalarSolve
  will still find the values of the scalar variables which
  reduce the number of terms in the equations as much as
  possible, and will additionally return the reduced equations.
```

Solving for an unknown Grassmann number

The function for solving for an unknown Grassmann number, or several unknown Grassmann numbers is `GrassmannSolve`.

? GrassmannSolve

```
GrassmannSolve[eqns,vars] attempts to solve an equation or set
  of equations for the Grassmann number variables vars. If
  the equations are not fully solvable, GrassmannSolve will
  still find the values of the Grassmann number variables
  which reduce the number of terms in the equations as much as
  possible, and will additionally return the reduced equations.
```

Suppose first that we have an equation involving an unknown Grassmann number, **A**, say. For example if **X** is a general number in 3-space, and we want to find its inverse **A** as we did in the Section 9.5 above, we can solve the following equation for **A**:

$\mathbb{V}_3$; X = CreateGrassmannNumber[ξ]

$$\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + e_3\,\xi_3 + \xi_4\,e_1\wedge e_2 + \xi_5\,e_1\wedge e_3 + \xi_6\,e_2\wedge e_3 + \xi_7\,e_1\wedge e_2\wedge e_3$$

R = GrassmannSolve[X ∧ A == 1, A]

$$\left\{\left\{A\to\frac{1}{\xi_0}-\frac{e_1\,\xi_1}{\xi_0^2}-\frac{e_2\,\xi_2}{\xi_0^2}-\frac{e_3\,\xi_3}{\xi_0^2}-\frac{\xi_4\,e_1\wedge e_2}{\xi_0^2}-\frac{\xi_5\,e_1\wedge e_3}{\xi_0^2}-\frac{\xi_6\,e_2\wedge e_3}{\xi_0^2}-\frac{(-2\,\xi_3\,\xi_4+2\,\xi_2\,\xi_5-2\,\xi_1\,\xi_6+\xi_0\,\xi_7)\,e_1\wedge e_2\wedge e_3}{\xi_0^3}\right\}\right\}$$

As usual, we verify the result.

$\mathcal{G}$[X ∧ A == 1 /. R]

{True}

In general, `GrassmannSolve` can solve the same sorts of equations that *Mathematica*'s inbuilt `Solve` routines can deal with because it uses `Solve` as its main calculation engine. In particular, it can handle powers of the unknowns. Further, it does not require the equations necessarily to be expressed in terms of basis elements. For example, we can take a quadratic equation in α:

```
Q = α ∧ α + 2 (1 + x + x ∧ y) ∧ α + (1 + 2 x + 2 x ∧ y) == 0;
```

```
S = GrassmannSolve[Q, α]
```

$$\{\{\alpha \to -1 + x\, C_1 + C_2\, x \wedge y\}\}$$

Here there are two arbitrary parameters C_1 and C_2 introduced by the `GrassmannSolve` function. (**C** is a symbol reserved by *Mathematica* for the generation of unknown coefficients in the solution of equations.)
Now test whether α is indeed a solution to the equation:

```
𝒢[Q /. S]
```

{True}

Note that here we have a double infinity of solutions, parametrized by the pair $\{C_1, C_2\}$. This means that for *any* values of C_1 and C_2, **S** is a solution. For example for various C_1 and C_2:

```
{𝒢[Q /. α → -1 - x], 𝒢[Q /. α → -1], 𝒢[Q /. α → -1 - 9 x ∧ y]}
```

{True, True, True}

`GrassmannSolve` can also take several equations in several unknowns. As an example we take the following pair of equations:

```
Q = {(1 + x) ∧ β + x ∧ y ∧ α == 3 + 2 x ∧ y,
    (1 - y) ∧ α ∧ α + (2 + y + x + 5 x ∧ y) == 7 - 5 x};
```

First, we solve for the two unknown Grassmann numbers α and β.

```
S = GrassmannSolve[Q, {α, β}]
```

$$\left\{\left\{\alpha \to -\sqrt{5} + \frac{3\,x}{\sqrt{5}} - \frac{2\,y}{\sqrt{5}} - \frac{x \wedge y}{2\sqrt{5}},\ \beta \to 3 - 3\,x + \left(2 + \sqrt{5}\right) x \wedge y\right\},\right.$$
$$\left.\left\{\alpha \to \sqrt{5} - \frac{3\,x}{\sqrt{5}} + \frac{2\,y}{\sqrt{5}} + \frac{x \wedge y}{2\sqrt{5}},\ \beta \to 3 - 3\,x + \left(2 - \sqrt{5}\right) x \wedge y\right\}\right\}$$

Here we get two solutions, which we verify with `GrassmannSimplify`.

```
𝒢[Q /. S]
```

{{True, True}, {True, True}}

Note that *GrassmannAlgebra* assumes that all other symbols (here, **x** and **y**) are 1-elements. We can add the symbol **x** to the list of Grassmann numbers to be solved for.

```
S = GrassmannSolve[Q, {α, β, x}]
```

$$\left\{\left\{\alpha \to C_2 + \frac{y\,(-31 - 36\,C_1 + 11\,C_2^2)}{12\,C_2},\ \beta \to -\frac{18}{-11 + C_2^2} + y\left(2 - C_2 - \frac{108\,C_1}{(-11 + C_2^2)^2} + \frac{12}{-11 + C_2^2} - \frac{6\,C_2}{-11 + C_2^2}\right),\ x \to y\,C_1 + \frac{1}{6}\,(5 - C_2^2)\right\},\ \left\{\alpha \to y\,C_3,\ \beta \to \frac{18}{11} + \frac{203\,y}{121},\ x \to \frac{5}{6} - \frac{31\,y}{36}\right\}\right\}$$

Again we get two solutions, but this time involving some arbitrary scalar constants.

```
𝒢[Q /. S] // Simplify
```

{{True, True}, {True, True}}

9.7 Exterior Division

Defining exterior quotients

An *exterior quotient* of two Grassmann numbers **X** and **Y** is a solution **Z** to the equation:

$$X == Y \wedge Z$$

or to the equation:

$$X == Z \wedge Y$$

To distinguish these two possibilities, we call the solution **Z** to the first equation **X==Y∧Z** the *left exterior quotient* of **X** by **Y** because the denominator **Y** multiplies the quotient **Z** *from the left*.

Correspondingly, we call the solution **Z** to the second equation **X==Z∧Y** the *right exterior quotient* of **X** by **Y** because the denominator **Y** multiplies the quotient **Z** *from the right*.

To solve for **Z** in either case, we can use the `GrassmannSolve` function. For example if **X** and **Y** are general elements in a 2-space, the left exterior quotient is obtained as:

```
𝕍2; X = CreateGrassmannNumber[ξ];
Y = CreateGrassmannNumber[ψ];

GrassmannSolve[X == Y ∧ Z, {Z}]
```

$$\left\{\left\{Z \to \frac{\xi_0}{\psi_0} - \frac{e_1\,(-\xi_1\,\psi_0 + \xi_0\,\psi_1)}{\psi_0^2} - \frac{e_2\,(-\xi_2\,\psi_0 + \xi_0\,\psi_2)}{\psi_0^2} - \frac{(-\xi_3\,\psi_0 + \xi_2\,\psi_1 - \xi_1\,\psi_2 + \xi_0\,\psi_3)\,e_1 \wedge e_2}{\psi_0^2}\right\}\right\}$$

Whereas the right exterior quotient is obtained as:

GrassmannSolve[X == Z ∧ Y, {Z}]

$$\left\{\left\{Z \to \frac{\xi_0}{\psi_0} - \frac{e_1 \, (-\xi_1 \, \psi_0 + \xi_0 \, \psi_1)}{\psi_0^2} - \frac{e_2 \, (-\xi_2 \, \psi_0 + \xi_0 \, \psi_2)}{\psi_0^2} - \frac{(-\xi_3 \, \psi_0 - \xi_2 \, \psi_1 + \xi_1 \, \psi_2 + \xi_0 \, \psi_3) \, e_1 \wedge e_2}{\psi_0^2}\right\}\right\}$$

Note the differences in signs in the $e_1 \wedge e_2$ term.

To obtain these quotients more directly, *GrassmannAlgebra* provides the functions `LeftGrassmannDivide` and `RightGrassmannDivide`.

? LeftGrassmannDivide

```
LeftGrassmannDivide[X,Y] calculates
  the Grassmann number Z such that X == Y∧Z.
```

? RightGrassmannDivide

```
RightGrassmannDivide[X,Y] calculates
  the Grassmann number Z such that X == Z∧Y.
```

A shorthand infix notation for the division operation is provided by the down-vector symbol `LeftDownVector` ↽ (`LeftGrassmannDivide`) and `RightDownArrow` ⇂ (`RightGrassmannDivide`). The previous examples would then take the form

X ↽ Y

$$\frac{\xi_0}{\psi_0} - \frac{e_1 \, (-\xi_1 \, \psi_0 + \xi_0 \, \psi_1)}{\psi_0^2} - \frac{e_2 \, (-\xi_2 \, \psi_0 + \xi_0 \, \psi_2)}{\psi_0^2} - \frac{(-\xi_3 \, \psi_0 + \xi_2 \, \psi_1 - \xi_1 \, \psi_2 + \xi_0 \, \psi_3) \, e_1 \wedge e_2}{\psi_0^2}$$

X ⇂ Y

$$\frac{\xi_0}{\psi_0} - \frac{e_1 \, (-\xi_1 \, \psi_0 + \xi_0 \, \psi_1)}{\psi_0^2} - \frac{e_2 \, (-\xi_2 \, \psi_0 + \xi_0 \, \psi_2)}{\psi_0^2} - \frac{(-\xi_3 \, \psi_0 - \xi_2 \, \psi_1 + \xi_1 \, \psi_2 + \xi_0 \, \psi_3) \, e_1 \wedge e_2}{\psi_0^2}$$

Special cases of exterior division

The inverse of a Grassmann number

The inverse of a Grassmann number may be obtained as either the left or right form quotient.

{1 ↽ X, 1 ⇂ X}

$$\left\{\frac{1}{\xi_0} - \frac{e_1 \, \xi_1}{\xi_0^2} - \frac{e_2 \, \xi_2}{\xi_0^2} - \frac{\xi_3 \, e_1 \wedge e_2}{\xi_0^2}, \frac{1}{\xi_0} - \frac{e_1 \, \xi_1}{\xi_0^2} - \frac{e_2 \, \xi_2}{\xi_0^2} - \frac{\xi_3 \, e_1 \wedge e_2}{\xi_0^2}\right\}$$

Division by a scalar

Division by a scalar is just the Grassmann number divided by the scalar as expected.

```
{X ⨼ a, X ⨽ a}
```

$$\left\{\frac{\xi_0}{a}+\frac{e_1\,\xi_1}{a}+\frac{e_2\,\xi_2}{a}+\frac{\xi_3\,e_1\wedge e_2}{a},\ \frac{\xi_0}{a}+\frac{e_1\,\xi_1}{a}+\frac{e_2\,\xi_2}{a}+\frac{\xi_3\,e_1\wedge e_2}{a}\right\}$$

Division of a Grassmann number by itself

Division of a Grassmann number by itself give unity.

```
{X ⨼ X, X ⨽ X}
```

$\{1,\ 1\}$

Division by scalar multiples

Division involving scalar multiples also gives the results expected.

```
{(a X) ⨼ X, X ⨽ (a X)}
```

$\left\{a,\ \frac{1}{a}\right\}$

Division by an even Grassmann number

Division by an even Grassmann number will give the same result for both division operations, since exterior multiplication by an even Grassmann number is commutative.

```
{X ⨼ (1 + e₁ ∧ e₂), X ⨽ (1 + e₁ ∧ e₂)}
```

$$\{\xi_0+e_1\,\xi_1+e_2\,\xi_2+(-\xi_0+\xi_3)\,e_1\wedge e_2,\ \xi_0+e_1\,\xi_1+e_2\,\xi_2+(-\xi_0+\xi_3)\,e_1\wedge e_2\}$$

The non-uniqueness of exterior division

Because of the nilpotency of the exterior product, the division of Grassmann numbers does not necessarily lead to a unique result. To take a simple case consider the division of a simple 2-element $x\wedge y$ by one of its 1-element factors x.

```
(x ∧ y) ⨼ x
```

$y+x\,C_1+C_2\,x\wedge y$

The quotient is returned as a linear combination of elements containing *arbitrary* scalar constants C_1 and C_2. To see that this is indeed a left quotient, we can multiply it by the denominator *from the left* and see that we get the numerator.

```
𝒢[x ∧ (y + x C₁ + C₂ x ∧ y)]
```

$x\wedge y$

We get a slightly different result for the right quotient which we verify by multiplying from the right.

```
(x ∧ y) ⊢ x
```

$-y + x\,C_1 + C_2\, x \wedge y$

```
𝒢[(-y + x C1 + C2 x ∧ y) ∧ x]
```

$x \wedge y$

Here is a slightly more complex example.

```
𝕍3; (x ∧ y ∧ z) ⊣ (x - 2 y + 3 z + x ∧ z)
```

$$z\left(\frac{3}{2} - \frac{9\,C_1}{2} - 3\,C_2 - \frac{3\,C_3}{2}\right) + x\left(\frac{1}{2} - \frac{3\,C_1}{2} - C_2 - \frac{C_3}{2}\right) + y\,(-1 + 3\,C_1 + 2\,C_2 + C_3) + C_1\, x \wedge y + C_2\, x \wedge z + C_3\, y \wedge z + C_4\, x \wedge y \wedge z$$

In some circumstances we may not want the most general result. Say, for example, we knew that the result we wanted had to be a 1-element. We can use the *GrassmannAlgebra* function `ExtractGrade`.

```
ExtractGrade[1][(x ∧ y ∧ z) ⊣ (x - 2 y + 3 z + x ∧ z)]
```

$$z\left(\frac{3}{2} - \frac{9\,C_1}{2} - 3\,C_2 - \frac{3\,C_3}{2}\right) + x\left(\frac{1}{2} - \frac{3\,C_1}{2} - C_2 - \frac{C_3}{2}\right) + y\,(-1 + 3\,C_1 + 2\,C_2 + C_3)$$

Should there be insufficient information to determine a result, the quotient will be returned unchanged. For example.

```
(x ∧ y) ⊣ z
```

$(x \wedge y) \dashv z$

9.8 Factorization of Grassmann Numbers

The non-uniqueness of factorization

Factorization of a Grassmann number will rarely be unique due to the nilpotency property of the exterior product. However, using the capability of `GrassmannScalarSolve` to generate general solutions with arbitrary constants we may be able to find a factorization in the form we require. The *GrassmannAlgebra* function which implements the attempt to factorize is `GrassmannFactor.`

```
? GrassmannFactor
```

```
GrassmannFactor[X,F] attempts to factorize the Grassmann number X
  into the form given by F. F must be an expression involving the
  symbols of X, together with sufficient scalar coefficients whose
  values are to be determined. GrassmannFactor[X,F,S]  attempts to
  factorize the Grassmann number X by determining the scalars S.
  If the scalars S are not already declared as scalars, they will
  be automatically declared. If no factorization can be effected
  in the form given, the original expression will be returned.
```

Example: Factorizing a Grassmann number in 2-space

First, consider a general Grassmann number in 2-space.

$\mathbb{V}_2$; **X = CreateGrassmannNumber[ξ]**

$$\xi_0 + e_1 \xi_1 + e_2 \xi_2 + \xi_3 e_1 \wedge e_2$$

By using `GrassmannFactor` we find a general factorization into two Grassmann numbers of lower maximum grade. We want to see if there is a factorization in the form $(a + b e_1 + c e_2) \wedge (f + g e_1 + h e_2)$, so we enter this as the second argument. The scalars that we want to find are **{a,b,c,f,g,h}**, so we enter this list as the third argument.

Xf = GrassmannFactor[X,
$(a + b e_1 + c e_2) \wedge (f + g e_1 + h e_2)$, {a, b, c, f, g, h}]

$$\left(\frac{\xi_0}{f} + \frac{e_2 (-h \xi_0 + f \xi_2)}{f^2} + \frac{e_1 (-h \xi_0 \xi_1 + f \xi_1 \xi_2 + f \xi_0 \xi_3)}{f^2 \xi_2}\right) \wedge \left(f + h e_2 + \frac{e_1 (h \xi_1 - f \xi_3)}{\xi_2}\right)$$

This factorization is valid for any values of the scalars **{a,b,c,f,g,h}** which appear in the result, as we can verify by applying `GrassmannSimplify` to the result.

𝒢[Xf]

$$\xi_0 + e_1 \xi_1 + e_2 \xi_2 + \xi_3 e_1 \wedge e_2$$

Example: Factorizing a 2-element in 3-space

Next, consider a general 2-element in 3-space. We know that 2-elements in 3-space are simple, hence we would expect to be able to obtain a factorization.

$\mathbb{V}_3$; **Y = CreateElement$[\underset{2}{\psi}]$**

$$\psi_1 e_1 \wedge e_2 + \psi_2 e_1 \wedge e_3 + \psi_3 e_2 \wedge e_3$$

By using `GrassmannFactor` we can determine a factorization. In this example we obtain two classes of solution, each with different parameters.

```
Xf = GrassmannFactor[Y,
  (a e1 + b e2 + c e3) ∧ (d e1 + e e2 + f e3), {a, b, c, d, e, f}]
```

$$\left\{\left(c\,e_3 + \frac{e_1\left(\psi_2 + \frac{c\,(-f\,\psi_1 + e\,\psi_2)}{\psi_3}\right)}{f} + \frac{e_2\,(c\,e + \psi_3)}{f}\right) \wedge \left(e\,e_2 + f\,e_3 + \frac{e_1\,(-f\,\psi_1 + e\,\psi_2)}{\psi_3}\right),\ \left(b\,e_2 + \frac{e_1\left(\psi_1 + \frac{b\,e\,\psi_2}{\psi_3}\right)}{e} - \frac{e_3\,\psi_3}{e}\right) \wedge \left(e\,e_2 + \frac{e\,e_1\,\psi_2}{\psi_3}\right)\right\}$$

Both may be verified as valid factorizations.

```
𝒢[Xf]
```

$$\{\psi_1\,e_1 \wedge e_2 + \psi_2\,e_1 \wedge e_3 + \psi_3\,e_2 \wedge e_3,\ \psi_1\,e_1 \wedge e_2 + \psi_2\,e_1 \wedge e_3 + \psi_3\,e_2 \wedge e_3\}$$

Example: Factorizing a 3-element in 4-space

Finally, consider a specific 3-element in 4-space.

```
𝕍4; X = -14 w ∧ x ∧ y + 26 w ∧ x ∧ z + 11 w ∧ y ∧ z + 13 x ∧ y ∧ z;
```

Since every (n–1)-element in an n-space is simple, we know the 3-element can be factorized. Suppose also that we would like the factorization in the following form:

```
J = (x α1 + y α2) ∧ (y β2 + z β3) ∧ (z γ3 + w γ4)
```

$$(x\,\alpha_1 + y\,\alpha_2) \wedge (y\,\beta_2 + z\,\beta_3) \wedge (z\,\gamma_3 + w\,\gamma_4)$$

By using `GrassmannFactor` we can explore if a factorization into such a form exists.

```
XJ = GrassmannFactor[X, J]
```

$$\left(x\,\alpha_1 + \frac{11\,y\,\alpha_1}{26}\right) \wedge \left(-\frac{14\,y}{\alpha_1\,\gamma_4} + \frac{26\,z}{\alpha_1\,\gamma_4}\right) \wedge \left(w\,\gamma_4 - \frac{13\,z\,\gamma_4}{14}\right)$$

We check that this is indeed a factorization by applying `GrassmannSimplify`.

```
𝒢[XJ]
```

$$-14\,x \wedge y \wedge w + 13\,x \wedge y \wedge z + 26\,x \wedge z \wedge w + 11\,y \wedge z \wedge w$$

9.9 Functions of Grassmann Numbers

The Taylor series formula

Functions of Grassmann numbers can be defined by power series as with ordinary numbers. Let **X** be a Grassmann number composed of a body ξ_0 and a soul X_s such that $X == \xi_0 + X_s$. Then the Taylor series expanded about the body ξ_0 (and truncated at the third term) is:

```
Normal[Series[f[X], {X, ξ0, 3}]]
```

$$f[\xi_0] + (X - \xi_0)\, f'[\xi_0] + \frac{1}{2}\,(X - \xi_0)^2\, f''[\xi_0] + \frac{1}{6}\,(X - \xi_0)^3\, f^{(3)}[\xi_0]$$

It has been shown in Section 9.5 that the degree p_{max} of the highest non-zero power of a Grassmann number in a space of *n* dimensions is given by $p_{max} = \frac{1}{4}\,(2\,n + 1 - (-1)^n)$.
If we expand the series up to the term in this power, it then becomes a formula for a function **f** of a Grassmann number in that space. The tables below give explicit expressions for a function **f[X]** of a Grassmann variable in spaces of dimensions 1 to 4.

n	p_{max}	f[X]
1	1	$f[\xi_0] + f'[\xi_0]\, X_s$
2	1	$f[\xi_0] + f'[\xi_0]\, X_s$
3	2	$f[\xi_0] + f'[\xi_0]\, X_s + \frac{1}{2}\, f''[\xi_0]\, X_s^2$
4	2	$f[\xi_0] + f'[\xi_0]\, X_s + \frac{1}{2}\, f''[\xi_0]\, X_s^2$

If we replace the square of the soul of **X** by its simpler formulation in terms of the components of **X** of the first and second grade we get a computationally more convenient expression in the case on dimensions 3 and 4.

n	p_{max}	f[X]
1	1	$f[\xi_0] + f'[\xi_0]\, X_s$
2	1	$f[\xi_0] + f'[\xi_0]\, X_s$
3	2	$f[\xi_0] + f'[\xi_0]\, X_s + f''[\xi_0]\, \sigma \wedge \underset{2}{\sigma}$
4	2	$f[\xi_0] + f'[\xi_0]\, X_s + \frac{1}{2}\, f''[\xi_0]\, \left(2\,\sigma + \underset{2}{\sigma}\right) \wedge \underset{2}{\sigma}$

The form of a function of a Grassmann number

To automatically generate the *form* of formula for a function of a Grassmann number, we can use the *GrassmannAlgebra* function `GrassmannFunctionForm`.

? GrassmannFunctionForm

```
GrassmannFunctionForm[{f[x],x},Xb,Xs] expresses the function f[x]
  of a Grassmann number in a space of the currently declared
  number of dimensions in terms of the symbols Xb and Xs, where
  Xb represents the body, and Xs represents the soul of the
  number. GrassmannFunctionForm[{f[x,y,z,...],{x,y,z,...}},{Xb,
  Yb,Zb,...},{Xs,Ys,Zs,...}] expresses the function f[x,y,z,...]
  of Grassmann numbers x,y,z,... in terms of the symbols Xb,Yb,
  Zb,... and Xs,Ys,Zs..., where  Xb,Yb,Zb,... represent the
  bodies, and Xs,Ys,Zs... represent the souls of the numbers.
  GrassmannFunctionForm[f[Xb,Xs]] expresses the function f[X]
  of a Grassmann number X=Xb+Xs in a space of the currently
  declared number of dimensions in terms of body and soul
  symbols Xb and Xs. GrassmannFunctionForm[f[{Xb,Yb,Zb,...},
  {Xs,Ys,Zs,...}]] is equivalent to the second form above.
```

GrassmannFunctionForm for a single variable

The form of a function will be different, depending on the dimension of the space it is in - the higher the dimension, the potentially more terms there are in the series. We can generate a table similar to the first of the pair above by using `GrassmannFunctionForm`.

**Table[{i, ($\mathbb{V}_i$; GrassmannFunctionForm[f[X_b, X_s]])},
{i, 1, 8}] // TableForm**

1	$f[X_b] + X_s f'[X_b]$
2	$f[X_b] + X_s f'[X_b]$
3	$f[X_b] + X_s f'[X_b] + \frac{1}{2} X_s^2 f''[X_b]$
4	$f[X_b] + X_s f'[X_b] + \frac{1}{2} X_s^2 f''[X_b]$
5	$f[X_b] + X_s f'[X_b] + \frac{1}{2} X_s^2 f''[X_b] + \frac{1}{6} X_s^3 f^{(3)}[X_b]$
6	$f[X_b] + X_s f'[X_b] + \frac{1}{2} X_s^2 f''[X_b] + \frac{1}{6} X_s^3 f^{(3)}[X_b]$
7	$f[X_b] + X_s f'[X_b] + \frac{1}{2} X_s^2 f''[X_b] + \frac{1}{6} X_s^3 f^{(3)}[X_b] + \frac{1}{24} X_s^4 f^{(4)}[X_b$
8	$f[X_b] + X_s f'[X_b] + \frac{1}{2} X_s^2 f''[X_b] + \frac{1}{6} X_s^3 f^{(3)}[X_b] + \frac{1}{24} X_s^4 f^{(4)}[X_b$

GrassmannFunctionForm for several variables

`GrassmannFunctionForm` will also give the form of the function for functions of several Grassmann variables. Due to their length, we will have to display them individually rather than in table form. We display only the odd dimensional results, since the following even dimensional ones are the same.

$\mathbb{V}_1$; GrassmannFunctionForm[f[{X_b, Y_b}, {X_s, Y_s}]]

$$f[X_b, Y_b] + Y_s f^{(0,1)}[X_b, Y_b] + X_s f^{(1,0)}[X_b, Y_b] + X_s \wedge Y_s f^{(1,1)}[X_b, Y_b]$$

```
𝕍3; GrassmannFunctionForm[f[{Xb, Yb}, {Xs, Ys}]]
```

$$f[X_b, Y_b] + Y_s\, f^{(0,1)}[X_b, Y_b] + \frac{1}{2} Y_s^2\, f^{(0,2)}[X_b, Y_b] + X_s\, f^{(1,0)}[X_b, Y_b] + X_s \wedge Y_s\, f^{(1,1)}[X_b, Y_b] + \frac{1}{2} X_s \wedge Y_s^2\, f^{(1,2)}[X_b, Y_b] + \frac{1}{2} X_s^2\, f^{(2,0)}[X_b, Y_b] + \frac{1}{2} X_s^2 \wedge Y_s\, f^{(2,1)}[X_b, Y_b] + \frac{1}{4} X_s^2 \wedge Y_s^2\, f^{(2,2)}[X_b, Y_b]$$

```
𝕍5; GrassmannFunctionForm[f[{Xb, Yb}, {Xs, Ys}]]
```

$$f[X_b, Y_b] + Y_s\, f^{(0,1)}[X_b, Y_b] + \frac{1}{2} Y_s^2\, f^{(0,2)}[X_b, Y_b] + \frac{1}{6} Y_s^3\, f^{(0,3)}[X_b, Y_b] + X_s\, f^{(1,0)}[X_b, Y_b] + X_s \wedge Y_s\, f^{(1,1)}[X_b, Y_b] + \frac{1}{2} X_s \wedge Y_s^2\, f^{(1,2)}[X_b, Y_b] + \frac{1}{6} X_s \wedge Y_s^3\, f^{(1,3)}[X_b, Y_b] + \frac{1}{2} X_s^2\, f^{(2,0)}[X_b, Y_b] + \frac{1}{2} X_s^2 \wedge Y_s\, f^{(2,1)}[X_b, Y_b] + \frac{1}{4} X_s^2 \wedge Y_s^2\, f^{(2,2)}[X_b, Y_b] + \frac{1}{12} X_s^2 \wedge Y_s^3\, f^{(2,3)}[X_b, Y_b] + \frac{1}{6} X_s^3\, f^{(3,0)}[X_b, Y_b] + \frac{1}{6} X_s^3 \wedge Y_s\, f^{(3,1)}[X_b, Y_b] + \frac{1}{12} X_s^3 \wedge Y_s^2\, f^{(3,2)}[X_b, Y_b] + \frac{1}{36} X_s^3 \wedge Y_s^3\, f^{(3,3)}[X_b, Y_b]$$

Calculating functions of Grassmann numbers

To *calculate* functions of Grassmann numbers, *GrassmannAlgebra* uses the `GrassmannFunction` operation based on `GrassmannFunctionForm`.

```
? GrassmannFunction
```

```
GrassmannFunction[{f[x],x},X] or GrassmannFunction[f[X]]
  computes the function f of a single Grassmann number X in
  a space of the currently declared number of dimensions.
  In the first form f[x] may be any formula; in the second,
  f must be a pure function. GrassmannFunction[{f[x,y,z,..],
  {x,y,z,..}},{X,Y,Z,..}] or GrassmannFunction[f[X,Y,Z,..]]
  computes the function f of Grassmann numbers X,Y,Z,...
  In the first form f[x,y,z,..] may be any formula (with
  xi corresponding to Xi); in the second, f must be a pure
  function. In both these forms the result of the function
  may be dependent on the ordering of the arguments X,Y,Z,...
```

In the sections below we explore the `GrassmannFunction` operation and give some examples. We will work in a 3-space with the general Grassmann number **X**.

```
𝕍3; X = CreateGrassmannNumber[ξ]
```

$$\xi_0 + e_1\, \xi_1 + e_2\, \xi_2 + e_3\, \xi_3 + \xi_4\, e_1 \wedge e_2 + \xi_5\, e_1 \wedge e_3 + \xi_6\, e_2 \wedge e_3 + \xi_7\, e_1 \wedge e_2 \wedge e_3$$

Powers of Grassmann numbers

The computation of powers of Grassmann numbers has already been discussed in Section 9.5. We can also calculate powers of Grassmann numbers using the `GrassmannFunction` operation. The syntax we will usually use is:

GrassmannFunction[X^a]

$$\xi_0^a + a\, e_1\, \xi_0^{-1+a}\, \xi_1 + a\, e_2\, \xi_0^{-1+a}\, \xi_2 + a\, e_3\, \xi_0^{-1+a}\, \xi_3 + a\, \xi_0^{-1+a}\, \xi_4\, e_1 \wedge e_2 + a\, \xi_0^{-1+a}\, \xi_5\, e_1 \wedge e_3 + a\, \xi_0^{-1+a}\, \xi_6\, e_2 \wedge e_3 + (\xi_0^{-2+a}\, ((a - a^2)\, \xi_2\, \xi_5 + (-a + a^2)\, (\xi_3\, \xi_4 + \xi_1\, \xi_6)) + a\, \xi_0^{-1+a}\, \xi_7)\, e_1 \wedge e_2 \wedge e_3$$

GrassmannFunction[X^-1]

$$\frac{1}{\xi_0} - \frac{e_1\, \xi_1}{\xi_0^2} - \frac{e_2\, \xi_2}{\xi_0^2} - \frac{e_3\, \xi_3}{\xi_0^2} - \frac{\xi_4\, e_1 \wedge e_2}{\xi_0^2} - \frac{\xi_5\, e_1 \wedge e_3}{\xi_0^2} - \frac{\xi_6\, e_2 \wedge e_3}{\xi_0^2} + \left(\frac{-2\, \xi_2\, \xi_5 + 2\, (\xi_3\, \xi_4 + \xi_1\, \xi_6)}{\xi_0^3} - \frac{\xi_7}{\xi_0^2}\right) e_1 \wedge e_2 \wedge e_3$$

But we can also use **GrassmannFunction[{x^a, x}, X]**, **(#^a &)[X] // GrassmannFunction** or **GrassmannPower[X,a]** to get the same result.

Exponential and logarithmic functions of Grassmann numbers

Exponential functions

Here is the exponential of a general Grassmann number in 3-space.

expX = GrassmannFunction[Exp[X]]

$$e^{\xi_0} + e^{\xi_0}\, e_1\, \xi_1 + e^{\xi_0}\, e_2\, \xi_2 + e^{\xi_0}\, e_3\, \xi_3 + e^{\xi_0}\, \xi_4\, e_1 \wedge e_2 + e^{\xi_0}\, \xi_5\, e_1 \wedge e_3 + e^{\xi_0}\, \xi_6\, e_2 \wedge e_3 + e^{\xi_0}\, (\xi_3\, \xi_4 - \xi_2\, \xi_5 + \xi_1\, \xi_6 + \xi_7)\, e_1 \wedge e_2 \wedge e_3$$

Here is the exponential of its negative.

expXn = GrassmannFunction[Exp[-X]]

$$e^{-\xi_0} - e^{-\xi_0}\, e_1\, \xi_1 - e^{-\xi_0}\, e_2\, \xi_2 - e^{-\xi_0}\, e_3\, \xi_3 - e^{-\xi_0}\, \xi_4\, e_1 \wedge e_2 - e^{-\xi_0}\, \xi_5\, e_1 \wedge e_3 - e^{-\xi_0}\, \xi_6\, e_2 \wedge e_3 + e^{-\xi_0}\, (\xi_3\, \xi_4 - \xi_2\, \xi_5 + \xi_1\, \xi_6 - \xi_7)\, e_1 \wedge e_2 \wedge e_3$$

We verify that they are indeed inverses of one another.

𝒢[expX ∧ expXn] // Simplify

1

Logarithmic functions

The logarithm of **X** is:

```
GrassmannFunction[Log[X]]
```

$$\mathrm{Log}[\xi_0]+\frac{e_1\,\xi_1}{\xi_0}+\frac{e_2\,\xi_2}{\xi_0}+\frac{e_3\,\xi_3}{\xi_0}+\frac{\xi_4\,e_1\wedge e_2}{\xi_0}+\frac{\xi_5\,e_1\wedge e_3}{\xi_0}+\frac{\xi_6\,e_2\wedge e_3}{\xi_0}+\left(\frac{-\xi_3\,\xi_4+\xi_2\,\xi_5-\xi_1\,\xi_6}{\xi_0^2}+\frac{\xi_7}{\xi_0}\right)e_1\wedge e_2\wedge e_3$$

We expect the logarithm of the previously calculated exponential to be the original number **X**.

```
Log[expX] // GrassmannFunction // Simplify // PowerExpand
```

$$\xi_0+e_1\,\xi_1+e_2\,\xi_2+e_3\,\xi_3+\xi_4\,e_1\wedge e_2+\xi_5\,e_1\wedge e_3+\xi_6\,e_2\wedge e_3+\xi_7\,e_1\wedge e_2\wedge e_3$$

Trigonometric functions of Grassmann numbers

Sines and cosines of a Grassmann number

We explore the classic trigonometric relation **Sin[X]² + Cos[X]² == 1** and show that even for Grassmann numbers this remains true.

```
s2X = GrassmannFunction[{Sin[x]^2, x}, X]
```

$$\mathrm{Sin}[\xi_0]^2+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,e_1\,\xi_1+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,e_2\,\xi_2+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,e_3\,\xi_3+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_4\,e_1\wedge e_2+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_5\,e_1\wedge e_3+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_6\,e_2\wedge e_3+((-2\,\mathrm{Cos}[\xi_0]^2+2\,\mathrm{Sin}[\xi_0]^2)\,\xi_2\,\xi_5+(2\,\mathrm{Cos}[\xi_0]^2-2\,\mathrm{Sin}[\xi_0]^2)\,(\xi_3\,\xi_4+\xi_1\,\xi_6)+2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_7)\,e_1\wedge e_2\wedge e_3$$

```
c2X = GrassmannFunction[{Cos[x]^2, x}, X]
```

$$\mathrm{Cos}[\xi_0]^2-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,e_1\,\xi_1-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,e_2\,\xi_2-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,e_3\,\xi_3-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_4\,e_1\wedge e_2-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_5\,e_1\wedge e_3-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_6\,e_2\wedge e_3+((2\,\mathrm{Cos}[\xi_0]^2-2\,\mathrm{Sin}[\xi_0]^2)\,\xi_2\,\xi_5+(-2\,\mathrm{Cos}[\xi_0]^2+2\,\mathrm{Sin}[\xi_0]^2)\,(\xi_3\,\xi_4+\xi_1\,\xi_6)-2\,\mathrm{Cos}[\xi_0]\,\mathrm{Sin}[\xi_0]\,\xi_7)\,e_1\wedge e_2\wedge e_3$$

```
𝒢[s2X + c2X == 1] // Simplify
```

```
True
```

The tangent of a Grassmann number

Finally we show that the tangent of a Grassmann number is the exterior product of its sine and its secant.

```
tanX = GrassmannFunction[Tan[X]]
```

$\mathrm{Sec}[\xi_0]^2\, e_1\, \xi_1 + \mathrm{Sec}[\xi_0]^2\, e_2\, \xi_2 + \mathrm{Sec}[\xi_0]^2\, e_3\, \xi_3 + \mathrm{Tan}[\xi_0] +$
$\mathrm{Sec}[\xi_0]^2\, \xi_4\, e_1 \wedge e_2 + \mathrm{Sec}[\xi_0]^2\, \xi_5\, e_1 \wedge e_3 + \mathrm{Sec}[\xi_0]^2\, \xi_6\, e_2 \wedge e_3 +$
$\mathrm{Sec}[\xi_0]^2\, (\xi_7 + (-2\, \xi_2\, \xi_5 + 2\, (\xi_3\, \xi_4 + \xi_1\, \xi_6))\, \mathrm{Tan}[\xi_0])\, e_1 \wedge e_2 \wedge e_3$

```
sinX = GrassmannFunction[Sin[X]]
```

$\mathrm{Sin}[\xi_0] + \mathrm{Cos}[\xi_0]\, e_1\, \xi_1 + \mathrm{Cos}[\xi_0]\, e_2\, \xi_2 + \mathrm{Cos}[\xi_0]\, e_3\, \xi_3 +$
$\mathrm{Cos}[\xi_0]\, \xi_4\, e_1 \wedge e_2 + \mathrm{Cos}[\xi_0]\, \xi_5\, e_1 \wedge e_3 + \mathrm{Cos}[\xi_0]\, \xi_6\, e_2 \wedge e_3 +$
$(\mathrm{Sin}[\xi_0]\, (-\xi_3\, \xi_4 + \xi_2\, \xi_5 - \xi_1\, \xi_6) + \mathrm{Cos}[\xi_0]\, \xi_7)\, e_1 \wedge e_2 \wedge e_3$

```
secX = GrassmannFunction[Sec[X]]
```

$\mathrm{Sec}[\xi_0] + \mathrm{Sec}[\xi_0]\, e_1\, \xi_1\, \mathrm{Tan}[\xi_0] + \mathrm{Sec}[\xi_0]\, e_2\, \xi_2\, \mathrm{Tan}[\xi_0] +$
$\mathrm{Sec}[\xi_0]\, e_3\, \xi_3\, \mathrm{Tan}[\xi_0] + \mathrm{Sec}[\xi_0]\, \xi_4\, \mathrm{Tan}[\xi_0]\, e_1 \wedge e_2 +$
$\mathrm{Sec}[\xi_0]\, \xi_5\, \mathrm{Tan}[\xi_0]\, e_1 \wedge e_3 + \mathrm{Sec}[\xi_0]\, \xi_6\, \mathrm{Tan}[\xi_0]\, e_2 \wedge e_3 +$
$(\mathrm{Sec}[\xi_0]^3\, (\xi_3\, \xi_4 - \xi_2\, \xi_5 + \xi_1\, \xi_6) + \mathrm{Sec}[\xi_0]$
$(\xi_7\, \mathrm{Tan}[\xi_0] + (\xi_3\, \xi_4 - \xi_2\, \xi_5 + \xi_1\, \xi_6)\, \mathrm{Tan}[\xi_0]^2))\, e_1 \wedge e_2 \wedge e_3$

```
tanX == 𝒢[sinX ∧ secX] // Simplify
```

True

Functions of several Grassmann numbers

GrassmannFunction can calculate functions of several Grassmann variables. It is important to realize however, that there may be several different results depending on the order in which the exterior products occur in the series expansion for the function, a factor that does not arise with functions of scalar variables. This means then, that the usual function notation for several variables is not adequate for specifically denoting precisely which form is required. For simplicity in what follows we work in 2-space.

```
𝕍2; {X = CreateGrassmannNumber[ξ],
 Y = CreateGrassmannNumber[ψ]}
```

$\{\xi_0 + e_1\, \xi_1 + e_2\, \xi_2 + \xi_3\, e_1 \wedge e_2,\ \psi_0 + e_1\, \psi_1 + e_2\, \psi_2 + \psi_3\, e_1 \wedge e_2\}$

Products of two Grassmann numbers

As a first example, we look at the product. Calculating the exterior product of two Grassmann numbers in 2-space gives:

```
𝒢[X ∧ Y]
```

$\xi_0\, \psi_0 + e_1\, (\xi_1\, \psi_0 + \xi_0\, \psi_1) + e_2\, (\xi_2\, \psi_0 + \xi_0\, \psi_2) +$
$(\xi_3\, \psi_0 - \xi_2\, \psi_1 + \xi_1\, \psi_2 + \xi_0\, \psi_3)\, e_1 \wedge e_2$

Using GrassmannFunction in the form below, we get the same result.

GrassmannFunction[{x y, {x, y}}, {X, Y}]

$\xi_0 \psi_0 + e_1 (\xi_1 \psi_0 + \xi_0 \psi_1) + e_2 (\xi_2 \psi_0 + \xi_0 \psi_2) + (\xi_3 \psi_0 - \xi_2 \psi_1 + \xi_1 \psi_2 + \xi_0 \psi_3) e_1 \wedge e_2$

However, to allow for the *two* different *exterior* products that **X** and **Y** can have together, the variables in the second argument **{X,Y}** may be interchanged. The parameters **x** and **y** of the first argument **{x y,{x,y}}** are simply dummy variables and can stay the same. Thus **Y∧X** may be obtained by evaluating:

𝒢[Y ∧ X]

$\xi_0 \psi_0 + e_1 (\xi_1 \psi_0 + \xi_0 \psi_1) + e_2 (\xi_2 \psi_0 + \xi_0 \psi_2) + (\xi_3 \psi_0 + \xi_2 \psi_1 - \xi_1 \psi_2 + \xi_0 \psi_3) e_1 \wedge e_2$

or using `GrassmannFunction` with **{Y,X}** as its second argument.

GrassmannFunction[{x y, {x, y}}, {Y, X}]

$\xi_0 \psi_0 + e_1 (\xi_1 \psi_0 + \xi_0 \psi_1) + e_2 (\xi_2 \psi_0 + \xi_0 \psi_2) + (\xi_3 \psi_0 + \xi_2 \psi_1 - \xi_1 \psi_2 + \xi_0 \psi_3) e_1 \wedge e_2$

The exponential of a sum

As a final example we show how `GrassmannFunction` manages the two different exterior product equivalents of the scalar identity **Exp[x+y] == Exp[x] Exp[y]**.

First calculate **Exp[X]** and **Exp[y]**.

expX = GrassmannFunction[Exp[X]]

$e^{\xi_0} + e^{\xi_0} e_1 \xi_1 + e^{\xi_0} e_2 \xi_2 + e^{\xi_0} \xi_3 e_1 \wedge e_2$

expY = GrassmannFunction[Exp[Y]]

$e^{\psi_0} + e^{\psi_0} e_1 \psi_1 + e^{\psi_0} e_2 \psi_2 + e^{\psi_0} \psi_3 e_1 \wedge e_2$

If we compute their product using the exterior product operation we observe that the order must be important, and that a different result is obtained when the order is reversed.

𝒢[expX ∧ expY]

$e^{\xi_0+\psi_0} + e^{\xi_0+\psi_0} e_1 (\xi_1 + \psi_1) + e^{\xi_0+\psi_0} e_2 (\xi_2 + \psi_2) + e^{\xi_0+\psi_0} (\xi_3 - \xi_2 \psi_1 + \xi_1 \psi_2 + \psi_3) e_1 \wedge e_2$

𝒢[expY ∧ expX]

$e^{\xi_0+\psi_0} + e^{\xi_0+\psi_0} e_1 (\xi_1 + \psi_1) + e^{\xi_0+\psi_0} e_2 (\xi_2 + \psi_2) + e^{\xi_0+\psi_0} (\xi_3 + \xi_2 \psi_1 - \xi_1 \psi_2 + \psi_3) e_1 \wedge e_2$

We can compute these two results also by using `GrassmannFunction`. Note that the second argument in the second computation has its components in reverse order.

```
GrassmannFunction[{Exp[x] Exp[y], {x, y}}, {X, Y}]
```

$$e^{\xi_0+\psi_0} + e^{\xi_0+\psi_0}\, e_1\, (\xi_1 + \psi_1) + e^{\xi_0+\psi_0}\, e_2\, (\xi_2 + \psi_2) + e^{\xi_0+\psi_0}\, (\xi_3 - \xi_2\, \psi_1 + \xi_1\, \psi_2 + \psi_3)\, e_1 \wedge e_2$$

```
GrassmannFunction[{Exp[x] Exp[y], {x, y}}, {Y, X}]
```

$$e^{\xi_0+\psi_0} + e^{\xi_0+\psi_0}\, e_1\, (\xi_1 + \psi_1) + e^{\xi_0+\psi_0}\, e_2\, (\xi_2 + \psi_2) + e^{\xi_0+\psi_0}\, (\xi_3 + \xi_2\, \psi_1 - \xi_1\, \psi_2 + \psi_3)\, e_1 \wedge e_2$$

On the other hand, if we wish to compute the exponential of the sum **X + Y** we *need* to use `GrassmannFunction`, because there can be two possibilities for its definition. That is, although **Exp[x+y]** appears to be independent of the order of its arguments, an interchange in the 'ordering' argument from **{X,Y}** to **{Y,X}** will change the signs in the result.

```
GrassmannFunction[{Exp[x + y], {x, y}}, {X, Y}]
```

$$e^{\xi_0+\psi_0} + e^{\xi_0+\psi_0}\, e_1\, (\xi_1 + \psi_1) + e^{\xi_0+\psi_0}\, e_2\, (\xi_2 + \psi_2) + e^{\xi_0+\psi_0}\, (\xi_3 - \xi_2\, \psi_1 + \xi_1\, \psi_2 + \psi_3)\, e_1 \wedge e_2$$

```
GrassmannFunction[{Exp[x + y], {x, y}}, {Y, X}]
```

$$e^{\xi_0+\psi_0} + e^{\xi_0+\psi_0}\, e_1\, (\xi_1 + \psi_1) + e^{\xi_0+\psi_0}\, e_2\, (\xi_2 + \psi_2) + e^{\xi_0+\psi_0}\, (\xi_3 + \xi_2\, \psi_1 - \xi_1\, \psi_2 + \psi_3)\, e_1 \wedge e_2$$

In sum: A function of several Grassmann variables may have different results depending on the ordering of the variables in the function, even if their usual (scalar) form is commutative.

10 Exploring the Generalized Grassmann Product

10.1 Introduction

In this chapter we define and explore a new product which we call the *generalized Grassmann product*. The generalized Grassmann product was originally developed by the author in order to treat the Clifford product of general elements in a succinct manner. The Clifford product of general elements can lead to quite complex expressions, however we will show that such expressions always reduce to simple sums of generalized Grassmann products. We discuss the Clifford product in Chapter 12.

In its own right, the generalized Grassmann product leads to a suite of useful identities between expressions involving exterior and interior products, and also has some useful geometric applications.

We have chosen to call it the *generalized* Grassmann product because, in specific cases, it reduces to either an exterior or an interior product.

For brevity in the rest of this book, where no confusion will arise, we may call the generalized Grassmann product simply the *generalized product*.

10.2 Defining the Generalized Product

Definition of the generalized product

The *generalized Grassmann product of order* λ of an m-element $\underset{m}{\alpha}$ and a *simple* k-element $\underset{k}{\beta}$ is denoted $\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}$ and *defined* by

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\underset{m}{\alpha}\ominus\underset{\lambda}{\beta^{j}}\right)\wedge\underset{k-\lambda}{\beta^{j}}$$
$$\underset{k}{\beta} == \underset{\lambda}{\beta^{1}}\wedge\underset{k-\lambda}{\beta^{1}} == \underset{\lambda}{\beta^{2}}\wedge\underset{k-\lambda}{\beta^{2}} == \cdots \qquad \mathbf{10.1}$$

Here, $\underset{k}{\beta}$ is expressed in all of the $\binom{k}{\lambda}$ *essentially different* arrangements of its 1-element factors into a λ-element and a (k-λ)-element. This pairwise decomposition of a simple exterior product is the same as that used in the Common Factor Theorem [3.28] and [6.28].

The *grade* of a generalized Grassmann product $\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}$ is therefore m + k - 2λ, and like the grades of the exterior and interior products in terms of which it is defined, is independent of the dimension of the underlying space.

Note the similarity to the form of the Interior Common Factor Theorem. However, in the definition of the generalized product there is no requirement for the interior product on the right hand side to be an inner product, since an exterior product has replaced the ordinary multiplication operator.

For brevity in the discussion that follows, we will assume without explicitly drawing attention to the fact, that an element undergoing a factorization is necessarily simple.

Case λ = 0: Reduction to the exterior product

When the order λ of a generalized product is zero, $\underset{\lambda}{\beta^{j}}$ reduces to a scalar, and the product reduces to the exterior product.

$$\underset{m}{\alpha}\underset{0}{\triangle}\underset{k}{\beta} == \underset{m}{\alpha}\wedge\underset{k}{\beta} \qquad \lambda == 0 \qquad \mathbf{10.2}$$

Case 0 < λ < Min[*m*, *k*]: Reduction to exterior and interior products

When the order λ of the generalized product is greater than zero but less than the smaller of the grades of the factors in the product, the product may be expanded in terms of *either* factor (provided it is simple). The formula for expansion in terms of the first factor is similar to the definition [10.1] which expands in terms of the second factor.

$$\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{k}{\beta} == \sum_{i=1}^{\binom{m}{\lambda}} \underset{m-\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^{i}}\right) \qquad 0 < \lambda < \mathrm{Min}[m, k] \tag{10.3}$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots$$

In Section 10.3 we will prove this alternative form, showing in the process that the product can be expanded in terms of *both* factors, thus underscoring the essential symmetry of the product.

Case λ = Min[*m*, *k*]: Reduction to the interior product

When the order of the product λ is equal to the smaller of the grades of the factors of the product, there is only one term in the sum and the generalized product reduces to the interior product.

$$\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{k}{\beta} == \underset{m}{\alpha} \ominus \underset{k}{\beta} \qquad \lambda == k \leq m$$

$$\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{k}{\beta} == \underset{k}{\beta} \ominus \underset{m}{\alpha} \qquad \lambda == m \leq k \tag{10.4}$$

This is a particularly enticing property of the generalized product. If the interior product of two elements is non-zero, but they are of unequal grade, an interchange in the order of their factors will give an interior product which is zero. If an interior product of two factors is to be non-zero, it must have the larger factor first. The interior product is non-commutative. By contrast, the generalized product form of the interior product of two elements of unequal grade is commutative and equal to that interior product which is non-zero, *whichever one it may be*.

This becomes evident if we put λ equal to *k* in the first of the formulae of the previous subsection, and λ equal to *m* in the second.

Case Min[*m*, *k*] < λ < Max[*m*, *k*]: Reduction to zero

When the order of the product λ is greater than the smaller of the grades of the factors of the product, but less than the larger of the grades, the generalized product may still be expanded in terms of the factors of the element of larger grade, leading however to a sum of terms which is zero.

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == 0 \qquad \text{Min}[m, k] < \lambda < \text{Max}[m, k] \tag{10.5}$$

This result, in which a sum of terms is identically zero, is a source of many useful identities between exterior and interior products. It will be explored in Section 10.4 below.

Case λ = Max[m, k]: Reduction to zero

When λ is equal to the larger of the grades of the factors, the generalized product reduces to a single interior product which is zero by virtue of its left factor being of lesser grade than its right factor.

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == 0 \qquad \lambda == \text{Max}[m, k] \tag{10.6}$$

Case λ > Max[m, k]: Undefined

When the order of the product λ is greater than the larger of the grades of the factors of the product, the generalized product cannot be expanded in terms of either of its factors and is therefore *undefined*.

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \text{Undefined} \qquad \lambda > \text{Max}[m, k] \tag{10.7}$$

10.3 The Symmetric Form of the Generalized Product

Expansion of the generalized product in terms of both factors

If $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are both simple we can express the generalized Grassmann product more symmetrically by converting the interior products with the Interior Common Factor Theorem [6.28].

To show this, we start with the definition [10.1] of the generalized Grassmann product.

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\underset{m}{\alpha} \ominus \underset{\lambda}{\beta^j}\right) \wedge \underset{k-\lambda}{\beta^j}$$

From the Interior Common Factor Theorem we can write the interior product $\underset{m}{\alpha} \ominus \underset{\lambda}{\beta^j}$ as:

$$\underset{m}{\alpha} \ominus \underset{\lambda}{\beta^j} == \sum_{i=1}^{\binom{m}{\lambda}} \left(\underset{\lambda}{\alpha^i} \ominus \underset{\lambda}{\beta^j}\right) \underset{m-\lambda}{\alpha^i}$$

Substituting this in the above expression for the generalized product gives:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\sum_{i=1}^{\binom{m}{\lambda}} \left(\underset{\lambda}{\alpha^i} \ominus \underset{\lambda}{\beta^j}\right) \underset{m-\lambda}{\alpha^i}\right) \wedge \underset{k-\lambda}{\beta^j}$$

The expansion of the generalized product in terms of both factors $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is thus given by:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{i=1}^{\binom{m}{\lambda}} \sum_{j=1}^{\binom{k}{\lambda}} \left(\underset{\lambda}{\alpha^i} \ominus \underset{\lambda}{\beta^j}\right) \underset{m-\lambda}{\alpha^i} \wedge \underset{k-\lambda}{\beta^j} \qquad 0 < \lambda < \mathrm{Min}[m, k] \tag{10.8}$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1} \wedge \underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2} \wedge \underset{m-\lambda}{\alpha^2} == \cdots, \qquad \underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots$$

The quasi-commutativity of the generalized product

From the symmetric form of the generalized product [10.8] we can show directly that the ordering of the factors in a product is immaterial *except perhaps for a sign*.

To show this we begin with the symmetric form [10.8], and rewrite it for the product of $\underset{k}{\beta}$ with $\underset{m}{\alpha}$, that is, $\underset{k}{\beta} \underset{\lambda}{\triangle} \underset{m}{\alpha}$. We then reverse the order of the exterior product to obtain the terms of $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}$ except for possibly a sign. The inner product is of course symmetric, and so can be written in either ordering.

$$\underset{k}{\beta} \underset{\lambda}{\triangle} \underset{m}{\alpha} == \sum_{j=1}^{\binom{k}{\lambda}} \sum_{i=1}^{\binom{m}{\lambda}} \left(\underset{\lambda}{\beta^j} \ominus \underset{\lambda}{\alpha^i}\right) \underset{k-\lambda}{\beta^j} \wedge \underset{m-\lambda}{\alpha^i}$$

$$== \sum_{j=1}^{\binom{k}{\lambda}} \sum_{i=1}^{\binom{m}{\lambda}} \left(\underset{\lambda}{\alpha^i} \ominus \underset{\lambda}{\beta^j}\right) (-1)^{(m-\lambda)\,(k-\lambda)} \underset{m-\lambda}{\alpha^i} \wedge \underset{k-\lambda}{\beta^j}$$

Comparison with [10.8] then gives:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == (-1)^{(m-\lambda)\,(k-\lambda)} \underset{k}{\beta} \underset{\lambda}{\triangle} \underset{m}{\alpha} \tag{10.9}$$

It is easy to see, by using the distributivity of the generalized product that this relationship holds also for *non-simple* $\underset{m}{\alpha}$ and $\underset{k}{\beta}$.

Expansion in terms of the other factor

We can now prove the alternative expression [10.3] for $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}$ in terms of the factors of a simple $\underset{m}{\alpha}$ by expanding the right hand side of the quasi-commutativity relationship [10.9].

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == (-1)^{(m-\lambda)\,(k-\lambda)}\ \underset{k}{\beta} \underset{\lambda}{\triangle} \underset{m}{\alpha} == (-1)^{(m-\lambda)\,(k-\lambda)} \sum_{i=1}^{\binom{m}{\lambda}} \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^{i}}\right) \wedge \underset{m-\lambda}{\alpha^{i}}$$

Interchanging the order of the terms of the exterior product then gives the required alternative expression:

$$\boxed{\begin{array}{c} \underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \displaystyle\sum_{i=1}^{\binom{m}{\lambda}} \underset{m-\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^{i}}\right) \qquad 0 < \lambda < \mathrm{Min}[m, k] \\ \underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots \end{array}}$$

10.4 Calculating with Generalized Products

Entering a generalized product

In *GrassmannAlgebra* you can enter a generalized product in the form:

```
GeneralizedProduct[λ][X, Y]
```

where λ is the order of the product and **X** and **Y** are the factors. On entering this expression into *Mathematica* it will display it in the form:

$$X \underset{\lambda}{\triangle} Y$$

Alternatively, you can click on the generalized product button $\square \underset{\square}{\triangle} \square$ on the $\mathcal{GA}$ palette, and then tab through the placeholders, entering the elements of the product as you go.

To enter multiple products, click repeatedly on the button. For example to enter a concatenated product of four elements, click three times:

$$\square \underset{\blacksquare}{\triangle} \blacksquare \underset{\blacksquare}{\triangle} \blacksquare \underset{\square}{\triangle} \square$$

If you select this expression and enter it, you will see how the inherently binary product is grouped.

$$\left(\left(\square \underset{\blacksquare}{\triangle} \blacksquare\right) \underset{\blacksquare}{\triangle} \blacksquare\right) \underset{\square}{\triangle} \square$$

Tabbing through the placeholders and entering factors and product orders, we might obtain something like:

$$\left(\left(\mathbf{X} \underset{4}{\triangle} \mathbf{Y}\right) \underset{3}{\triangle} \mathbf{Z}\right) \underset{2}{\triangle} \mathbf{W}$$

Here, **X**, **Y**, **Z**, and **W** represent any Grassmann expressions. Using *Mathematica*'s `FullForm` operation shows how this is internally represented.

$$\mathbf{FullForm}\left[\left(\left(\mathbf{X} \underset{4}{\triangle} \mathbf{Y}\right) \underset{1}{\triangle} \mathbf{Z}\right) \underset{0}{\triangle} \mathbf{W}\right]$$

```
GeneralizedProduct[0][
 GeneralizedProduct[1][GeneralizedProduct[4][X, Y], Z], W]
```

You can edit the expression at any stage to group the products in a different order.

$$\mathbf{FullForm}\left[\mathbf{X} \underset{4}{\triangle} \left(\left(\mathbf{Y} \underset{3}{\triangle} \mathbf{Z}\right) \underset{2}{\triangle} \mathbf{W}\right)\right]$$

```
GeneralizedProduct[4][X,
 GeneralizedProduct[2][GeneralizedProduct[3][Y, Z], W]]
```

Reduction to interior products

If the generalized product $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}$ of simple elements is expanded in terms of the second factor $\underset{k}{\beta}$ it will result in a sum of $\binom{k}{\lambda}$ terms ($0 \le \lambda \le k$) involving just exterior and interior products. If it is expanded in terms of the first factor $\underset{m}{\alpha}$ it will result in a sum of $\binom{m}{\lambda}$ terms ($0 \le \lambda \le m$).

These sums, although appearing different because expressed in terms of interior products, are of course equal due to the result in the previous section.

The *GrassmannAlgebra* function `ToInteriorProducts` will take any generalized product and convert it to a sum involving exterior and interior products by expanding the second factor as in the definition. If the elements are given in symbolic grade-underscripted form, `ToInteriorProducts` will first create a corresponding simple exterior product before expanding.

For example, suppose the generalized product is given in symbolic form as $\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}$. Applying `ToInteriorProducts` expands with respect to $\underset{3}{\beta}$ and gives:

$$\mathbf{A} = \mathbf{ToInteriorProducts}\left[\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}\right]$$

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2) \wedge \beta_3 - (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3) \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3) \wedge \beta_1$$

To expand with respect to $\underset{m}{\alpha}$ we reverse the order of the generalized product to give $\underset{3}{\beta} \underset{2}{\triangle} \underset{4}{\alpha}$ and multiply by $(-1)^{(m-\lambda)\,(k-\lambda)}$ (which we note for this case to be 1) to give:

B = ToInteriorProducts$\left[\underset{3}{\beta} \underset{2}{\triangle} \underset{4}{\alpha}\right]$

$$(\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_1 \wedge \alpha_2) \wedge \alpha_3 \wedge \alpha_4 - (\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_1 \wedge \alpha_3) \wedge \alpha_2 \wedge \alpha_4 + (\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_1 \wedge \alpha_4) \wedge \alpha_2 \wedge \alpha_3 + (\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_2 \wedge \alpha_3) \wedge \alpha_1 \wedge \alpha_4 - (\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_2 \wedge \alpha_4) \wedge \alpha_1 \wedge \alpha_3 + (\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_3 \wedge \alpha_4) \wedge \alpha_1 \wedge \alpha_2$$

Note that in the expansion **A** there are $\binom{3}{2}$ (= 3) terms while in expansion **B** there are $\binom{4}{2}$ (= 6) terms.

Reduction to inner products

At this stage the results do not look similar. Next we develop the inner product forms:

A1 = ToInnerProducts[A]

$$(\alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \beta_1 - (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \beta_3 - (\alpha_2 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \beta_1 + (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_3 \wedge \beta_3 + (\alpha_2 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_4 \wedge \beta_1 - (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_4 \wedge \beta_2 + (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \beta_1 - (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \beta_2 + (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_3 \wedge \beta_3 - (\alpha_1 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_4 \wedge \beta_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_4 \wedge \beta_2 - (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \beta_2 \wedge \beta_3)\, \alpha_3 \wedge \alpha_4 \wedge \beta_1 - (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_3 \wedge \alpha_4 \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\, \alpha_3 \wedge \alpha_4 \wedge \beta_3$$

B1 = ToInnerProducts[B]

$$(\alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \beta_1 - (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \beta_3 - (\alpha_2 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \beta_1 + (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_3 \wedge \beta_3 + (\alpha_2 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_4 \wedge \beta_1 - (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_4 \wedge \beta_2 + (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \beta_1 - (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \beta_2 + (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_3 \wedge \beta_3 - (\alpha_1 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_4 \wedge \beta_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_4 \wedge \beta_2 - (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \beta_2 \wedge \beta_3)\, \alpha_3 \wedge \alpha_4 \wedge \beta_1 - (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_3 \wedge \alpha_4 \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\, \alpha_3 \wedge \alpha_4 \wedge \beta_3$$

By inspection we see that these two expressions are the same. Calculating the difference in *Mathematica* verifies this.

A1 – B1

0

The identity of the forms **A1** and **B1** is an example of the expansion of a generalized product in terms of either factor yielding the same result.

Example: Case Min[*m*, *k*] < λ < Max[*m*, *k*]: Reduction to zero

When the order of the product λ is greater than the smaller of the grades of the factors of the product, but less than the larger of the grades, the generalized product may still be expanded in terms of the factors of the element of larger grade. This leads however to a sum of terms which is zero.

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == 0 \qquad \text{Min}[m, k] < \lambda < \text{Max}[m, k]$$

When λ is equal to the larger of the grades of the factors, the generalized product reduces to a single interior product which is zero by virtue of its left factor being of lesser grade than its right factor. Suppose λ = *k* > *m*. Then:

$$\underset{m}{\alpha} \underset{k}{\triangle} \underset{k}{\beta} == \underset{m}{\alpha} \ominus \underset{k}{\beta} == 0 \qquad \lambda = k > m$$

These relationships are the source of an interesting suite of identities relating exterior and interior products. We take some examples; in each case we verify that the result is zero by converting the expression to its scalar product form.

Example 1

A_{123} = ToInteriorProductsB$[\alpha \underset{2}{\triangle} \underset{3}{\beta}]$

$$\alpha \wedge \beta_1 \ominus \beta_2 \wedge \beta_3 - \alpha \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 + \alpha \wedge \beta_3 \ominus \beta_1 \wedge \beta_2$$

ToScalarProducts[A_{123}]

0

Example 2

A_{235} = ToInteriorProductsB$[\underset{2}{\alpha} \underset{3}{\triangle} \underset{5}{\beta}]$

$$\alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \ominus \beta_3 \wedge \beta_4 \wedge \beta_5 - \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_3 \ominus \beta_2 \wedge \beta_4 \wedge \beta_5 +$$
$$\alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_4 \ominus \beta_2 \wedge \beta_3 \wedge \beta_5 - \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_5 \ominus \beta_2 \wedge \beta_3 \wedge \beta_4 +$$
$$\alpha_1 \wedge \alpha_2 \wedge \beta_2 \wedge \beta_3 \ominus \beta_1 \wedge \beta_4 \wedge \beta_5 - \alpha_1 \wedge \alpha_2 \wedge \beta_2 \wedge \beta_4 \ominus \beta_1 \wedge \beta_3 \wedge \beta_5 +$$
$$\alpha_1 \wedge \alpha_2 \wedge \beta_2 \wedge \beta_5 \ominus \beta_1 \wedge \beta_3 \wedge \beta_4 + \alpha_1 \wedge \alpha_2 \wedge \beta_3 \wedge \beta_4 \ominus \beta_1 \wedge \beta_2 \wedge \beta_5 -$$
$$\alpha_1 \wedge \alpha_2 \wedge \beta_3 \wedge \beta_5 \ominus \beta_1 \wedge \beta_2 \wedge \beta_4 + \alpha_1 \wedge \alpha_2 \wedge \beta_4 \wedge \beta_5 \ominus \beta_1 \wedge \beta_2 \wedge \beta_3$$

Expand[ToScalarProducts[A_{235}]]

0

Example 3

```
A245 = ToInteriorProductsB[α △ β]
                           2 4 5
```

$\alpha_1 \wedge \alpha_2 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3 \wedge \beta_4 \wedge \beta_5 - \alpha_1 \wedge \alpha_2 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 \wedge \beta_4 \wedge \beta_5 + \alpha_1 \wedge \alpha_2 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2 \wedge \beta_4 \wedge \beta_5 - \alpha_1 \wedge \alpha_2 \wedge \beta_4 \ominus \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_5 + \alpha_1 \wedge \alpha_2 \wedge \beta_5 \ominus \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \beta_4$

```
ToScalarProducts[A245]
```

0

10.5 The Generalized Product Theorem

The A and B forms of a generalized product

There is a surprising alternative form for the expansion of the generalized product in terms of exterior and interior products. We distinguish the two forms by calling the first (definitional) form the *A form* and the new second form the *B form*.

$$\mathbf{A:}\quad \underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\underset{m}{\alpha} \ominus \underset{\lambda}{\beta^j}\right) \wedge \underset{k-\lambda}{\beta^j}$$

$$\mathbf{B:}\quad \underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\underset{m}{\alpha} \wedge \underset{k-\lambda}{\beta^j}\right) \ominus \underset{\lambda}{\beta^j} \qquad \mathbf{10.10}$$

$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots$$

The identity between the A form and the B form is the source of many useful relations in the Grassmann and Clifford algebras. We call it the Generalized Product Theorem.

In the previous sections, the identity between the expansions in terms of either of the two factors was shown to be at the *inner* product level. The identity between the A form and the B form is at a further level of complexity - that of the *scalar* product. That is, in order to show the identity between the two forms, a generalized product may need to be reduced to an expression involving exterior and scalar products.

Example: Verification of the Generalized Product Theorem

As an example we take the same generalized product $\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}$ that we explored in section [10.4]. First we convert it to interior products as we did in that section.

A = ToInteriorProducts$\left[\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}\right]$

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2) \wedge \beta_3 - (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3) \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3) \wedge \beta_1$$

Next, we convert the same generalized product to its B form expression by using a modified version of the *GrassmannAlgebra* function `ToInteriorProducts`. This modified version is termed `ToInteriorProductsB`. (Note the 'B' at the end of the function name).

B = ToInteriorProductsB$\left[\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}\right]$

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2$$

Note that the A form and the B form at this first (interior product) level of their expansion both have the same number of terms (in this case $\binom{3}{2} = 3$).

We now convert these two expressions to their inner product forms.

A1 = ToInnerProducts[A]

$$\begin{aligned}&(\alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \beta_1 - (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \beta_2 + \\ &(\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_2 \wedge \beta_3 - (\alpha_2 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \beta_1 + \\ &(\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_3 \wedge \beta_3 + \\ &(\alpha_2 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_4 \wedge \beta_1 - (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_4 \wedge \beta_2 + \\ &(\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \beta_1 - \\ &(\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \beta_2 + (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_3 \wedge \beta_3 - \\ &(\alpha_1 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_4 \wedge \beta_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_4 \wedge \beta_2 - \\ &(\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \beta_2 \wedge \beta_3)\ \alpha_3 \wedge \alpha_4 \wedge \beta_1 - \\ &(\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_3 \wedge \alpha_4 \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\ \alpha_3 \wedge \alpha_4 \wedge \beta_3\end{aligned}$$

```
B1 = ToInnerProducts[B]
```

$$(\alpha_4 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - (\alpha_4 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + (\alpha_4 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - (\alpha_3 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + (\alpha_3 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_4 - (\alpha_3 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + (\alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \beta_1 - (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \beta_3 + (\alpha_2 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_2 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_2 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_2 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \beta_1 + (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_3 \wedge \beta_3 + (\alpha_2 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\, \alpha_1 \wedge \alpha_4 \wedge \beta_1 - (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\, \alpha_1 \wedge \alpha_4 \wedge \beta_2 + (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_4 \wedge \beta_3 - (\alpha_1 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_1 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_1 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_1 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \beta_1 - (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_3 \wedge \beta_2 + (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_3 \wedge \beta_3 - (\alpha_1 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\, \alpha_2 \wedge \alpha_4 \wedge \beta_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\, \alpha_2 \wedge \alpha_4 \wedge \beta_2 - (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \beta_2 \wedge \beta_3)\, \alpha_3 \wedge \alpha_4 \wedge \beta_1 - (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_3)\, \alpha_3 \wedge \alpha_4 \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\, \alpha_3 \wedge \alpha_4 \wedge \beta_3$$

It can be seen that at this inner product level, these two expressions are not of the same form: **B1** contains additional terms to those of **A1**. The interior products of **A1** are only of the form $(\alpha_i \wedge \alpha_j \ominus \beta_r \wedge \beta_s)$, whereas the extra terms of **B1** are of the form $(\alpha_i \wedge \beta_j \ominus \beta_r \wedge \beta_s)$. Calculating their difference (and simplifying by using the *GrassmannAlgebra* function `CollectTerms`) gives:

```
AB = CollectTerms[B1 - A1]
```

$$(\alpha_4 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3 - \alpha_4 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 + \alpha_4 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + (-(\alpha_3 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3) + \alpha_3 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 - \alpha_3 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + (\alpha_2 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3 - \alpha_2 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 + \alpha_2 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + (-(\alpha_1 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3) + \alpha_1 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 - \alpha_1 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\, \alpha_2 \wedge \alpha_3 \wedge \alpha_4$$

To verify the equality of the two forms we need only show that this difference **AB** is zero. This is equivalent to showing that the coefficient of each of the exterior products is zero. We can do this most directly by converting the whole expression to its scalar product form.

```
ToScalarProducts[AB]
```

$$0$$

The fact that the expression **AB** is zero has shown us that certain sums of inner products are zero. We generalize this result in Section 10.7.

Verification that the B form may be expanded in terms of either factor

In this section we verify that just as for the A form, the B form may be expanded in terms of either of its factors.

The principal difference between an expansion using the A form, and one using the B form is that the A form needs only to be expanded to the inner product level to show the identity between two equal expressions. On the other hand, expansion of the B form requires expansion to the *scalar* product level, often involving significantly more computation.

We take the same generalized product $\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}$ that we discussed in the previous section, and show that we need to expand the product to the scalar product level before the identity of the two expansions become evident.

Reduction to interior products

A = ToInteriorProductsB$\left[\underset{4}{\alpha} \underset{2}{\triangle} \underset{3}{\beta}\right]$

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2$$

To expand with respect to $\underset{4}{\alpha}$ we reverse the order of the generalized product to give $\underset{3}{\beta} \underset{2}{\triangle} \underset{4}{\alpha}$ and multiply by $(-1)^{(m-\lambda)(k-\lambda)}$ (which we note for this case to be 1) to give

B = ToInteriorProductsB$\left[\underset{3}{\beta} \underset{2}{\triangle} \underset{4}{\alpha}\right]$

$$\alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_3 \wedge \alpha_4 - \alpha_1 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_2 \wedge \alpha_4 + \alpha_1 \wedge \alpha_4 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_2 \wedge \alpha_3 + \alpha_2 \wedge \alpha_3 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_4 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_1 \wedge \alpha_3 + \alpha_3 \wedge \alpha_4 \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \alpha_1 \wedge \alpha_2$$

Reduction to inner products

A1 = ToInnerProducts[A]

$$(\alpha_4 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - (\alpha_4 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + (\alpha_4 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - (\alpha_3 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + (\alpha_3 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \alpha_4 - (\alpha_3 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_2 \wedge \alpha_4 + (\alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \beta_1 - (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_2 \wedge \beta_3 + (\alpha_2 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_2 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_2 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_2 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \beta_1 + (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \beta_2 - (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_3 \wedge \beta_3 + (\alpha_2 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_4 \wedge \beta_1 - (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_4 \wedge \beta_2 + (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_4 \wedge \beta_3 - (\alpha_1 \wedge \beta_1 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_1 \wedge \beta_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_1 \wedge \beta_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_1 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \beta_1 - (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \beta_2 + (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_3 \wedge \beta_3 - (\alpha_1 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_4 \wedge \beta_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_4 \wedge \beta_2 - (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \beta_2 \wedge \beta_3)\ \alpha_3 \wedge \alpha_4 \wedge \beta_1 - (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_3 \wedge \alpha_4 \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\ \alpha_3 \wedge \alpha_4 \wedge \beta_3$$

B1 = ToInnerProducts[B]

$$
\begin{aligned}
&(\alpha_3 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \beta_1 - \\
&(\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_2 \wedge \beta_2 + (\alpha_3 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_2 \wedge \beta_3 - \\
&(\alpha_2 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \beta_1 + (\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_3 \wedge \beta_2 - \\
&(\alpha_2 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_3 \wedge \beta_3 + (\alpha_2 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\ \alpha_1 \wedge \alpha_4 \wedge \beta_1 - \\
&(\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\ \alpha_1 \wedge \alpha_4 \wedge \beta_2 + (\alpha_2 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_1 \wedge \alpha_4 \wedge \beta_3 + \\
&(\alpha_2 \wedge \alpha_3 \ominus \alpha_4 \wedge \beta_3)\ \alpha_1 \wedge \beta_1 \wedge \beta_2 - (\alpha_2 \wedge \alpha_4 \ominus \alpha_3 \wedge \beta_3)\ \alpha_1 \wedge \beta_1 \wedge \beta_2 + \\
&(\alpha_2 \wedge \beta_3 \ominus \alpha_3 \wedge \alpha_4)\ \alpha_1 \wedge \beta_1 \wedge \beta_2 - (\alpha_2 \wedge \alpha_3 \ominus \alpha_4 \wedge \beta_2)\ \alpha_1 \wedge \beta_1 \wedge \beta_3 + \\
&(\alpha_2 \wedge \alpha_4 \ominus \alpha_3 \wedge \beta_2)\ \alpha_1 \wedge \beta_1 \wedge \beta_3 - (\alpha_2 \wedge \beta_2 \ominus \alpha_3 \wedge \alpha_4)\ \alpha_1 \wedge \beta_1 \wedge \beta_3 + \\
&(\alpha_2 \wedge \alpha_3 \ominus \alpha_4 \wedge \beta_1)\ \alpha_1 \wedge \beta_2 \wedge \beta_3 - (\alpha_2 \wedge \alpha_4 \ominus \alpha_3 \wedge \beta_1)\ \alpha_1 \wedge \beta_2 \wedge \beta_3 + \\
&(\alpha_2 \wedge \beta_1 \ominus \alpha_3 \wedge \alpha_4)\ \alpha_1 \wedge \beta_2 \wedge \beta_3 + (\alpha_1 \wedge \alpha_4 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \beta_1 - \\
&(\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_3 \wedge \beta_2 + (\alpha_1 \wedge \alpha_4 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_3 \wedge \beta_3 - \\
&(\alpha_1 \wedge \alpha_3 \ominus \beta_2 \wedge \beta_3)\ \alpha_2 \wedge \alpha_4 \wedge \beta_1 + (\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_3)\ \alpha_2 \wedge \alpha_4 \wedge \beta_2 - \\
&(\alpha_1 \wedge \alpha_3 \ominus \beta_1 \wedge \beta_2)\ \alpha_2 \wedge \alpha_4 \wedge \beta_3 - (\alpha_1 \wedge \alpha_3 \ominus \alpha_4 \wedge \beta_3)\ \alpha_2 \wedge \beta_1 \wedge \beta_2 + \\
&(\alpha_1 \wedge \alpha_4 \ominus \alpha_3 \wedge \beta_3)\ \alpha_2 \wedge \beta_1 \wedge \beta_2 - (\alpha_1 \wedge \beta_3 \ominus \alpha_3 \wedge \alpha_4)\ \alpha_2 \wedge \beta_1 \wedge \beta_2 + \\
&(\alpha_1 \wedge \alpha_3 \ominus \alpha_4 \wedge \beta_2)\ \alpha_2 \wedge \beta_1 \wedge \beta_3 - (\alpha_1 \wedge \alpha_4 \ominus \alpha_3 \wedge \beta_2)\ \alpha_2 \wedge \beta_1 \wedge \beta_3 + \\
&(\alpha_1 \wedge \beta_2 \ominus \alpha_3 \wedge \alpha_4)\ \alpha_2 \wedge \beta_1 \wedge \beta_3 - (\alpha_1 \wedge \alpha_3 \ominus \alpha_4 \wedge \beta_1)\ \alpha_2 \wedge \beta_2 \wedge \beta_3 + \\
&(\alpha_1 \wedge \alpha_4 \ominus \alpha_3 \wedge \beta_1)\ \alpha_2 \wedge \beta_2 \wedge \beta_3 - (\alpha_1 \wedge \beta_1 \ominus \alpha_3 \wedge \alpha_4)\ \alpha_2 \wedge \beta_2 \wedge \beta_3 + \\
&(\alpha_1 \wedge \alpha_2 \ominus \beta_2 \wedge \beta_3)\ \alpha_3 \wedge \alpha_4 \wedge \beta_1 - (\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_3)\ \alpha_3 \wedge \alpha_4 \wedge \beta_2 + \\
&(\alpha_1 \wedge \alpha_2 \ominus \beta_1 \wedge \beta_2)\ \alpha_3 \wedge \alpha_4 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \alpha_4 \wedge \beta_3)\ \alpha_3 \wedge \beta_1 \wedge \beta_2 - \\
&(\alpha_1 \wedge \alpha_4 \ominus \alpha_2 \wedge \beta_3)\ \alpha_3 \wedge \beta_1 \wedge \beta_2 + (\alpha_1 \wedge \beta_3 \ominus \alpha_2 \wedge \alpha_4)\ \alpha_3 \wedge \beta_1 \wedge \beta_2 - \\
&(\alpha_1 \wedge \alpha_2 \ominus \alpha_4 \wedge \beta_2)\ \alpha_3 \wedge \beta_1 \wedge \beta_3 + (\alpha_1 \wedge \alpha_4 \ominus \alpha_2 \wedge \beta_2)\ \alpha_3 \wedge \beta_1 \wedge \beta_3 - \\
&(\alpha_1 \wedge \beta_2 \ominus \alpha_2 \wedge \alpha_4)\ \alpha_3 \wedge \beta_1 \wedge \beta_3 + (\alpha_1 \wedge \alpha_2 \ominus \alpha_4 \wedge \beta_1)\ \alpha_3 \wedge \beta_2 \wedge \beta_3 - \\
&(\alpha_1 \wedge \alpha_4 \ominus \alpha_2 \wedge \beta_1)\ \alpha_3 \wedge \beta_2 \wedge \beta_3 + (\alpha_1 \wedge \beta_1 \ominus \alpha_2 \wedge \alpha_4)\ \alpha_3 \wedge \beta_2 \wedge \beta_3 - \\
&(\alpha_1 \wedge \alpha_2 \ominus \alpha_3 \wedge \beta_3)\ \alpha_4 \wedge \beta_1 \wedge \beta_2 + (\alpha_1 \wedge \alpha_3 \ominus \alpha_2 \wedge \beta_3)\ \alpha_4 \wedge \beta_1 \wedge \beta_2 - \\
&(\alpha_1 \wedge \beta_3 \ominus \alpha_2 \wedge \alpha_3)\ \alpha_4 \wedge \beta_1 \wedge \beta_2 + (\alpha_1 \wedge \alpha_2 \ominus \alpha_3 \wedge \beta_2)\ \alpha_4 \wedge \beta_1 \wedge \beta_3 - \\
&(\alpha_1 \wedge \alpha_3 \ominus \alpha_2 \wedge \beta_2)\ \alpha_4 \wedge \beta_1 \wedge \beta_3 + (\alpha_1 \wedge \beta_2 \ominus \alpha_2 \wedge \alpha_3)\ \alpha_4 \wedge \beta_1 \wedge \beta_3 - \\
&(\alpha_1 \wedge \alpha_2 \ominus \alpha_3 \wedge \beta_1)\ \alpha_4 \wedge \beta_2 \wedge \beta_3 + (\alpha_1 \wedge \alpha_3 \ominus \alpha_2 \wedge \beta_1)\ \alpha_4 \wedge \beta_2 \wedge \beta_3 - \\
&(\alpha_1 \wedge \beta_1 \ominus \alpha_2 \wedge \alpha_3)\ \alpha_4 \wedge \beta_2 \wedge \beta_3 + 2\ (\alpha_1 \wedge \alpha_2 \ominus \alpha_3 \wedge \alpha_4)\ \beta_1 \wedge \beta_2 \wedge \beta_3 - \\
&2\ (\alpha_1 \wedge \alpha_3 \ominus \alpha_2 \wedge \alpha_4)\ \beta_1 \wedge \beta_2 \wedge \beta_3 + 2\ (\alpha_1 \wedge \alpha_4 \ominus \alpha_2 \wedge \alpha_3)\ \beta_1 \wedge \beta_2 \wedge \beta_3
\end{aligned}
$$

By observation we can see that **A1** and **B1** contain some common terms. We compute the difference of the two inner product forms and collect terms with the same exterior product factor:

```
AB = CollectTerms[B1 - A1]
```

$$\begin{aligned}&(-(\alpha_4\wedge\beta_1\ominus\beta_2\wedge\beta_3)+\alpha_4\wedge\beta_2\ominus\beta_1\wedge\beta_3-\alpha_4\wedge\beta_3\ominus\beta_1\wedge\beta_2)\ \alpha_1\wedge\alpha_2\wedge\alpha_3+\\
&(\alpha_3\wedge\beta_1\ominus\beta_2\wedge\beta_3-\alpha_3\wedge\beta_2\ominus\beta_1\wedge\beta_3+\alpha_3\wedge\beta_3\ominus\beta_1\wedge\beta_2)\ \alpha_1\wedge\alpha_2\wedge\alpha_4+\\
&(-(\alpha_2\wedge\beta_1\ominus\beta_2\wedge\beta_3)+\alpha_2\wedge\beta_2\ominus\beta_1\wedge\beta_3-\alpha_2\wedge\beta_3\ominus\beta_1\wedge\beta_2)\ \alpha_1\wedge\alpha_3\wedge\alpha_4+\\
&(\alpha_2\wedge\alpha_3\ominus\alpha_4\wedge\beta_3-\alpha_2\wedge\alpha_4\ominus\alpha_3\wedge\beta_3+\alpha_2\wedge\beta_3\ominus\alpha_3\wedge\alpha_4)\ \alpha_1\wedge\beta_1\wedge\beta_2+\\
&(-(\alpha_2\wedge\alpha_3\ominus\alpha_4\wedge\beta_2)+\alpha_2\wedge\alpha_4\ominus\alpha_3\wedge\beta_2-\alpha_2\wedge\beta_2\ominus\alpha_3\wedge\alpha_4)\ \alpha_1\wedge\beta_1\wedge\beta_3+\\
&(\alpha_2\wedge\alpha_3\ominus\alpha_4\wedge\beta_1-\alpha_2\wedge\alpha_4\ominus\alpha_3\wedge\beta_1+\alpha_2\wedge\beta_1\ominus\alpha_3\wedge\alpha_4)\ \alpha_1\wedge\beta_2\wedge\beta_3+\\
&(\alpha_1\wedge\beta_1\ominus\beta_2\wedge\beta_3-\alpha_1\wedge\beta_2\ominus\beta_1\wedge\beta_3+\alpha_1\wedge\beta_3\ominus\beta_1\wedge\beta_2)\ \alpha_2\wedge\alpha_3\wedge\alpha_4+\\
&(-(\alpha_1\wedge\alpha_3\ominus\alpha_4\wedge\beta_3)+\alpha_1\wedge\alpha_4\ominus\alpha_3\wedge\beta_3-\alpha_1\wedge\beta_3\ominus\alpha_3\wedge\alpha_4)\ \alpha_2\wedge\beta_1\wedge\beta_2+\\
&(\alpha_1\wedge\alpha_3\ominus\alpha_4\wedge\beta_2-\alpha_1\wedge\alpha_4\ominus\alpha_3\wedge\beta_2+\alpha_1\wedge\beta_2\ominus\alpha_3\wedge\alpha_4)\ \alpha_2\wedge\beta_1\wedge\beta_3+\\
&(-(\alpha_1\wedge\alpha_3\ominus\alpha_4\wedge\beta_1)+\alpha_1\wedge\alpha_4\ominus\alpha_3\wedge\beta_1-\alpha_1\wedge\beta_1\ominus\alpha_3\wedge\alpha_4)\ \alpha_2\wedge\beta_2\wedge\beta_3+\\
&(\alpha_1\wedge\alpha_2\ominus\alpha_4\wedge\beta_3-\alpha_1\wedge\alpha_4\ominus\alpha_2\wedge\beta_3+\alpha_1\wedge\beta_3\ominus\alpha_2\wedge\alpha_4)\ \alpha_3\wedge\beta_1\wedge\beta_2+\\
&(-(\alpha_1\wedge\alpha_2\ominus\alpha_4\wedge\beta_2)+\alpha_1\wedge\alpha_4\ominus\alpha_2\wedge\beta_2-\alpha_1\wedge\beta_2\ominus\alpha_2\wedge\alpha_4)\ \alpha_3\wedge\beta_1\wedge\beta_3+\\
&(\alpha_1\wedge\alpha_2\ominus\alpha_4\wedge\beta_1-\alpha_1\wedge\alpha_4\ominus\alpha_2\wedge\beta_1+\alpha_1\wedge\beta_1\ominus\alpha_2\wedge\alpha_4)\ \alpha_3\wedge\beta_2\wedge\beta_3+\\
&(-(\alpha_1\wedge\alpha_2\ominus\alpha_3\wedge\beta_3)+\alpha_1\wedge\alpha_3\ominus\alpha_2\wedge\beta_3-\alpha_1\wedge\beta_3\ominus\alpha_2\wedge\alpha_3)\ \alpha_4\wedge\beta_1\wedge\beta_2+\\
&(\alpha_1\wedge\alpha_2\ominus\alpha_3\wedge\beta_2-\alpha_1\wedge\alpha_3\ominus\alpha_2\wedge\beta_2+\alpha_1\wedge\beta_2\ominus\alpha_2\wedge\alpha_3)\ \alpha_4\wedge\beta_1\wedge\beta_3+\\
&(-(\alpha_1\wedge\alpha_2\ominus\alpha_3\wedge\beta_1)+\alpha_1\wedge\alpha_3\ominus\alpha_2\wedge\beta_1-\alpha_1\wedge\beta_1\ominus\alpha_2\wedge\alpha_3)\ \alpha_4\wedge\beta_2\wedge\beta_3+\\
&(-2\ (\alpha_1\wedge\alpha_3\ominus\alpha_2\wedge\alpha_4)+2\ (\alpha_1\wedge\alpha_2\ominus\alpha_3\wedge\alpha_4+\alpha_1\wedge\alpha_4\ominus\alpha_2\wedge\alpha_3))\\
&\ \beta_1\wedge\beta_2\wedge\beta_3\end{aligned}$$

Reduction to scalar products

In order to show the equality of the two forms, we need to show that the above difference is zero. As in the previous section we may do this by converting the inner products to scalar products. Alternatively we may observe that the coefficients of the terms are zero by virtue of the Zero Interior Sum Theorem discussed Section 10.8.

```
ToScalarProducts[AB]
```

0

This is an example of how the B form of the generalized product may be expanded in terms of either factor.

10.6 The Zero Interior Sum Theorem

Generalized Grassmann products with the unit scalar

Suppose in the Generalized Product Theorem that $\underset{m}{\alpha}$ is unity. Then we can write:

$$1 \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(1 \ominus \underset{\lambda}{\beta^j}\right) \wedge \underset{k-\lambda}{\beta^j} == \sum_{j=1}^{\binom{k}{\lambda}} \left(1 \wedge \underset{k-\lambda}{\beta^j}\right) \ominus \underset{\lambda}{\beta^j}$$

When λ *is equal to zero* we have the trivial identity that:

$$1 \underset{0}{\triangle} \underset{k}{\beta} == \underset{k}{\beta}$$

When λ *greater than zero*, the A form is clearly zero, and hence the B form is zero also. Thus we have the immediate result that

$$\sum_{j=1}^{\binom{k}{\lambda}} \beta^j_{k-\lambda} \ominus \beta^j_{\lambda} == 0$$
$$\underset{k}{\beta} == \beta^1_{\lambda} \wedge \beta^1_{k-\lambda} == \beta^2_{\lambda} \wedge \beta^2_{k-\lambda} == \cdots \qquad 10.11$$

We might express this more mnemonically as

$$\underset{k}{\beta} == \beta^1_{\lambda} \wedge \beta^1_{k-\lambda} == \beta^2_{\lambda} \wedge \beta^2_{k-\lambda} == \beta^3_{\lambda} \wedge \beta^3_{k-\lambda} == \cdots$$
$$\Longrightarrow \quad \beta^1_{k-\lambda} \ominus \beta^1_{\lambda} + \beta^2_{k-\lambda} \ominus \beta^2_{\lambda} + \beta^2_{k-\lambda} \ominus \beta^3_{\lambda} + \cdots == 0 \qquad 10.12$$

Generalized Grassmann products with the unit *n*-element

In a development *mutatis mutandis* of formulae 10.11 and 10.12 we determine the consequences of one of the elements being the unit *n*-element $\overline{1}$.

Suppose in the Generalized Product Theorem that $\underset{m}{\alpha}$ is the unit *n*-element $\overline{1}$.

$$\overline{1} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\overline{1} \ominus \beta^j_{\lambda}\right) \wedge \beta^j_{k-\lambda} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\overline{1} \wedge \beta^j_{k-\lambda}\right) \ominus \beta^j_{\lambda}$$

The interior product of the unit *n*-element with an arbitrary element is just the complement of that element, since:

$$\overline{1} \ominus \underset{k}{\beta} == \overline{1} \vee \overline{\underset{k}{\beta}} == \overline{1 \wedge \underset{k}{\beta}} == \overline{\underset{k}{\beta}}$$

Thus we see that *when λ is equal to k* we have the trivial identity that:

$$\overline{1} \underset{k}{\triangle} \underset{k}{\beta} == \overline{\underset{k}{\beta}}$$

When λ *is less than k*, the B form is clearly zero, and hence the A form is zero also. Note also that the following expressions are equal, apart from a possible sign.

$$\left(\overline{1} \ominus \beta^j_{\lambda}\right) \wedge \beta^j_{k-\lambda} \equiv \overline{\beta^j_{\lambda}} \wedge \beta^j_{k-\lambda} \equiv \overline{\beta^j_{\lambda} \ominus \beta^j_{k-\lambda}}$$

Since if the complement of an expression is zero, then the expression itself is zero, we have

$$\sum_{j=1}^{\binom{k}{\lambda}} \underset{\lambda}{\beta^j} \ominus \underset{k-\lambda}{\beta^j} == 0$$
$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots \qquad 10.13$$

Note that this is the same result as in [10.11] and [10.12] *except* that the elements of grade λ have exchanged places with those of grade $k-\lambda$ in the interior products. We might express this more mnemonically as:

$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \underset{\lambda}{\beta^3} \wedge \underset{k-\lambda}{\beta^3} == \cdots$$
$$\Longrightarrow \quad \underset{\lambda}{\beta^1} \ominus \underset{k-\lambda}{\beta^1} + \underset{\lambda}{\beta^2} \ominus \underset{k-\lambda}{\beta^2} + \underset{\lambda}{\beta^2} \ominus \underset{k-\lambda}{\beta^3} + \cdots == 0 \qquad 10.14$$

The Zero Interior Sum Theorem

These two results are collected together and referred to as the Zero Interior Sum Theorem. It is valid for $0 < \lambda < k$.

$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \underset{\lambda}{\beta^3} \wedge \underset{k-\lambda}{\beta^3} == \cdots$$
$$\Longrightarrow \quad \underset{\lambda}{\beta^1} \ominus \underset{k-\lambda}{\beta^1} + \underset{\lambda}{\beta^2} \ominus \underset{k-\lambda}{\beta^2} + \underset{\lambda}{\beta^2} \ominus \underset{k-\lambda}{\beta^3} + \cdots == 0 \qquad 10.15$$
$$\Longrightarrow \quad \underset{k-\lambda}{\beta^1} \ominus \underset{\lambda}{\beta^1} + \underset{k-\lambda}{\beta^2} \ominus \underset{\lambda}{\beta^2} + \underset{k-\lambda}{\beta^2} \ominus \underset{\lambda}{\beta^3} + \cdots == 0$$

Generating the zero interior sum

We can generate the zero interior sum by applying `ToInteriorProductsB` to the generalized product of an element with unity. For example:

Example: 2-elements

$$\text{ToInteriorProductsB}\left[1 \underset{1}{\triangle} \underset{2}{\beta}\right]$$

0

Example: 3-elements

$$\text{ToInteriorProductsB}\left[1 \underset{1}{\triangle} \underset{3}{\beta}\right]$$

$$\beta_1 \wedge \beta_2 \ominus \beta_3 - \beta_1 \wedge \beta_3 \ominus \beta_2 + \beta_2 \wedge \beta_3 \ominus \beta_1$$

ToInteriorProductsB$\left[1 \underset{2}{\triangle} \underset{3}{\beta}\right]$

$\beta_1 \ominus \beta_2 \wedge \beta_3 - \beta_2 \ominus \beta_1 \wedge \beta_3 + \beta_3 \ominus \beta_1 \wedge \beta_2$

Example: 4-elements

Alternatively we can use the *GrassmannAlgebra* function `InteriorSum`.

InteriorSum[1]$\left[\underset{4}{\beta}\right]$

$\beta_1 \ominus \beta_2 \wedge \beta_3 \wedge \beta_4 - \beta_2 \ominus \beta_1 \wedge \beta_3 \wedge \beta_4 + \beta_3 \ominus \beta_1 \wedge \beta_2 \wedge \beta_4 - \beta_4 \ominus \beta_1 \wedge \beta_2 \wedge \beta_3$

InteriorSum[2]$\left[\underset{4}{\beta}\right]$

$2 (\beta_1 \wedge \beta_2 \ominus \beta_3 \wedge \beta_4) - 2 (\beta_1 \wedge \beta_3 \ominus \beta_2 \wedge \beta_4) + 2 (\beta_1 \wedge \beta_4 \ominus \beta_2 \wedge \beta_3)$

InteriorSum[3]$\left[\underset{4}{\beta}\right]$

$\beta_1 \wedge \beta_2 \wedge \beta_3 \ominus \beta_4 - \beta_1 \wedge \beta_2 \wedge \beta_4 \ominus \beta_3 + \beta_1 \wedge \beta_3 \wedge \beta_4 \ominus \beta_2 - \beta_2 \wedge \beta_3 \wedge \beta_4 \ominus \beta_1$

We can verify that these expressions are indeed zero by converting them to their scalar product form. For example:

ToScalarProducts$\left[\textbf{InteriorSum[3]}\left[\underset{4}{\beta}\right]\right]$

0

10.7 The Zero Generalized Sum

The zero generalized sum conjecture

The zero generalized sum conjecture conjectures that if the interior product operation is replaced in the Zero Interior Sum Theorem by a generalized product of order other than zero (that is, other than by the exterior product), then the theorem still holds. That is

$$\begin{gathered}
\underset{k}{\beta} == \underset{p}{\beta^1} \wedge \underset{k-p}{\beta^1} == \underset{p}{\beta^2} \wedge \underset{k-p}{\beta^2} == \underset{p}{\beta^3} \wedge \underset{k-p}{\beta^3} == \cdots \\
\Longrightarrow \quad \underset{p}{\beta^1} \underset{\lambda}{\triangle} \underset{k-p}{\beta^1} + \underset{p}{\beta^2} \underset{\lambda}{\triangle} \underset{k-p}{\beta^2} + \underset{p}{\beta^3} \underset{\lambda}{\triangle} \underset{k-p}{\beta^3} + \cdots == 0 \qquad \lambda \neq 0 \\
\Longrightarrow \quad \underset{k-p}{\beta^1} \underset{\lambda}{\triangle} \underset{p}{\beta^1} + \underset{k-p}{\beta^2} \underset{\lambda}{\triangle} \underset{p}{\beta^2} + \underset{k-p}{\beta^3} \underset{\lambda}{\triangle} \underset{p}{\beta^3} + \cdots == 0 \qquad \lambda \neq 0
\end{gathered} \tag{10.16}$$

Generating the zero generalized sum

In *GrassmannAlgebra* we can use **GeneralizedSum[p,λ][$\beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \cdots$]** to generate the zero generalized sum expression. Here *p* is the grade of the first factor. We obtain an equivalent sum if *p* is replaced by *p*–λ since for each *i*:

$$\underset{k-p}{\beta^i} \underset{\lambda}{\triangle} \underset{p}{\beta^i} == (-1)^{(p-\lambda)\,(k-p-\lambda)}\, \underset{p}{\beta^i} \underset{\lambda}{\triangle} \underset{k-p}{\beta^i}$$

```
GeneralizedSum[2, 1][β1 ∧ β2 ∧ β3 ∧ β4]
```

$$\beta_1 \wedge \beta_2 \underset{1}{\triangle} \beta_3 \wedge \beta_4 - \beta_1 \wedge \beta_3 \underset{1}{\triangle} \beta_2 \wedge \beta_4 + \beta_1 \wedge \beta_4 \underset{1}{\triangle} \beta_2 \wedge \beta_3 + \beta_2 \wedge \beta_3 \underset{1}{\triangle} \beta_1 \wedge \beta_4 - \beta_2 \wedge \beta_4 \underset{1}{\triangle} \beta_1 \wedge \beta_3 + \beta_3 \wedge \beta_4 \underset{1}{\triangle} \beta_1 \wedge \beta_2$$

We can use the graded variable $\underset{4}{\beta}$ as an argument to get the same result.

```
GeneralizedSum[2, 1][β_4]
```

$$\beta_1 \wedge \beta_2 \underset{1}{\triangle} \beta_3 \wedge \beta_4 - \beta_1 \wedge \beta_3 \underset{1}{\triangle} \beta_2 \wedge \beta_4 + \beta_1 \wedge \beta_4 \underset{1}{\triangle} \beta_2 \wedge \beta_3 + \beta_2 \wedge \beta_3 \underset{1}{\triangle} \beta_1 \wedge \beta_4 - \beta_2 \wedge \beta_4 \underset{1}{\triangle} \beta_1 \wedge \beta_3 + \beta_3 \wedge \beta_4 \underset{1}{\triangle} \beta_1 \wedge \beta_2$$

Exploring the conjecture

We can use *GrassmannAlgebra* and *Mathematica* to explore the conjecture by tabulating cases to see if the result is zero. Here we tabulate the first 50 cases.

```
Flatten[Table[ToScalarProducts[GeneralizedSum[p, λ][x_k]],
  {k, 2, 8}, {p, 1, k - 1}, {λ, 1, Min[p, k - p]}]]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

10.8 Nilpotent Generalized Products

Nilpotent products of simple elements

In the case $\underset{k}{\beta}$ is equal to $\underset{m}{\alpha}$ and $\underset{m}{\alpha}$ is simple and *m* is not zero, form B becomes:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{m}{\alpha} == \sum_{j=1}^{\binom{k}{\lambda}} \left(\underset{m}{\alpha} \wedge \underset{m-\lambda}{\alpha^j}\right) \ominus \underset{\lambda}{\alpha^j}$$

We can see from the right hand side that $\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{m}{\alpha}$ is zero for all λ *except* λ equal to m (where the generalized product reduces to the interior (and in this case inner) product). Thus

$$\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{m}{\alpha} == 0, \quad \lambda \neq m \qquad 10.17$$

For example, we can calculate the generalized products of all orders for the 3-element $\underset{3}{\alpha}$, to see that they are all zero except for λ equal to 3.

$$\text{Table}\left[\text{ToInnerProducts}\left[\underset{3}{\alpha} \underset{\lambda}{\Delta} \underset{3}{\alpha}\right], \{\lambda, 0, 3\}\right]$$

$$\{0, 0, 0, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_1 \wedge \alpha_2 \wedge \alpha_3\}$$

This result gives rise to a series of relationships among the exterior and interior products of factors of a simple element. These relationships are independent of the dimension of the space concerned. For example, applying the result to simple 2, 3 and 4-elements gives the following relationships at the interior product level:

Example: 2-elements

$$\text{ToInteriorProducts}\left[\underset{2}{\alpha} \underset{1}{\Delta} \underset{2}{\alpha}\right] == 0$$

$$(\alpha_1 \wedge \alpha_2 \ominus \alpha_1) \wedge \alpha_2 - (\alpha_1 \wedge \alpha_2 \ominus \alpha_2) \wedge \alpha_1 == 0$$

Example: 3-elements

$$\text{ToInteriorProducts}\left[\underset{3}{\alpha} \underset{1}{\Delta} \underset{3}{\alpha}\right] == 0$$

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_1) \wedge \alpha_2 \wedge \alpha_3 - (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_2) \wedge \alpha_1 \wedge \alpha_3 + (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_3) \wedge \alpha_1 \wedge \alpha_2 == 0$$

$$\text{ToInteriorProducts}\left[\underset{3}{\alpha} \underset{2}{\Delta} \underset{3}{\alpha}\right] == 0$$

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_1 \wedge \alpha_2) \wedge \alpha_3 - (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_1 \wedge \alpha_3) \wedge \alpha_2 + (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \ominus \alpha_2 \wedge \alpha_3) \wedge \alpha_1 == 0$$

Example: 4-elements

$$\text{ToInteriorProducts}\left[\underset{4}{\alpha} \underset{1}{\Delta} \underset{4}{\alpha}\right] == 0$$

$$(\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \alpha_1) \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 - (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \alpha_2) \wedge \alpha_1 \wedge \alpha_3 \wedge \alpha_4 + (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \alpha_3) \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_4 - (\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \ominus \alpha_4) \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 == 0$$

$$\text{ToInteriorProducts}\left[\underset{4}{\alpha}\underset{2}{\triangle}\underset{4}{\alpha}\right] == 0$$

$$(\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_1\wedge\alpha_2)\wedge\alpha_3\wedge\alpha_4 - (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_1\wedge\alpha_3)\wedge\alpha_2\wedge\alpha_4 + (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_1\wedge\alpha_4)\wedge\alpha_2\wedge\alpha_3 + (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_2\wedge\alpha_3)\wedge\alpha_1\wedge\alpha_4 - (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_2\wedge\alpha_4)\wedge\alpha_1\wedge\alpha_3 + (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_3\wedge\alpha_4)\wedge\alpha_1\wedge\alpha_2 == 0$$

$$\text{ToInteriorProducts}\left[\underset{4}{\alpha}\underset{3}{\triangle}\underset{4}{\alpha}\right] == 0$$

$$(\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_1\wedge\alpha_2\wedge\alpha_3)\wedge\alpha_4 - (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_1\wedge\alpha_2\wedge\alpha_4)\wedge\alpha_3 + (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_1\wedge\alpha_3\wedge\alpha_4)\wedge\alpha_2 - (\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4\ominus\alpha_2\wedge\alpha_3\wedge\alpha_4)\wedge\alpha_1 == 0$$

At the inner product level, some of the generalized products yield further relationships. For example, $\underset{4}{\alpha}\underset{2}{\triangle}\underset{4}{\alpha}$ confirms the Zero Interior Sum Theorem.

$$\text{Simplify}\left[\text{CollectTerms}\left[\text{ToInnerProducts}\left[\underset{4}{\alpha}\underset{2}{\triangle}\underset{4}{\alpha}\right]\right]\right]$$

$$2\,(\alpha_1\wedge\alpha_2\ominus\alpha_3\wedge\alpha_4 - \alpha_1\wedge\alpha_3\ominus\alpha_2\wedge\alpha_4 + \alpha_1\wedge\alpha_4\ominus\alpha_2\wedge\alpha_3)\,\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\alpha_4$$

Nilpotent products of non-simple elements

From the axioms for the exterior product it is straightforward to show that for a general (not necessarily simple) *m*-element $\underset{m}{\alpha}$ that $\underset{m}{\alpha}\wedge\underset{m}{\alpha} == (-1)^{m}\,\underset{m}{\alpha}\wedge\underset{m}{\alpha}$. This means, of course, that the exterior product of a general *m*-element with itself is zero if *m* is odd.

In a similar manner, the formula for the quasi-commutativity of the generalized product is:

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta} == (-1)^{(m-\lambda)\,(k-\lambda)}\,\underset{k}{\beta}\underset{\lambda}{\triangle}\underset{m}{\alpha}$$

This shows that the generalized product of a general *m*-element with itself is:

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{m}{\alpha} == (-1)^{(m-\lambda)}\,\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{m}{\alpha}$$

Thus the generalization of the above result for the exterior product is that the generalized product of a general *m*-element with itself is zero if *m*-λ is odd.

$$\boxed{\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{m}{\alpha} == 0 \qquad m-\lambda \in \text{OddIntegers}} \quad \mathbf{10.18}$$

10.9 Properties of the Generalized Product

Summary of properties

The following simple results may be obtained from the definition and results derived above.

◆ **The generalized product is distributive with respect to addition**

$$\left(\underset{m}{\alpha}+\underset{k}{\beta}\right)\underset{\lambda}{\triangle}\underset{p}{\gamma} = \underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{p}{\gamma} + \underset{k}{\beta}\underset{\lambda}{\triangle}\underset{p}{\gamma} \qquad 10.19$$

$$\underset{p}{\gamma}\underset{\lambda}{\triangle}\left(\underset{m}{\alpha}+\underset{k}{\beta}\right) = \underset{p}{\gamma}\underset{\lambda}{\triangle}\underset{m}{\alpha} + \underset{p}{\gamma}\underset{\lambda}{\triangle}\underset{k}{\beta} \qquad 10.20$$

◆ **The generalized product with a zero element is zero**

$$0\underset{\lambda}{\triangle}\underset{m}{\alpha} = \underset{m}{\alpha}\underset{\lambda}{\triangle}0 = 0 \qquad 10.21$$

◆ **The generalized product with a scalar reduces to the field product**

The only non-zero generalized product with a scalar is of order zero, leading to the ordinary field product.

$$a\underset{0}{\triangle}\underset{m}{\alpha} = \underset{m}{\alpha}\underset{0}{\triangle}a = a\wedge\underset{m}{\alpha} = \underset{m}{\alpha}\ominus a = a\,\underset{m}{\alpha} \qquad 10.22$$

$$a\underset{0}{\triangle}b = a\wedge b = a\ominus b = a\,b \qquad 10.23$$

◆ **The generalized products of a scalar with an m-element are zero**

For λ greater than zero

$$a\underset{\lambda}{\triangle}\underset{m}{\alpha} = \underset{m}{\alpha}\underset{\lambda}{\triangle}a = 0 \qquad \lambda > 0 \qquad 10.24$$

◆ **Scalars may be factored out of generalized products**

$$\left(a\,\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\underset{k}{\beta} == \underset{m}{\alpha}\underset{\lambda}{\triangle}\left(a\,\underset{k}{\beta}\right) == a\left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right) \qquad 10.25$$

◆ **The generalized product is quasi-commutative**

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta} == (-1)^{(m-\lambda)\,(k-\lambda)}\,\underset{k}{\beta}\underset{\lambda}{\triangle}\underset{m}{\alpha} \qquad 10.26$$

This relationship was proven in Section 10.3.

10.10 The Triple Generalized Sum Conjecture

In this section we discuss an interesting conjecture associated with our definition of the Clifford product in Chapter 12. As already mentioned, we have defined the generalized Grassmann product to facilitate the definition of the Clifford product of general Grassmann expressions. As is well known, the Clifford product is associative. But in general, a Clifford product will involve both exterior and interior products. And the interior product is not associative. In this section we look at a conjecture, which, if established, will prove the validity of our definition of the Clifford product in terms of generalized Grassmann products.

The generalized product is not associative

We take a simple example to show that the generalized product is not associative. First we set up the assertion:

$$H = \left(x\underset{0}{\triangle}y\right)\underset{1}{\triangle}z \neq x\underset{0}{\triangle}\left(y\underset{1}{\triangle}z\right);$$

Then we convert the generalized products to scalar products.

```
ToScalarProducts[H]
```

$$y\,(x\ominus z) - x\,(y\ominus z) \neq x\,(y\ominus z)$$

Clearly the two sides of the relation are not in general equal. Hence the generalized product is not in general associative.

The triple generalized sum

Triple generalized products

We define a *triple generalized product* to be a product either of the form $\left(\underset{m}{x}\underset{\lambda}{\triangle}\underset{k}{y}\right)\underset{\mu}{\triangle}\underset{p}{z}$ or of the form $\underset{m}{x}\underset{\lambda}{\triangle}\left(\underset{k}{y}\underset{\mu}{\triangle}\underset{p}{z}\right)$. Since we have shown above that these forms will in general be different, we will refer to the first one as the A form, and to the second as the B form.

Signed triple generalized products

We define the *signed* triple generalized product of form A to be the triple generalized product of form A multiplied by the factor $(-1)^{s_A}$, where $s_A =$ $m\lambda+\frac{1}{2}\lambda(1+\lambda)+(k+m)\mu+\frac{1}{2}\mu(1+\mu)$.

We define the *signed* triple generalized product of form B to be the triple generalized product of form B multiplied by the factor $(-1)^{s_B}$, where $s_B = m\lambda+\frac{1}{2}\lambda(1+\lambda)+k\mu+\frac{1}{2}\mu(1+\mu)$.

Triple generalized sums

We define a *triple generalized sum of form* A to be the sum of all the signed triple generalized products of form A of the same elements and which have the same grade n.

$$\sum_{\lambda=0}^{n}(-1)^{s_A}\left(\underset{m}{x}\underset{\lambda}{\triangle}\underset{k}{y}\right)\underset{n-\lambda}{\triangle}\underset{p}{z}$$

where $s_A = m\lambda+\frac{1}{2}\lambda(\lambda+1)+(m+k)(n-\lambda)+\frac{1}{2}(n-\lambda)(n-\lambda+1)$.

We define a *triple generalized sum of form* B to be the sum of all the signed triple generalized products of form B of the same elements and which have the same grade n.

$$\sum_{\lambda=0}^{n}(-1)^{s_B}\underset{m}{x}\underset{\lambda}{\triangle}\left(\underset{k}{y}\underset{n-\lambda}{\triangle}\underset{p}{z}\right)$$

where $s_B = m\lambda+\frac{1}{2}\lambda(\lambda+1)+k(n-\lambda)+\frac{1}{2}(n-\lambda)(n-\lambda+1)$.

For example, we tabulate below the triple generalized sums of form A for grades of 0, 1 and 2:

```
Table[{i, TripleGeneralizedSumA[i][x_m, y_k, z_p]}, {i, 0, 2}] //
 TableForm
```

0	$\left(\underset{m}{x}\underset{0}{\triangle}\underset{k}{y}\right)\underset{0}{\triangle}\underset{p}{z}$
1	$(-1)^{1+m}\left(\left(\underset{m}{x}\underset{1}{\triangle}\underset{k}{y}\right)\underset{0}{\triangle}\underset{p}{z}\right)+(-1)^{1+k+m}\left(\left(\underset{m}{x}\underset{0}{\triangle}\underset{k}{y}\right)\underset{1}{\triangle}\underset{p}{z}\right)$
2	$-\left(\left(\underset{m}{x}\underset{2}{\triangle}\underset{k}{y}\right)\underset{0}{\triangle}\underset{p}{z}\right)+(-1)^{k}\left(\left(\underset{m}{x}\underset{1}{\triangle}\underset{k}{y}\right)\underset{1}{\triangle}\underset{p}{z}\right)-\left(\underset{m}{x}\underset{0}{\triangle}\underset{k}{y}\right)\underset{2}{\triangle}\underset{p}{z}$

Similarly the triple generalized sums of form B for grades of 0, 1 and 2 are

```
Table[{i, TripleGeneralizedSumB[i][x_m, y_k, z_p]}, {i, 0, 2}] //
 TableForm
```

0	$\underset{m}{x}\underset{0}{\triangle}\left(\underset{k}{y}\underset{0}{\triangle}\underset{p}{z}\right)$
1	$(-1)^{1+k}\left(\underset{m}{x}\underset{0}{\triangle}\left(\underset{k}{y}\underset{1}{\triangle}\underset{p}{z}\right)\right)+(-1)^{1+m}\left(\underset{m}{x}\underset{1}{\triangle}\left(\underset{k}{y}\underset{0}{\triangle}\underset{p}{z}\right)\right)$
2	$-\left(\underset{m}{x}\underset{0}{\triangle}\left(\underset{k}{y}\underset{2}{\triangle}\underset{p}{z}\right)\right)+(-1)^{2+k+m}\left(\underset{m}{x}\underset{1}{\triangle}\left(\underset{k}{y}\underset{1}{\triangle}\underset{p}{z}\right)\right)-\underset{m}{x}\underset{2}{\triangle}\left(\underset{k}{y}\underset{0}{\triangle}\underset{p}{z}\right)$

The triple generalized sum conjecture

As we have shown above, a *single* triple generalized product is not in general associative. That is, the A form and the B form are not in general equal. However we *conjecture* that the *triple generalized sum* of form A is equal to the *triple generalized sum* of form B.

$$\boxed{\sum_{\lambda=0}^{n}(-1)^{s_A}\left(\underset{m}{x}\underset{\lambda}{\triangle}\underset{k}{y}\right)\underset{n-\lambda}{\triangle}\underset{p}{z}==\sum_{\lambda=0}^{n}(-1)^{s_B}\underset{m}{x}\underset{\lambda}{\triangle}\left(\underset{k}{y}\underset{n-\lambda}{\triangle}\underset{p}{z}\right)} \qquad 10.27$$

where $s_A = m\lambda+\frac{1}{2}\lambda(\lambda+1)+(m+k)(n-\lambda)+\frac{1}{2}(n-\lambda)(n-\lambda+1)$
and $\quad s_B = m\lambda+\frac{1}{2}\lambda(\lambda+1)+k(n-\lambda)+\frac{1}{2}(n-\lambda)(n-\lambda+1)$.

If this conjecture can be shown to be true, then the associativity of the Clifford product of a general Grassmann expression can be straightforwardly proven using the definition of the Clifford product in terms of exterior and interior products (or, what is equivalent, in terms of generalized products). That is, the rules of Clifford algebra may be entirely determined by the axioms and theorems of the Grassmann algebra, making it unnecessary, and indeed potentially inconsistent, to introduce any special Clifford algebra axioms.

Exploring the triple generalized sum conjecture

We can explore the triple generalized sum conjecture by converting the generalized products into scalar products.

For example first we generate the triple generalized sums.

```
A = TripleGeneralizedSumA[2][x_3, y_2, z]
```

$$-\left(\left(\underset{3}{x}\underset{2}{\triangle}\underset{2}{y}\right)\underset{0}{\triangle}z\right)+\left(\underset{3}{x}\underset{1}{\triangle}\underset{2}{y}\right)\underset{1}{\triangle}z-\left(\underset{3}{x}\underset{0}{\triangle}\underset{2}{y}\right)\underset{2}{\triangle}z$$

B = TripleGeneralizedSumB[2]$\left[\underset{3}{x}, \underset{2}{y}, z\right]$

$$-\left(\underset{3}{x} \underset{0}{\triangle} \left(\underset{2}{y} \underset{2}{\triangle} z\right)\right) - \underset{3}{x} \underset{1}{\triangle} \left(\underset{2}{y} \underset{1}{\triangle} z\right) - \underset{3}{x} \underset{2}{\triangle} \left(\underset{2}{y} \underset{0}{\triangle} z\right)$$

We could demonstrate the equality of these two sums directly by converting their difference into scalar product form, thus allowing terms to cancel.

Expand[ToScalarProducts[A - B]]

0

However it is more instructive to convert each of the terms in the difference separately to scalar products.

X = List @@ (A - B)

$$\begin{array}{l}\left\{\underset{3}{x} \underset{0}{\triangle} \left(\underset{2}{y} \underset{2}{\triangle} z\right), -\left(\left(\underset{3}{x} \underset{2}{\triangle} \underset{2}{y}\right) \underset{0}{\triangle} z\right), \underset{3}{x} \underset{1}{\triangle} \left(\underset{2}{y} \underset{1}{\triangle} z\right),\right. \\ \quad \left.\left(\underset{3}{x} \underset{1}{\triangle} \underset{2}{y}\right) \underset{1}{\triangle} z, \underset{3}{x} \underset{2}{\triangle} \left(\underset{2}{y} \underset{0}{\triangle} z\right), -\left(\left(\underset{3}{x} \underset{0}{\triangle} \underset{2}{y}\right) \underset{2}{\triangle} z\right)\right\}\end{array}$$

X1 = Expand[ToScalarProducts[X]]

$$\begin{array}{l}
\{0, -(x_2 \ominus y_2)\,(x_3 \ominus y_1)\, z \wedge x_1 + (x_2 \ominus y_1)\,(x_3 \ominus y_2)\, z \wedge x_1 + \\
\quad (x_1 \ominus y_2)\,(x_3 \ominus y_1)\, z \wedge x_2 - (x_1 \ominus y_1)\,(x_3 \ominus y_2)\, z \wedge x_2 - \\
\quad (x_1 \ominus y_2)\,(x_2 \ominus y_1)\, z \wedge x_3 + (x_1 \ominus y_1)\,(x_2 \ominus y_2)\, z \wedge x_3, \\
\; -(z \ominus y_2)\,(x_3 \ominus y_1)\, x_1 \wedge x_2 + (z \ominus y_1)\,(x_3 \ominus y_2)\, x_1 \wedge x_2 + \\
\quad (z \ominus y_2)\,(x_2 \ominus y_1)\, x_1 \wedge x_3 - (z \ominus y_1)\,(x_2 \ominus y_2)\, x_1 \wedge x_3 - \\
\quad (z \ominus y_2)\,(x_1 \ominus y_1)\, x_2 \wedge x_3 + (z \ominus y_1)\,(x_1 \ominus y_2)\, x_2 \wedge x_3, \\
\; (z \ominus y_2)\,(x_3 \ominus y_1)\, x_1 \wedge x_2 - (z \ominus y_1)\,(x_3 \ominus y_2)\, x_1 \wedge x_2 - \\
\quad (z \ominus y_2)\,(x_2 \ominus y_1)\, x_1 \wedge x_3 + (z \ominus y_1)\,(x_2 \ominus y_2)\, x_1 \wedge x_3 - \\
\quad (z \ominus x_3)\,(x_2 \ominus y_2)\, x_1 \wedge y_1 + (z \ominus x_2)\,(x_3 \ominus y_2)\, x_1 \wedge y_1 + \\
\quad (z \ominus x_3)\,(x_2 \ominus y_1)\, x_1 \wedge y_2 - (z \ominus x_2)\,(x_3 \ominus y_1)\, x_1 \wedge y_2 + \\
\quad (z \ominus y_2)\,(x_1 \ominus y_1)\, x_2 \wedge x_3 - (z \ominus y_1)\,(x_1 \ominus y_2)\, x_2 \wedge x_3 + \\
\quad (z \ominus x_3)\,(x_1 \ominus y_2)\, x_2 \wedge y_1 - (z \ominus x_1)\,(x_3 \ominus y_2)\, x_2 \wedge y_1 - \\
\quad (z \ominus x_3)\,(x_1 \ominus y_1)\, x_2 \wedge y_2 + (z \ominus x_1)\,(x_3 \ominus y_1)\, x_2 \wedge y_2 - \\
\quad (z \ominus x_2)\,(x_1 \ominus y_2)\, x_3 \wedge y_1 + (z \ominus x_1)\,(x_2 \ominus y_2)\, x_3 \wedge y_1 + \\
\quad (z \ominus x_2)\,(x_1 \ominus y_1)\, x_3 \wedge y_2 - (z \ominus x_1)\,(x_2 \ominus y_1)\, x_3 \wedge y_2, \\
\; (x_2 \ominus y_2)\,(x_3 \ominus y_1)\, z \wedge x_1 - (x_2 \ominus y_1)\,(x_3 \ominus y_2)\, z \wedge x_1 - \\
\quad (x_1 \ominus y_2)\,(x_3 \ominus y_1)\, z \wedge x_2 + (x_1 \ominus y_1)\,(x_3 \ominus y_2)\, z \wedge x_2 + \\
\quad (x_1 \ominus y_2)\,(x_2 \ominus y_1)\, z \wedge x_3 - (x_1 \ominus y_1)\,(x_2 \ominus y_2)\, z \wedge x_3 + \\
\quad (z \ominus x_3)\,(x_2 \ominus y_2)\, x_1 \wedge y_1 - (z \ominus x_2)\,(x_3 \ominus y_2)\, x_1 \wedge y_1 - \\
\quad (z \ominus x_3)\,(x_2 \ominus y_1)\, x_1 \wedge y_2 + (z \ominus x_2)\,(x_3 \ominus y_1)\, x_1 \wedge y_2 - \\
\quad (z \ominus x_3)\,(x_1 \ominus y_2)\, x_2 \wedge y_1 + (z \ominus x_1)\,(x_3 \ominus y_2)\, x_2 \wedge y_1 + \\
\quad (z \ominus x_3)\,(x_1 \ominus y_1)\, x_2 \wedge y_2 - (z \ominus x_1)\,(x_3 \ominus y_1)\, x_2 \wedge y_2 + \\
\quad (z \ominus x_2)\,(x_1 \ominus y_2)\, x_3 \wedge y_1 - (z \ominus x_1)\,(x_2 \ominus y_2)\, x_3 \wedge y_1 - \\
\quad (z \ominus x_2)\,(x_1 \ominus y_1)\, x_3 \wedge y_2 + (z \ominus x_1)\,(x_2 \ominus y_1)\, x_3 \wedge y_2, 0\}
\end{array}$$

Converting back to a sum shows that the terms add to zero.

```
X2 = Plus @@ X1
```

```
0
```

An algorithm to test the conjecture

It is simple to automate testing the conjecture by collecting the steps above in a function:

```
TestTripleGeneralizedSumConjecture[n_][x_,y_,z_]:=
Expand[ToScalarProducts[TripleGeneralizedSumA[n][x,y,z]-
TripleGeneralizedSumB[n][x,y,z]]]
```

As an example of this procedure we run through the first 320 cases. A value of zero will validate the conjecture for that case.

```
Table[TestTripleGeneralizedSumConjecture[n][x_m, y_k, z_p],
  {n, 0, 4}, {m, 0, 3}, {k, 0, 3}, {p, 0, 3}] // Flatten
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

10.11 Exploring Conjectures

As we have already shown, it is easy to explore conjectures using *Mathematica* to compute individual cases. By 'explore' of course, we mean the generation of cases which either disprove the conjecture or increase our confidence that the conjecture is correct.

A conjecture

As an example we suggest a conjecture for a relationship amongst generalized products of order 1 of the following form, valid for any m, k, and p.

$$\left(\underset{m}{\alpha}\underset{1}{\triangle}\underset{k}{\beta}\right)\underset{1}{\triangle}\underset{p}{\gamma}+(-1)^{\xi}\left(\underset{k}{\beta}\underset{1}{\triangle}\underset{p}{\gamma}\right)\underset{1}{\triangle}\underset{m}{\alpha}+(-1)^{\psi}\left(\underset{p}{\gamma}\underset{1}{\triangle}\underset{m}{\alpha}\right)\underset{1}{\triangle}\underset{k}{\beta}=0 \qquad \mathbf{10.28}$$

The signs $(-1)^{\xi}$ and $(-1)^{\psi}$ are at this point unknown, but if the conjecture is true we expect them to be simple functions of *m*, *k*, *p* and their products. We conjecture therefore that in any particular case they will be either +1 or –1.

Exploring the conjecture

The first step therefore is to test some cases to see if the formula fails for some combination of signs. We do that by setting up a function that calculates the formula for each of the possible sign combinations, returning `True` if any one of them satisfies the formula, and `False` otherwise. Then we create a table of cases. Below we let the grades *m*, *k*, and *p* range from 0 to 3 (that is, two instances of odd parity and two instances of even parity each, in all combinations). Although we have written the function as printing the intermediate results as they are calculated, we do not show this output as the results are summarized in the final list.

```
TestPostulate[{m_, k_, p_}] :=
 Module[{X, Y, Z, α, β, γ}, X = ToScalarProducts[(α_m △_1 β_k) △_1 γ_p];
  Y = ToScalarProducts[(β_k △_1 γ_p) △_1 α_m];
  Z = ToScalarProducts[(γ_p △_1 α_m) △_1 β_k];
  TrueQ[X + Y + Z == 0 || X + Y - Z == 0 || X - Y + Z == 0 || X - Y - Z == 0]]
```

```
Table[T = TestPostulate[{m, k, p}]; Print[{{m, k, p}, T}];
  T, {m, 0, 3}, {k, 0, 3}, {p, 0, 3}] // Flatten
```

```
{True, True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True, True}
```

It can be seen that the formula is valid for some values of ξ and ψ in the first 64 cases. This gives us some confidence that our efforts may not be wasted if we wished to prove the formula and/or find the values of ξ and ψ an terms of *m*, *k*, and *p*.

10.12 The Generalized Product of Intersecting Elements

The case $\lambda < p$

Consider three simple elements $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$. The elements $\underset{p}{\gamma} \wedge \underset{m}{\alpha}$ and $\underset{p}{\gamma} \wedge \underset{k}{\beta}$ may then be considered elements with an intersection $\underset{p}{\gamma}$. Generalized products of such intersecting elements are zero whenever the grade λ of the product is less than the grade of the intersection.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == 0 \qquad \lambda < p \qquad 10.29$$

We can test this by reducing the expression to scalar products and tabulating cases.

```
Flatten[Table[A = ToScalarProducts[(γ_p ∧ α_m) △_λ (γ_p ∧ β_k)];
  Print[{{m, k, p, λ}, A}]; A, {m, 0, 3},
  {k, 0, 3}, {p, 1, 3}, {λ, 0, p - 1}]]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Of course, this result is also valid in the case that either or both the grades of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is zero.

$$\underset{p}{\gamma} \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\triangle} \underset{p}{\gamma} == \underset{p}{\gamma} \underset{\lambda}{\triangle} \underset{p}{\gamma} == 0 \qquad \lambda < p \qquad 10.30$$

The case $\lambda \geq p$

For the case where λ is equal to, or greater than the grade of the intersection p, the generalized product of such intersecting elements may be expressed as the interior product of the common factor $\underset{p}{\gamma}$ with a generalized product of the remaining factors. This generalized product is of order lower by the grade of the common factor. Hence the formula is only valid for $\lambda \geq p$ (since the generalized product has only been defined for non-negative orders).

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{p\,(\lambda-p)}\left(\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda-p}{\triangle}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\right)$$
$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{p\,m}\left(\left(\underset{m}{\alpha}\underset{\lambda-p}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma}\right) \qquad \mathbf{10.31}$$

We can test these formulae by reducing the expressions on either side to scalar products and tabulating cases. For $\lambda > p$, the first formula gives

$$\texttt{Flatten}\Big[\texttt{Table}\Big[\texttt{A = ToScalarProducts}\Big[\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) - (-1)^{p\,(\lambda-p)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda-p}{\triangle}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\Big];$$
$$\texttt{Print[\{\{m, k, p, }\lambda\texttt{\}, A\}]; A, \{m, 0, 3\}, \{k, 0, 3\}, \{p, 0, 3\}, \{}\lambda\texttt{, p + 1, Min[m, k]\}}\Big]\Big]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The case of $\lambda = p$ is tabulated in the next section.

The second formula can be derived from the first by using the quasi-commutativity of the generalized product. To check, we tabulate the first few cases:

$$\texttt{Flatten}\Big[\texttt{Table}\Big[\texttt{ToScalarProducts}\Big[\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) - (-1)^{p\,m}\left(\underset{m}{\alpha}\underset{\lambda-p}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma}\Big],$$
$$\texttt{\{m, 1, 2\}, \{k, 1, m\}, \{p, 0, m\}, \{}\lambda\texttt{, p + 1, Min[p + m, p + k]\}}\Big]\Big]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Finally, we can put $\mu = \lambda - p$, and rearrange the above to give:

$$\left(\underset{m}{\alpha}\underset{\mu}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} == (-1)^{p\,\mu}\left(\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\underset{\mu}{\triangle}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\right) \qquad \mathbf{10.32}$$

$$\texttt{Flatten}\Big[\texttt{Table}\Big[\texttt{ToScalarProducts}\Big[\left(\underset{m}{\alpha}\underset{\mu}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} - (-1)^{p\,\mu}\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\underset{\mu}{\triangle}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\Big],$$
$$\texttt{\{m, 0, 2\}, \{k, 0, m\}, \{p, 0, m\}, \{}\mu\texttt{, 0, Min[m, k]\}}\Big]\Big]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The special case of $\lambda = p$

By putting $\lambda = p$ into the first formula of [10.31] we get:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{p}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)=\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}$$

By interchanging $\underset{p}{\gamma}$, $\underset{m}{\alpha}$, and $\underset{k}{\beta}$ we also have two simple alternate expressions

$$\begin{gathered}\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{p}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)=\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\\ \left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\underset{p}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)=\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\\ \left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\underset{p}{\triangle}\left(\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)=\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma}\end{gathered}\qquad\textbf{10.33}$$

Again we can demonstrate this identity by converting to scalar products.

$$\begin{aligned}&\texttt{Flatten}\Big[\\&\quad\texttt{Table}\Big[\texttt{A = ToScalarProducts}\Big[\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{p}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)-\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\Big];\\&\qquad\texttt{Print[\{\{m, k, p\}, A\}]; A, \{m, 0, 3\}, \{k, 0, 3\}, \{p, 0, 3\}}\Big]\Big]\end{aligned}$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

10.13 The Generalized Product of Orthogonal Elements

The generalized product of totally orthogonal elements

It is simple to see from the definition of the generalized product that if $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal, *and λ is not zero*, then their generalized product is zero.

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}=0\qquad \alpha_i\ominus\beta_j=0\qquad \lambda>0\qquad\textbf{10.34}$$

We can verify this easily:

$$\begin{aligned}&\texttt{Flatten}\Big[\texttt{Table}\Big[\texttt{ToScalarProducts}\Big[\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\Big]\ \texttt{/.}\\&\quad\texttt{OrthogonalSimplificationRules}\Big[\Big\{\Big\{\underset{m}{\alpha},\ \underset{k}{\beta}\Big\}\Big\}\Big],\\&\quad\texttt{\{m, 0, 4\}, \{k, 0, m\}, \{}\lambda\texttt{, 1, Min[m, k]\}}\Big]\Big]\end{aligned}$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The *GrassmannAlgebra* function `OrthogonalSimplificationRules` generates a list of rules which put to zero all the scalar products of a 1-element from $\underset{m}{\alpha}$ with a 1-element from $\underset{k}{\beta}$.

The generalized product of partially orthogonal elements

Consider the generalized product $\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)$ of an element $\underset{m}{\alpha}$ and another element $\underset{p}{\gamma}\wedge\underset{k}{\beta}$ in which $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal, but $\underset{p}{\gamma}$ is arbitrary.

But since $\alpha_i \ominus \beta_j == 0$, the interior products of $\underset{m}{\alpha}$ and any factors of $\underset{p}{\gamma}\wedge\underset{k}{\beta}$ in the expansion of the generalized product will be zero whenever they contain any factors of $\underset{k}{\beta}$. That is, the only non-zero terms in the expansion of the generalized product are of the form $\left(\underset{m}{\alpha}\ominus\underset{\lambda}{\gamma^i}\right)\wedge\underset{p-\lambda}{\gamma^i}\wedge\underset{k}{\beta}$. Hence:

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{p}{\gamma}\right)\wedge\underset{k}{\beta} \qquad \alpha_i\ominus\beta_j == 0 \qquad \lambda \le \mathrm{Min}[m, p] \tag{10.35}$$

$$\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == 0 \qquad \alpha_i\ominus\beta_j == 0 \qquad \lambda > \mathrm{Min}[m, p] \tag{10.36}$$

$$\text{Flatten}\left[\text{Table}\left[\text{ToScalarProducts}\left[\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)-\left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{p}{\gamma}\right)\wedge\underset{k}{\beta}\right] /. \text{OrthogonalSimplificationRules}\left[\left\{\left\{\underset{m}{\alpha}, \underset{k}{\beta}\right\}\right\}\right], \{m, 0, 3\}, \{k, 0, m\}, \{p, 0, m\}, \{\lambda, 0, \text{Min}[m, p]\}\right]\right]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

By using the quasi-commutativity of the generalized and exterior products, we can transform the above results to:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\underset{\lambda}{\triangle}\underset{k}{\beta} == (-1)^{m\lambda}\,\underset{m}{\alpha}\wedge\left(\underset{p}{\gamma}\underset{\lambda}{\triangle}\underset{k}{\beta}\right) \qquad \alpha_i\ominus\beta_j == 0 \qquad \lambda \le \mathrm{Min}[k, p] \tag{10.37}$$

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\underset{k}{\beta} == 0 \qquad \alpha_i\ominus\beta_j == 0 \qquad \lambda > \mathrm{Min}[k, p] \tag{10.38}$$

```
Flatten[
 Table[ToScalarProducts[(α_m ∧ γ_p) △_λ β_k - (-1)^(m λ) α_m ∧ (γ_p △_λ β_k)] /.
   OrthogonalSimplificationRules[{{α_m, β_k}}],
  {m, 0, 2}, {k, 0, m}, {p, 0, m}, {λ, 0, Min[k, p]}]]
```

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

10.14 The Generalized Product of Intersecting Orthogonal Elements

The case $\lambda < p$

Consider three simple elements $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$. The elements $\underset{p}{\gamma} \wedge \underset{m}{\alpha}$ and $\underset{p}{\gamma} \wedge \underset{k}{\beta}$ may then be considered elements with an intersection $\underset{p}{\gamma}$. As has been shown in Section 10.12, generalized products of such intersecting elements are zero whenever the grade λ of the product is less than the grade of the intersection. Hence the product will be zero for $\lambda < p$, *independent* of the orthogonality relationships of its factors.

The case $\lambda \geq p$

Consider now the case of three simple elements $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$ where $\underset{p}{\gamma}$ is *totally orthogonal* to both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ (and hence to $\underset{m}{\alpha} \wedge \underset{k}{\beta}$). A simple element $\underset{p}{\gamma}$ is totally orthogonal to an element $\underset{m}{\alpha}$ if and only if $\underset{m}{\alpha} \ominus \gamma_i == 0$ for all γ_i belonging to $\underset{p}{\gamma}$.

The generalized product of the elements $\underset{p}{\gamma} \wedge \underset{m}{\alpha}$ and $\underset{p}{\gamma} \wedge \underset{k}{\beta}$ of order $\lambda \geq p$ is a scalar factor $\underset{p}{\gamma} \ominus \underset{p}{\gamma}$ times the generalized product of the factors $\underset{m}{\alpha}$ and $\underset{k}{\beta}$, but of an order lower by the grade of the common factor; that is, of order $\lambda - p$.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \underset{\lambda-p}{\triangle} \underset{k}{\beta}\right)$$
$$\underset{p}{\gamma} = \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_p \qquad \underset{m}{\alpha} \ominus \gamma_i == \underset{k}{\beta} \ominus \gamma_i == 0 \qquad \lambda \geq p \qquad \textbf{10.39}$$

Note here that the factors of $\underset{p}{\gamma}$ are not necessarily orthogonal to each other. Neither are the factors of $\underset{m}{\alpha}$. Nor the factors of $\underset{k}{\beta}$.

We can test this out by making a table of cases. Here we look at the first 25 cases.

$$\texttt{Flatten}\Big[\texttt{Table}\Big[\texttt{ToScalarProducts}\Big[\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)-\left(\underset{p}{\gamma}\ominus\underset{p}{\gamma}\right)\left(\underset{m}{\alpha}\underset{\lambda-p}{\triangle}\underset{k}{\beta}\right)\Big]\;/.$$
$$\texttt{OrthogonalSimplificationRules}\left[\left\{\left\{\underset{m}{\alpha},\underset{p}{\gamma}\right\},\left\{\underset{k}{\beta},\underset{p}{\gamma}\right\}\right\}\right],$$
$$\{m,0,2\},\{k,0,m\},\{p,1,m+1\},\{\lambda,p,p+\texttt{Min}[m,k]\}\Big]\Big]$$

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

By combining the results of the previous section with those of this section, we see that we can also write that:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\underset{k}{\beta}\;\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)$$

$$\boxed{\begin{gathered}\left(\underset{p}{\gamma}\ominus\underset{p}{\gamma}\right)\left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right)==\\(-1)^{p\lambda}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\lambda}{\triangle}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}==(-1)^{pm}\left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma}\\\underset{p}{\gamma}=\gamma_1\wedge\gamma_2\wedge\cdots\wedge\gamma_p\qquad\underset{m}{\alpha}\ominus\gamma_i==\underset{k}{\beta}\ominus\gamma_i==0\end{gathered}}\qquad\textbf{10.40}$$

We check the second rule by taking the first 25 cases.

$$\texttt{Flatten}\Big[\texttt{Table}\Big[$$
$$\texttt{ToScalarProducts}\Big[\left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma}-(-1)^{pm}\left(\underset{p}{\gamma}\ominus\underset{p}{\gamma}\right)\left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right)\Big]\;/.$$
$$\texttt{OrthogonalSimplificationRules}\left[\left\{\left\{\underset{m}{\alpha},\underset{p}{\gamma}\right\},\left\{\underset{k}{\beta},\underset{p}{\gamma}\right\}\right\}\right],$$
$$\{m,0,2\},\{k,0,m\},\{p,1,m+1\},\{\lambda,0,\texttt{Min}[m,k]\}\Big]\Big]$$

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

10.15 Generalized Products in Lower Dimensional Spaces

Generalized products in 0, 1, and 2-spaces

In this section we summarize the properties of generalized products for spaces of dimension 0, 1 and 2.

In spaces of all dimensions, generalized products in which one of the factors is a scalar will reduce to either the usual (field or linear space) product by the scalar quantity (when the order of the product is zero) or else to zero (when the order of the product is greater than zero). Except for a brief discussion of the implications of this for 0-space, further discussions will assume that the factors of the generalized products are not scalar.

In spaces of dimension 0, 1, or 2, all generalized products reduce either to the ordinary field product, the exterior product, the interior product, or zero. Hence they bring little new to a study of these spaces.

These results will be clear upon reference to the simple properties of generalized products discussed in previous sections. The only result that may not be immediately obvious is that of the generalized product of order one of two 2-elements in a 2-space, for example $\underset{2}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}$. In a 2-space, $\underset{2}{\alpha}$ and $\underset{2}{\beta}$ must be congruent (differ only by (possibly) a scalar factor), and hence by formula 10.17 is zero.

If one of the factors is a scalar, then the only non-zero generalized product is that of order zero, equivalent to the ordinary field product (and, incidentally, also to the exterior and interior products).

0-space

In a space of zero dimensions (the underlying field of the Grassmann algebra), the generalized product reduces to the exterior product and hence to the usual scalar (field) product.

$$\mathbf{ToScalarProducts}\left[a \underset{0}{\triangle} b\right]$$

$$a\, b$$

Higher order products (for example $a \underset{1}{\triangle} b$) are of course zero since the order of the product is greater than the minimum of the grades of the factors (which in this case is zero).

In sum: The only non-zero generalized product in a space of zero dimensions is the product of order zero, equivalent to the underlying field product.

1-space

Products of zero order

In a space of one dimension there is only one basis element, so the product of zero order (that is, the exterior product) of any two elements is zero:

$$(a\, e_1) \underset{0}{\triangle} (b\, e_1) == a\, b\, e_1 \wedge e_1 == 0$$

$$\mathbf{ToScalarProducts}\left[(a\, e_1) \underset{0}{\triangle} (b\, e_1)\right]$$

$$0$$

Products of the first order

The product of the first order reduces to the scalar product.

$$(a\,e_1) \underset{1}{\triangle} (b\,e_1) == a\,b\,(e_1 \ominus e_1)$$

$$\texttt{ToScalarProducts}\left[(a\,e_1) \underset{1}{\triangle} (b\,e_1)\right]$$

$$a\,b\,(e_1 \ominus e_1)$$

In sum: The only non-zero generalized product of elements (other than where one is a scalar) in a space of one dimension is the product of order one, equivalent to the scalar product.

2-space

Products of zero order

The product of zero order of two elements reduces to their exterior product. Hence in a 2-space, the only non-zero products (apart from when one is a scalar) is the exterior product of two 1-elements.

$$(a\,e_1 + b\,e_2) \underset{0}{\triangle} (c\,e_1 + d\,e_2) == (a\,e_1 + b\,e_2) \wedge (c\,e_1 + d\,e_2)$$

Products of the first order

The product of the first order of two 1-elements is commutative and reduces to their scalar product.

$$(a\,e_1 + b\,e_2) \underset{1}{\triangle} (c\,e_1 + d\,e_2) == (a\,e_1 + b\,e_2) \ominus (c\,e_1 + d\,e_2)$$

$$\texttt{ToScalarProducts}\left[(a\,e_1 + b\,e_2) \underset{1}{\triangle} (c\,e_1 + d\,e_2)\right]$$

$$a\,c\,(e_1 \ominus e_1) + b\,c\,(e_1 \ominus e_2) + a\,d\,(e_1 \ominus e_2) + b\,d\,(e_2 \ominus e_2)$$

The product of the first order of a 1-element and a 2-element is also commutative and reduces to their interior product.

$$(a\,e_1 + b\,e_2) \underset{1}{\triangle} (c\,e_1 \wedge e_2) ==$$
$$(c\,e_1 \wedge e_2) \underset{1}{\triangle} (a\,e_1 + b\,e_2) == (c\,e_1 \wedge e_2) \ominus (a\,e_1 + b\,e_2)$$

$$\texttt{ToScalarProducts}\left[(a\,e_1 + b\,e_2) \underset{1}{\triangle} (c\,e_1 \wedge e_2)\right]$$

$$-a\,c\,(e_1 \ominus e_2)\,e_1 - b\,c\,(e_2 \ominus e_2)\,e_1 + a\,c\,(e_1 \ominus e_1)\,e_2 + b\,c\,(e_1 \ominus e_2)\,e_2$$

The product of the first order of two 2-elements is zero. This can be determined directly from [10.17].

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{m}{\alpha} == 0, \quad \lambda \neq m$$

$\mathbf{ToScalarProducts}\left[(\mathbf{a}\,\mathbf{e_1}\wedge\mathbf{e_2})\underset{1}{\triangle}(\mathbf{b}\,\mathbf{e_1}\wedge\mathbf{e_2})\right]$

0

Products of the second order

Products of the second order of two 1-elements, or of a 1-element and a 2-element are necessarily zero since the order of the product is greater than the minimum of the grades of the factors.

$\mathbf{ToScalarProducts}\left[(\mathbf{a}\,\mathbf{e_1}+\mathbf{b}\,\mathbf{e_2})\underset{2}{\triangle}(\mathbf{c}\,\mathbf{e_1}+\mathbf{d}\,\mathbf{e_2})\right]$

0

Products of the second order of two 2-elements reduce to their inner product:

$(\mathbf{a}\,\mathbf{e_1}\wedge\mathbf{e_2})\underset{2}{\triangle}(\mathbf{b}\,\mathbf{e_1}\wedge\mathbf{e_2}) == (\mathbf{a}\,\mathbf{e_1}\wedge\mathbf{e_2})\ominus(\mathbf{b}\,\mathbf{e_1}\wedge\mathbf{e_2})$

$\mathbf{ToInnerProducts}\left[(\mathbf{a}\,\mathbf{e_1}\wedge\mathbf{e_2})\underset{2}{\triangle}(\mathbf{b}\,\mathbf{e_1}\wedge\mathbf{e_2})\right]$

$a\,e_1\wedge e_2\ominus b\,e_1\wedge e_2$

In sum: The only non-zero generalized products of elements (other than where one is a scalar) in a space of two dimensions reduce to exterior, interior, inner or scalar products.

10.16 Generalized Products in 3-Space

To be completed

11 Exploring Hypercomplex Algebra

11.1 Introduction

In this chapter we explore the relationship of Grassmann algebra to the hypercomplex algebras. Typical examples of hypercomplex algebras are the algebras of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{Q}$, octonions (or Cayley numbers) $\mathbb{O}$ and Clifford algebras $\mathcal{C}$. We will show that each of these algebras can be generated as an algebra of Grassmann numbers under a new product which we take the liberty of calling the *hypercomplex product*.

This hypercomplex product of an *m*-element $\underset{m}{\alpha}$ and a *k*-element $\underset{k}{\beta}$ is denoted $\underset{m}{\alpha} \circ \underset{k}{\beta}$ and defined as a sum of signed generalized products of $\underset{m}{\alpha}$ with $\underset{k}{\beta}$.

$$\underset{m}{\alpha} \circ \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \overset{m,\lambda,k}{\sigma}\; \underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} \qquad 11.1$$

Here, the sign $\overset{m,\lambda,k}{\sigma}$ is a function of the grades m and k of the factors and the order λ of the generalized product and takes the values +1 or -1 depending on the type of hypercomplex product being defined.

Note particularly that this is an 'invariant' approach, not requiring the algebra to be defined in terms of generators or basis elements. We will see in what follows that this leads to considerably more insight on the nature of the hypercomplex numbers than afforded by the usual approach. Because the generalized Grassmann products on the right-hand side may represent the only geometrically valid constructs from the factors on the left, it is enticing to postulate that all algebras with a geometric interpretation can be defined by such a product of Grassmann numbers.

It turns out that the real numbers are hypercomplex numbers in a space of zero dimensions, the complex numbers are hypercomplex numbers in a space of one dimension, the quaternions are hypercomplex numbers in a space of two dimensions, and the octonions are hypercomplex numbers in a space of three dimensions.

In Chapter 12: Exploring Clifford Algebra we will also show that the Clifford product may be defined in a space of any number of dimensions as a hypercomplex product with signs:

$$\overset{m,\lambda,k}{\sigma} == (-1)^{\lambda (m-\lambda)+\frac{1}{2} \lambda (\lambda-1)}$$

We could also explore the algebras generated by products of the type defined by definition 11.1, but which have some of the $\sigma^{m,\lambda,k}$ defined to be zero. For example the relations,

$$\sigma^{m,\lambda,k} == 1,\ \lambda == 0 \qquad \sigma^{m,\lambda,k} == 0,\ \lambda > 0$$

defines the algebra with the hypercomplex product reducing to the exterior product, and

$$\sigma^{m,\lambda,k} == 1,\ \lambda = \mathrm{Min}[m, k] \qquad \sigma^{m,\lambda,k} == 0,\ \lambda \neq \mathrm{Min}[m, k]$$

defines the algebra with the hypercomplex product reducing to the interior product.

Both of these definitions however lead to products having zero divisors, that is, some products can be zero even though neither of its factors is zero. Because one of the principal characteristics of the hypercomplex product is that it has no zero divisors, we shall limit ourselves in this chapter to exploring just those algebras. That is, we shall always assume that $\sigma^{m,\lambda,k}$ is not zero.

We begin by exploring the implications of definition 11.1 to the generation of various hypercomplex algebras by considering spaces of increasing dimension, starting with a space of zero dimensions generating the real numbers. In order to embed the algebras defined on lower dimensional spaces within those defined on higher dimensional spaces we will maintain the lower dimensional relations we determine for the $\sigma^{m,\lambda,k}$ when we explore the higher dimensional spaces.

Throughout the development we will see that the various hypercomplex algebras could be developed in alternative fashions by various combinations of signs $\sigma^{m,\lambda,k}$ and metrics. However, we shall for simplicity generally restrict the discussion to constraining the signs only and keeping products of the form $e_i \ominus e_i$ positive.

Although in this chapter we develop the hypercomplex product by determining more and more constraints on the signs $\sigma^{m,\lambda,k}$ that give us the properties we want of the number systems $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{O}$, it can be seen that this is only one direction out of the multitude possible. Many other consistent algebras could be developed by adopting other constraints amongst the signs $\sigma^{m,\lambda,k}$.

$$e_i \ominus e_i$$

11.2 Some Initial Definitions and Properties

The conjugate

The conjugate of a Grassmann number X is denoted X^c and is defined as the body X_b (scalar part) of X *minus* the soul X_s (non-scalar part) of X.

$$X == X_b + X_s \qquad\qquad X^c == X_b - X_s \tag{11.2}$$

Distributivity

The first property we define of the Hypercomplex product is its distributivity.

$$\left(\sum_i \underset{m_i}{\alpha}\right) \circ \underset{k}{\beta} == \sum_i \left(\underset{m_i}{\alpha} \circ \underset{k}{\beta}\right) \tag{11.3}$$

$$\underset{k}{\beta} \circ \left(\sum_i \underset{m_i}{\alpha}\right) == \sum_i \left(\underset{k}{\beta} \circ \underset{m_i}{\alpha}\right) \tag{11.4}$$

The norm

The norm of a Grassmann number **X** is denoted $n[X]$ and is defined as the hypercomplex product of **X** with its conjugate.

$$n[X] == X \circ X^c == (X_b + X_s) \circ (X_b - X_s) == {X_b}^2 - X_s \circ X_s \tag{11.5}$$

Factorization of scalars

From the definition of the hypercomplex product we can use the properties of the generalized Grassmann product to show that scalars may be factorized out of any hypercomplex products:

$$\left(a\, \underset{m}{\alpha}\right) \circ \underset{k}{\beta} == \underset{m}{\alpha} \circ \left(a\, \underset{k}{\beta}\right) == a \left(\underset{m}{\alpha} \circ \underset{k}{\beta}\right) \tag{11.6}$$

Multiplication by scalars

The next property we will require of the hypercomplex product is that it behaves as expected when one of the elements is a scalar. That is:

$$\underset{m}{\alpha} \circ b == b \circ \underset{m}{\alpha} == b \underset{m}{\alpha} \qquad 11.7$$

From the relations 11.7 and the definition 11.1 we can determine the constraints on the $\overset{m,\lambda,k}{\sigma}$ which will accomplish this.

$$\underset{m}{\alpha} \circ b == \sum_{\lambda=0}^{\mathrm{Min}[m,0]} \overset{m,\lambda,0}{\sigma} \underset{m}{\alpha} \underset{\lambda}{\triangle} b == \overset{m,0,0}{\sigma} \underset{m}{\alpha} \wedge b == \overset{m,0,0}{\sigma} b \underset{m}{\alpha}$$

$$b \circ \underset{m}{\alpha} == \sum_{\lambda=0}^{\mathrm{Min}[0,m]} \overset{0,\lambda,m}{\sigma} b \underset{\lambda}{\triangle} \underset{m}{\alpha} == \overset{0,0,m}{\sigma} b \wedge \underset{m}{\alpha} == \overset{0,0,m}{\sigma} b \underset{m}{\alpha}$$

Hence the first constraints we impose on the $\overset{m,\lambda,k}{\sigma}$ to ensure the properties we require for multiplication by scalars are that:

$$\overset{m,0,0}{\sigma} == \overset{0,0,m}{\sigma} == 1 \qquad 11.8$$

The real numbers ℝ

The real numbers are a simple consequence of the relations determined above for scalar multiplication. When the grades of both the factors are zero we have that $\overset{0,0,0}{\sigma} == 1$. Hence:

$$a \circ b == b \circ a == a\, b \qquad 11.9$$

The hypercomplex product in a space of zero dimensions is therefore equivalent to the (usual) real field product of the underlying linear space. Hypercomplex numbers in a space of zero dimensions are therefore (isomorphic to) the real numbers.

11.3 The Complex Numbers ℂ

Constraints on the hypercomplex signs

Complex numbers may be viewed as hypercomplex numbers in a space of one dimension.

In one dimension all 1-elements are of the form **ae**, where **a** is a scalar and **e** is the basis element.

Let α be a 1-element. From the definition of the hypercomplex product we see that the hypercomplex product of a 1-element with itself is the (possibly signed) scalar product.

$$\alpha \circ \alpha == \sum_{\lambda=0}^{\mathrm{Min}[1,1]} \overset{1,\lambda,1}{\sigma}\ \alpha \underset{\lambda}{\triangle} \alpha == \overset{1,0,1}{\sigma}\ \alpha \wedge \alpha + \overset{1,1,1}{\sigma}\ \alpha \ominus \alpha == \overset{1,1,1}{\sigma}\ \alpha \ominus \alpha$$

In the usual notation, the product of two complex numbers would be written:

$$(a + i\, b)\ (c + i\, d) == (a\, c - b\, d) + i\ (b\, c + a\, d)$$

In the general hypercomplex notation we have:

$$(a + b\, e) \circ (c + d\, e) == a \circ c + a \circ (d\, e) + (b\, e) \circ c + (b\, e) \circ (d\, e)$$

Simplifying this using the relations 11.6 and 11.7 above gives:

$$(a\, c + b\, d\, e \circ e) + (b\, c + a\, d)\ e ==$$
$$\left(a\, c + b\, d\ \overset{1,1,1}{\sigma}\ e \ominus e\right) + (b\, c + a\, d)\ e$$

Isomorphism with the complex numbers is then obtained by constraining $\overset{1,1,1}{\sigma}\ e \ominus e$ to be -1.

One immediate interpretation that we can explore to satisfy this is that e is a unit 1-element (with $e \ominus e == 1$), and $\overset{1,1,1}{\sigma} == -1$.

This then is the constraint we will impose to allow the incorporation of complex numbers in the hypercomplex structure.

$$\overset{1,1,1}{\sigma} == -1 \qquad 11.10$$

Complex numbers as Grassmann numbers under the hypercomplex product

The imaginary unit i is then equivalent to a unit basis element (e, say) of the 1-space under the hypercomplex product.

$$i == e \qquad e \ominus e == 1 \qquad 11.11$$

$$i \circ i == -i \ominus i == -1 \qquad 11.12$$

In this interpretation, instead of the i being a new entity with the special property $i^2 == -1$, the focus is shifted to interpret it as a unit 1-element acting under a new product operation $\circ$. Complex numbers are then interpreted as Grassmann numbers in a space of one dimension under the hypercomplex product operation.

For example, the norm of a complex number $a+\beta == a+be == a+b\,i$ can be calculated as:

$$n[a+\beta] == (a+\beta)\circ(a-\beta) == a^2 - \beta\circ\beta == a^2 + \beta\ominus\beta == a^2 + b^2$$

$$\boxed{n[a+\beta] == a^2 + \beta\ominus\beta == a^2 + b^2} \qquad 11.13$$

11.4 The Hypercomplex Product in a 2-Space

Tabulating products in 2-space

From our previous interpretation of complex numbers as Grassmann numbers in a space of one dimension under the hypercomplex product operation, we shall require any one dimensional subspace of the 2-space under consideration to have the same properties.

The product table for the types of hypercomplex products that can be encountered in a 2-space can be entered by using the *GrassmannAlgebra* function `HypercomplexProductTable`.

T = HypercomplexProductTable[{1, α, β, α∧β}]

$$\{\{1\circ 1,\ 1\circ\alpha,\ 1\circ\beta,\ 1\circ(\alpha\wedge\beta)\},\ \{\alpha\circ 1,\ \alpha\circ\alpha,\ \alpha\circ\beta,\ \alpha\circ(\alpha\wedge\beta)\},\ \{\beta\circ 1,\ \beta\circ\alpha,\ \beta\circ\beta,\ \beta\circ(\alpha\wedge\beta)\},\ \{(\alpha\wedge\beta)\circ 1,\ (\alpha\wedge\beta)\circ\alpha,\ (\alpha\wedge\beta)\circ\beta,\ (\alpha\wedge\beta)\circ(\alpha\wedge\beta)\}\}$$

To make this easier to read we format this with the *GrassmannAlgebra* function `TableTemplate`.

TableTemplate[T]

$1\circ 1$	$1\circ\alpha$	$1\circ\beta$	$1\circ(\alpha\wedge\beta)$
$\alpha\circ 1$	$\alpha\circ\alpha$	$\alpha\circ\beta$	$\alpha\circ(\alpha\wedge\beta)$
$\beta\circ 1$	$\beta\circ\alpha$	$\beta\circ\beta$	$\beta\circ(\alpha\wedge\beta)$
$(\alpha\wedge\beta)\circ 1$	$(\alpha\wedge\beta)\circ\alpha$	$(\alpha\wedge\beta)\circ\beta$	$(\alpha\wedge\beta)\circ(\alpha\wedge\beta)$

Products where one of the factors is a scalar have already been discussed. Products of a 1-element with itself have been discussed in the section on complex numbers. Applying these results to the table above enables us to simplify it to:

1	α	β	$\alpha\wedge\beta$
α	$-(\alpha\ominus\alpha)$	$\alpha\circ\beta$	$\alpha\circ(\alpha\wedge\beta)$
β	$\beta\circ\alpha$	$-(\beta\ominus\beta)$	$\beta\circ(\alpha\wedge\beta)$
$\alpha\wedge\beta$	$(\alpha\wedge\beta)\circ\alpha$	$(\alpha\wedge\beta)\circ\beta$	$(\alpha\wedge\beta)\circ(\alpha\wedge\beta)$

In this table there are four essentially different new products which we have not yet discussed.

$$\alpha\circ\beta \qquad \alpha\circ(\alpha\wedge\beta) \qquad (\alpha\wedge\beta)\circ\alpha \qquad (\alpha\wedge\beta)\circ(\alpha\wedge\beta)$$

In the next three subsections we will take each of these products and, with a view to developing the quaternion algebra, show how they may be expressed in terms of exterior and interior products.

The hypercomplex product of two 1-elements

From the definition, and the constraint $\overset{1,1,1}{\sigma} == -1$ derived above from the discussion on complex numbers we can write the hypercomplex product of two (possibly) distinct 1-elements as

$$\alpha\circ\beta == \overset{1,0,1}{\sigma}\,\alpha\wedge\beta + \overset{1,1,1}{\sigma}\,(\alpha\ominus\beta) == \overset{1,0,1}{\sigma}\,\alpha\wedge\beta - (\alpha\ominus\beta)$$

The hypercomplex product $\beta\circ\alpha$ can be obtained by reversing the sign of the exterior product, since the scalar product is symmetric.

$$\begin{gathered}\alpha\circ\beta == \overset{1,0,1}{\sigma}\,\alpha\wedge\beta - (\alpha\ominus\beta)\\ \beta\circ\alpha == -\overset{1,0,1}{\sigma}\,\alpha\wedge\beta - (\alpha\ominus\beta)\end{gathered} \qquad 11.14$$

The hypercomplex product of a 1-element and a 2-element

From the definition of the hypercomplex product 11.1 we can obtain expressions for the hypercomplex products of a 1-element and a 2-element in a 2-space. Since the space is only of two dimensions, the 2-element may be represented without loss of generality as a product which incorporates the 1-element as one of its factors.

$$\alpha\circ(\alpha\wedge\beta) == \sum_{\lambda=0}^{\mathrm{Min}[1,2]} \overset{1,\lambda,2}{\sigma}\,\alpha\underset{\lambda}{\triangle}(\alpha\wedge\beta) ==$$
$$\overset{1,0,2}{\sigma}\,\alpha\wedge(\alpha\wedge\beta) + \overset{1,1,2}{\sigma}\,(\alpha\wedge\beta)\ominus\alpha$$

$$(\alpha \wedge \beta) \circ \alpha == \sum_{\lambda=0}^{\mathrm{Min}[2,1]} \overset{2,\lambda,1}{\sigma} \; (\alpha \wedge \beta) \underset{\lambda}{\triangle} \alpha ==$$

$$\overset{2,0,1}{\sigma} \; (\alpha \wedge \beta) \wedge \alpha + \overset{2,1,1}{\sigma} \; (\alpha \wedge \beta) \ominus \alpha$$

Since the first term involving only exterior products is zero, we obtain:

$$\alpha \circ (\alpha \wedge \beta) == \overset{1,1,2}{\sigma} \; (\alpha \wedge \beta) \ominus \alpha$$
$$(\alpha \wedge \beta) \circ \alpha == \overset{2,1,1}{\sigma} \; (\alpha \wedge \beta) \ominus \alpha \qquad 11.15$$

The hypercomplex square of a 2-element

All 2-elements in a space of 2 dimensions differ by only a scalar factor. We need only therefore consider the hypercomplex product of a 2-element with itself.

The generalized product of two identical elements is zero in all cases except for that which reduces to the inner product. From this fact and the definition of the hypercomplex product we obtain

$$(\alpha \wedge \beta) \circ (\alpha \wedge \beta) ==$$
$$\sum_{\lambda=0}^{\mathrm{Min}[2,2]} \overset{2,\lambda,2}{\sigma} \; (\alpha \wedge \beta) \underset{\lambda}{\triangle} (\alpha \wedge \beta) == \overset{2,2,2}{\sigma} \; (\alpha \wedge \beta) \ominus (\alpha \wedge \beta)$$

$$(\alpha \wedge \beta) \circ (\alpha \wedge \beta) == \overset{2,2,2}{\sigma} \; (\alpha \wedge \beta) \ominus (\alpha \wedge \beta) \qquad 11.16$$

The product table in terms of exterior and interior products

Collecting together the results above, we can rewrite the hypercomplex product table solely in terms of exterior and interior products and some (as yet) undetermined signs.

	α	β	$\alpha \wedge \beta$
	$-(\alpha \ominus \alpha)$	$\overset{1,0,1}{\sigma} \; \alpha \wedge \beta - (\alpha \ominus \beta)$	$\overset{1,1,2}{\sigma} \; (\alpha \wedge \beta) \ominus$
	$-\overset{1,0,1}{\sigma} \; \alpha \wedge \beta - (\alpha \ominus \beta)$	$-(\beta \ominus \beta)$	$\overset{1,1,2}{\sigma} \; (\alpha \wedge \beta) \ominus$
β	$\overset{2,1,1}{\sigma} \; (\alpha \wedge \beta) \ominus \alpha$	$\overset{2,1,1}{\sigma} \; (\alpha \wedge \beta) \ominus \beta$	$\overset{2,2,2}{\sigma} \; (\alpha \wedge \beta) \ominus (\alpha$

11.17

11.5 The Quaternions $\mathbb{Q}$

The product table for orthonormal elements

Suppose now that α and β are orthonormal. Then:

$$\alpha \ominus \alpha == \beta \ominus \beta == 1$$

$$\alpha \ominus \beta == 0 \qquad (\alpha \wedge \beta) \ominus \alpha == (\alpha \ominus \alpha)\ \beta \qquad (\alpha \wedge \beta) \ominus (\alpha \wedge \beta) == (\alpha \ominus \alpha)\ (\beta \ominus \beta)$$

The hypercomplex product table then becomes:

1	α	β	$\alpha \wedge \beta$
α	-1	$\overset{1,0,1}{\sigma}\ \alpha \wedge \beta$	$\overset{1,1,2}{\sigma}\ \beta$
β	$-\overset{1,0,1}{\sigma}\ \alpha \wedge \beta$	-1	$-\overset{1,1,2}{\sigma}\ \alpha$
$\alpha \wedge \beta$	$\overset{2,1,1}{\sigma}\ \beta$	$-\overset{2,1,1}{\sigma}\ \alpha$	$\overset{2,2,2}{\sigma}$

Generating the quaternions

Exploring a possible isomorphism with the quaternions leads us to the correspondence:

$$\boxed{\alpha \longrightarrow \mathbb{i} \qquad \beta \longrightarrow \mathbb{j} \qquad \alpha \wedge \beta \longrightarrow \mathbb{k}} \qquad \mathbf{11.18}$$

In terms of the quaternion units, the product table becomes:

1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
$\mathbb{i}$	-1	$\overset{1,0,1}{\sigma}\ \mathbb{k}$	$\overset{1,1,2}{\sigma}\ \mathbb{j}$
$\mathbb{j}$	$-\overset{1,0,1}{\sigma}\ \mathbb{k}$	-1	$-\overset{1,1,2}{\sigma}\ \mathbb{i}$
$\mathbb{k}$	$\overset{2,1,1}{\sigma}\ \mathbb{j}$	$-\overset{2,1,1}{\sigma}\ \mathbb{i}$	$\overset{2,2,2}{\sigma}$

To obtain the product table for the quaternions we therefore require:

$$\boxed{\overset{1,0,1}{\sigma} == 1 \qquad \overset{1,1,2}{\sigma} == -1 \qquad \overset{2,1,1}{\sigma} == 1 \qquad \overset{2,2,2}{\sigma} == -1} \qquad \mathbf{11.19}$$

These values give the quaternion table as expected.

1	$\mathbb{i}$	$\mathbb{j}$	$\mathbb{k}$
$\mathbb{i}$	-1	$\mathbb{k}$	$-\mathbb{j}$
$\mathbb{j}$	$-\mathbb{k}$	-1	$\mathbb{i}$
$\mathbb{k}$	$\mathbb{j}$	$-\mathbb{i}$	-1

11.20

Substituting these values back into the original table 11.17 gives a hypercomplex product table in terms only of exterior and interior products. This table defines the real, complex, and quaternion product operations.

1	α	β	$\alpha \wedge \beta$
α	$-(\alpha \ominus \alpha)$	$\alpha \wedge \beta - (\alpha \ominus \beta)$	$-(\alpha \wedge \beta) \ominus \alpha$
β	$-\alpha \wedge \beta - (\alpha \ominus \beta)$	$-(\beta \ominus \beta)$	$-(\alpha \wedge \beta) \ominus \beta$
$\alpha \wedge \beta$	$(\alpha \wedge \beta) \ominus \alpha$	$(\alpha \wedge \beta) \ominus \beta$	$-(\alpha \wedge \beta) \ominus (\alpha \wedge \beta)$

11.21

The norm of a quaternion

Let $\mathbf{Q}$ be a quaternion given in basis-free form as an element of a 2-space under the hypercomplex product.

$$\mathbf{Q} == \mathbf{a} + \alpha + \alpha \wedge \beta \qquad 11.22$$

Here, $\mathbf{a}$ is a scalar, α is a 1-element and $\alpha \wedge \beta$ is congruent to any 2-element of the space.

The norm of $\mathbf{Q}$ is denoted $n[\mathbf{Q}]$ and given as the hypercomplex product of $\mathbf{Q}$ with its conjugate $\mathbf{Q}^{c}$. expanding using formula 11.5 gives:

$$n[\mathbf{Q}] == (\mathbf{a} + \alpha + \alpha \wedge \beta) \circ (\mathbf{a} - \alpha - \alpha \wedge \beta) ==$$
$$\mathbf{a}^2 - (\alpha + \alpha \wedge \beta) \circ (\alpha + \alpha \wedge \beta)$$

Expanding the last term gives:

$$n[\mathbf{Q}] == \mathbf{a}^2 - \alpha \circ \alpha - \alpha \circ (\alpha \wedge \beta) - (\alpha \wedge \beta) \circ \alpha - (\alpha \wedge \beta) \circ (\alpha \wedge \beta)$$

From table 11.21 we see that:

$$\alpha \circ (\alpha \wedge \beta) == -(\alpha \wedge \beta) \ominus \alpha \qquad (\alpha \wedge \beta) \circ \alpha == (\alpha \wedge \beta) \ominus \alpha$$

Whence:

$$\alpha \circ (\alpha \wedge \beta) == -(\alpha \wedge \beta) \circ \alpha$$

Using table 11.21 again then allows us to write the norm of a quaternion either in terms of the hypercomplex product or the inner product.

$$n[\mathbf{Q}] == \mathbf{a}^2 - \alpha \circ \alpha - (\alpha \wedge \beta) \circ (\alpha \wedge \beta) \qquad \mathbf{11.23}$$

$$n[\mathbf{Q}] == \mathbf{a}^2 + \alpha \ominus \alpha + (\alpha \wedge \beta) \ominus (\alpha \wedge \beta) \qquad \mathbf{11.24}$$

In terms of $\mathbb{i}$, $\mathbb{j}$, and $\mathbb{k}$

$$\mathbf{Q} = \mathbf{a} + \mathbf{b}\,\mathbb{i} + \mathbf{c}\,\mathbb{j} + \mathbf{d}\,\mathbb{k}$$

$$\begin{aligned} n[\mathbf{Q}] &== \mathbf{a}^2 - (\mathbf{b}\,\mathbb{i} + \mathbf{c}\,\mathbb{j}) \circ (\mathbf{b}\,\mathbb{i} + \mathbf{c}\,\mathbb{j}) - (\mathbf{d}\,\mathbb{k}) \circ (\mathbf{d}\,\mathbb{k}) \\ &== \mathbf{a}^2 - \mathbf{b}^2\,(\mathbb{i} \circ \mathbb{i}) - \mathbf{c}^2\,(\mathbb{j} \circ \mathbb{j}) - \mathbf{d}^2\,(\mathbb{k} \circ \mathbb{k}) \\ &== \mathbf{a}^2 + \mathbf{b}^2\,(\mathbb{i} \ominus \mathbb{i}) + \mathbf{c}^2\,(\mathbb{j} \ominus \mathbb{j}) + \mathbf{d}^2\,(\mathbb{k} \ominus \mathbb{k}) \\ &== \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{d}^2 \end{aligned}$$

The Cayley-Dickson algebra

If we set α and β to be orthogonal in the general hypercomplex product table 11.21, we retrieve the multiplication table for the Cayley-Dickson algebra with 4 generators and 2 parameters.

$\mathbf{1}$	α	β	$\alpha \wedge \beta$
α	$-(\alpha \ominus \alpha)$	$\alpha \wedge \beta$	$-(\alpha \ominus \alpha)\ \beta$
β	$-\alpha \wedge \beta$	$-(\beta \ominus \beta)$	$(\beta \ominus \beta)\ \alpha$
$\alpha \wedge \beta$	$(\alpha \ominus \alpha)\ \beta$	$-(\beta \ominus \beta)\ \alpha$	$-(\alpha \ominus \alpha)\ (\beta \ominus \beta)$

11.25

In the notation we have used, the generators are $\mathbf{1}$, α, β, and $\alpha \wedge \beta$. The parameters are $\alpha \ominus \alpha$ and $\beta \ominus \beta$.

11.6 The Norm of Grassmann number

The norm

The first property that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{O}$ possess in common is that of being normed.

The *norm* of a Grassmann number $\mathbf{X}$ is denoted $n[\mathbf{X}]$ and has been defined in 11.5 as the hypercomplex product of $\mathbf{X}$ with its conjugate.

$$n[\mathbf{X}] == \mathbf{X} \circ \mathbf{X}^{\mathbf{c}} == (\mathbf{X}_{\mathbf{b}} + \mathbf{X}_{\mathbf{s}}) \circ (\mathbf{X}_{\mathbf{b}} - \mathbf{X}_{\mathbf{s}}) == \mathbf{X}_{\mathbf{b}}{}^2 - \mathbf{X}_{\mathbf{s}} \circ \mathbf{X}_{\mathbf{s}}$$

The norm of an m-element $\underset{m}{\alpha}$ is then denoted $n[\underset{m}{\alpha}]$ and defined as the hypercomplex product of $\underset{m}{\alpha}$ with its conjugate $\underset{m}{\alpha}^c$. If $\underset{m}{\alpha}$ is not scalar, then $\underset{m}{\alpha}^c$ is simply $-\underset{m}{\alpha}$. Hence

$$n[\underset{m}{\alpha}] == \underset{m}{\alpha} \circ \underset{m}{\alpha}^c == -\underset{m}{\alpha} \circ \underset{m}{\alpha} \qquad m \neq 0 \qquad 11.26$$

We require that the norm of any element be a *positive scalar* quantity.

The norm of a simple *m*-element

If $\underset{m}{\alpha}$ is *simple*, then the generalized Grassmann product $\underset{m}{\alpha} \underset{\lambda}{\Delta} \underset{m}{\alpha}$ is zero for all λ except λ equal to m, in which case the generalized product (and hence the hypercomplex product) becomes an inner product. Thus for m not zero we have:

$$\underset{m}{\alpha} \circ \underset{m}{\alpha} == \overset{m,m,m}{\sigma} \underset{m}{\alpha} \ominus \underset{m}{\alpha} \qquad m \neq 0$$

Equation 11.26 then implies that for the norm to be a positive scalar quantity we must have $\overset{m,m,m}{\sigma} == -1$.

$$n[\underset{m}{\alpha}] == \underset{m}{\alpha} \ominus \underset{m}{\alpha} \qquad \underset{m}{\alpha} \text{ simple} \qquad 11.27$$

$$\overset{m,m,m}{\sigma} == -1 \qquad 11.28$$

The norm of a scalar a is just $a^2 == a \ominus a$, so this formula 11.27 applies also for m equal to zero.

The skew-symmetry of products of elements of different grade

We consider a general Grassmann number $X == a+A$, where a is its body (scalar component) and A is its soul (non-scalar component). We define the norm $n[X]$ of X in the standard way as the hypercomplex product of X with its conjugate X^c.

$$n[X] == X \circ X_c == (a + A) \circ (a - A) == a^2 - A \circ A$$

The norm of X can be written in terms of X as:

$$n[X] == \text{Body}[X]^2 - \text{Soul}[X] \circ \text{Soul}[X]$$

To investigate the norm further we look at the product $A \circ A$. Suppose A is the sum of two (not necessarily simple) elements of different grade: an m-element $\underset{m}{\alpha}$ and a k-element $\underset{k}{\beta}$. Then $A \circ A$ becomes:

$$\mathbf{A}\circ\mathbf{A} == \left(\underset{m}{\alpha}+\underset{k}{\beta}\right)\circ\left(\underset{m}{\alpha}+\underset{k}{\beta}\right) == \underset{m}{\alpha}\circ\underset{m}{\alpha}+\underset{m}{\alpha}\circ\underset{k}{\beta}+\underset{k}{\beta}\circ\underset{m}{\alpha}+\underset{k}{\beta}\circ\underset{k}{\beta}$$

We would *like* the norm of a general Grassmann number to be a scalar quantity. But none of the generalized product components of $\underset{m}{\alpha}\circ\underset{k}{\beta}$ or $\underset{k}{\beta}\circ\underset{m}{\alpha}$ can be scalar if *m* and *k* are different, so we choose to make $\underset{k}{\beta}\circ\underset{m}{\alpha}$ equal to $-\underset{m}{\alpha}\circ\underset{k}{\beta}$ as a fundamental defining axiom of the hypercomplex product, thus eliminating both products from the expression for the norm. This requirement of skew-symmetry can always be satisfied because, since the grades are different, the two products are always distinguishable.

$$\boxed{\underset{m}{\alpha}\circ\underset{k}{\beta} == -\underset{k}{\beta}\circ\underset{m}{\alpha}} \qquad \mathbf{11.29}$$

The norm of a Grassmann number in terms of hypercomplex products

More generally, suppose **A** were the sum of several elements of different grade.

$$\mathbf{A} == \underset{m}{\alpha}+\underset{k}{\beta}+\underset{p}{\gamma}+\ldots$$

The skew-symmetry axiom then allows us to write **A**∘**A** as the sum of hypercomplex squares of the components of different grade of **A**.

$$\mathbf{A}\circ\mathbf{A} == \left(\underset{m}{\alpha}+\underset{k}{\beta}+\underset{p}{\gamma}+\ldots\right)\circ\left(\underset{m}{\alpha}+\underset{k}{\beta}+\underset{p}{\gamma}+\ldots\right) == \underset{m}{\alpha}\circ\underset{m}{\alpha}+\underset{k}{\beta}\circ\underset{k}{\beta}+\underset{p}{\gamma}\circ\underset{p}{\gamma}+\ldots$$

And hence the norm is expressed simply as

$$n[\mathtt{X}] == \mathbf{a}^2-\underset{m}{\alpha}\circ\underset{m}{\alpha}-\underset{k}{\beta}\circ\underset{k}{\beta}-\underset{p}{\gamma}\circ\underset{p}{\gamma}-\ldots$$

$$\boxed{n\left[\mathbf{a}+\underset{m}{\alpha}+\underset{k}{\beta}+\underset{p}{\gamma}+\ldots\right] == \mathbf{a}^2-\underset{m}{\alpha}\circ\underset{m}{\alpha}-\underset{k}{\beta}\circ\underset{k}{\beta}-\underset{p}{\gamma}\circ\underset{p}{\gamma}-\ldots} \qquad \mathbf{11.30}$$

The norm of a Grassmann number of simple components

Consider a Grassmann number **A** in which the elements $\underset{m}{\alpha}$, $\underset{k}{\beta}$, … of each grade are simple.

$$\mathbf{A} == \underset{m}{\alpha}+\underset{k}{\beta}+\underset{p}{\gamma}+\ldots$$

Then by 11.28 and 11.30 we have that the norm of **A** is positive and may be written as the sum of interior products of the individual simple elements.

$$n\left[a + \underset{m}{\alpha} + \underset{k}{\beta} + \underset{p}{\gamma} + \ldots\right] == a^2 + \underset{m}{\alpha} \ominus \underset{m}{\alpha} + \underset{k}{\beta} \ominus \underset{k}{\beta} + \underset{p}{\gamma} \ominus \underset{p}{\gamma} + \ldots \qquad 11.31$$

The norm of a non-simple element

We now turn our attention to the case in which one of the component elements of **X** may not be simple. This is the general case for Grassmann numbers in spaces of dimension greater than 3, and indeed is the reason why the suite of normed hypercomplex numbers ends with the Octonions. An element of any grade in a space of any dimension 3 or less must be simple. So in spaces of dimension 0, 1, 2, and 3 the norm of an element will always be a scalar.

We now focus our attention on the product $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ (where $\underset{m}{\alpha}$ may or may not be simple). The following analysis will be critical to our understanding of how to generalize the notion of hypercomplex numbers to spaces of dimension greater than 3.

Suppose $\underset{m}{\alpha}$ is the sum of two simple elements $\underset{m}{\alpha_1}$ and $\underset{m}{\alpha_2}$. The product $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ then becomes:

$$\underset{m}{\alpha} \circ \underset{m}{\alpha} == \left(\underset{m}{\alpha_1} + \underset{m}{\alpha_2}\right) \circ \left(\underset{m}{\alpha_1} + \underset{m}{\alpha_2}\right) == \underset{m}{\alpha_1} \circ \underset{m}{\alpha_1} + \underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_1} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_2}$$

Since $\underset{m}{\alpha_1}$ and $\underset{m}{\alpha_2}$ are simple, $\underset{m}{\alpha_1} \circ \underset{m}{\alpha_1}$ and $\underset{m}{\alpha_2} \circ \underset{m}{\alpha_2}$ are scalar, and can be written:

$$\underset{m}{\alpha_1} \circ \underset{m}{\alpha_1} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_2} == -\underset{m}{\alpha_1} \ominus \underset{m}{\alpha_1} - \underset{m}{\alpha_2} \ominus \underset{m}{\alpha_2}$$

The expression of the remaining terms $\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2}$ and $\underset{m}{\alpha_2} \circ \underset{m}{\alpha_1}$ in terms of exterior and interior products is more interesting and complex. We discuss it in the next section.

11.7 Products of two different elements of the same grade

The symmetrized sum of two *m*-elements

We now investigate the remaining terms, products of *two different elements of the same grade*. We have left this discussion to last as these types of term introduce the most complexity into the nature of the hypercomplex product. It is their potential to introduce non-scalar terms into the (usually defined) norms of hypercomplex numbers of order higher than the Octonions that makes them of most interest.

Returning to our definition of the hypercomplex product and specializing it to the case where both elements are of the same grade we get:

$$\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} == \sum_{\lambda=0}^{m} \overset{m,\lambda,m}{\sigma} \underset{m}{\alpha_1} \underset{\lambda}{\triangle} \underset{m}{\alpha_2}$$

Reversing the order of the factors in the hypercomplex product gives:

$$\underset{m}{\alpha_2} \circ \underset{m}{\alpha_1} == \sum_{\lambda=0}^{m} \overset{m,\lambda,m}{\sigma} \underset{m}{\alpha_2} \underset{\lambda}{\triangle} \underset{m}{\alpha_1}$$

The generalized products on the right may now be reversed together with the change of sign $(-1)^{(m-\lambda)\,(m-\lambda)} = (-1)^{(m-\lambda)}$.

$$\underset{m}{\alpha_2} \circ \underset{m}{\alpha_1} == \sum_{\lambda=0}^{m} (-1)^{(m-\lambda)} \overset{m,\lambda,m}{\sigma} \underset{m}{\alpha_1} \underset{\lambda}{\triangle} \underset{m}{\alpha_2}$$

The sum $\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_1}$ may then be written finally as:

$$\boxed{\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_1} == \sum_{\lambda=0}^{m} \left(1 + (-1)^{(m-\lambda)}\right) \overset{m,\lambda,m}{\sigma} \underset{m}{\alpha_1} \underset{\lambda}{\triangle} \underset{m}{\alpha_2}} \quad \mathbf{11.32}$$

Because we will need to refer to this sum of products subsequently, we will call it the *symmetrized sum* of two *m*-elements (because the sum of products does not change if the order of the elements is reversed).

Symmetrized sums for elements of different grades

For two (in general, different) simple elements of the same grade $\underset{m}{\alpha_1}$ and $\underset{m}{\alpha_2}$ we have shown above that:

$$\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_1} == \sum_{\lambda=0}^{m} \left(1 + (-1)^{m-\lambda}\right) \overset{m,\lambda,m}{\sigma} \underset{m}{\alpha_1} \underset{\lambda}{\triangle} \underset{m}{\alpha_2}$$

The term $1 + (-1)^{m-\lambda}$ will be 2 for *m* and λ both even or both odd, and zero otherwise. To get a clearer picture of what this formula entails, we write it out explicitly for the lower values of *m*. Note that we use the fact already established that $\overset{m,m,m}{\sigma} == -1$.

◆ **m = 0**

$$\boxed{a \circ b + b \circ a == 2\,a\,b} \quad \mathbf{11.33}$$

◆ **m = 1**

$$\alpha_1 \circ \alpha_2 + \alpha_2 \circ \alpha_1 == -2\, \alpha_1 \ominus \alpha_2 \qquad 11.34$$

◆ **m = 2**

$$\underset{2}{\alpha_1} \circ \underset{2}{\alpha_2} + \underset{2}{\alpha_2} \circ \underset{2}{\alpha_1} == 2 \left(\overset{2,0,2}{\sigma}\; \underset{2}{\alpha_1} \wedge \underset{2}{\alpha_2} - \underset{2}{\alpha_1} \ominus \underset{2}{\alpha_2} \right) \qquad 11.35$$

◆ **m = 3**

$$\underset{3}{\alpha_1} \circ \underset{3}{\alpha_2} + \underset{3}{\alpha_2} \circ \underset{3}{\alpha_1} == 2 \left(\overset{3,1,3}{\sigma}\; \underset{3}{\alpha_1} \underset{1}{\triangle} \underset{3}{\alpha_2} - \underset{3}{\alpha_1} \ominus \underset{3}{\alpha_2} \right) \qquad 11.36$$

◆ **m = 4**

$$\underset{4}{\alpha_1} \circ \underset{4}{\alpha_2} + \underset{4}{\alpha_2} \circ \underset{4}{\alpha_1} == 2 \left(\overset{4,0,4}{\sigma}\; \underset{4}{\alpha_1} \wedge \underset{4}{\alpha_2} + \overset{4,2,4}{\sigma}\; \underset{4}{\alpha_1} \underset{2}{\triangle} \underset{4}{\alpha_2} - \underset{4}{\alpha_1} \ominus \underset{4}{\alpha_2} \right) \qquad 11.37$$

The body of a symmetrized sum

We now break this sum up into its body (scalar component) and soul (non-scalar components).

The body of $\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_1}$ is the inner product term in which λ becomes equal to m, that is: $2\, \overset{m,m,m}{\sigma} \left(\underset{m}{\alpha_1} \ominus \underset{m}{\alpha_2} \right)$.

And since constraint 11.28 requires that $\overset{m,m,m}{\sigma} == -1$, the body of $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ becomes:

$$\text{Body}\left[\underset{m}{\alpha} \circ \underset{m}{\alpha}\right] == \underset{m}{\alpha_1} \circ \underset{m}{\alpha_1} + \text{Body}\left[\underset{m}{\alpha_1} \circ \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_1}\right] + \underset{m}{\alpha_2} \circ \underset{m}{\alpha_2}$$
$$== -\left(\underset{m}{\alpha_1} \ominus \underset{m}{\alpha_1} + 2\, \underset{m}{\alpha_1} \ominus \underset{m}{\alpha_2} + \underset{m}{\alpha_2} \ominus \underset{m}{\alpha_2} \right) == -\, \underset{m}{\alpha} \ominus \underset{m}{\alpha}$$

Hence, even if a component element like $\underset{m}{\alpha}$ is not simple, the body of its hypercomplex square is given by the same expression as if it were simple, that is, its inner product square.

$$\mathbf{Body}\left[\underset{m}{\alpha}\circ\underset{m}{\alpha}\right] == -\underset{m}{\alpha}\ominus\underset{m}{\alpha} \qquad \mathbf{11.38}$$

The soul of a symmetrized sum

The soul of $\underset{m}{\alpha}\circ\underset{m}{\alpha}$ is then the non-scalar residue given by the formula:

$$\mathbf{Soul}\left[\underset{m}{\alpha}\circ\underset{m}{\alpha}\right] == \sum_{\lambda=0}^{m-2}\left(1+(-1)^{(m-\lambda)}\right)\overset{m,\lambda,m}{\sigma}\;\underset{m}{\alpha_1}\underset{\lambda}{\triangle}\underset{m}{\alpha_2} \qquad \mathbf{11.39}$$

(Here we have used the fact that because of the coefficient $\left(1+(-1)^{(m-\lambda)}\right)$, the terms with λ equal to m–1 is always zero, and we only sum to m–2).

For convenience in generating examples of this expression, we temporarily define the soul of $\underset{m}{\alpha}\circ\underset{m}{\alpha}$ as a function of m, and denote it **η[m]**. To make it easier to read we convert the generalized products where possible to exterior and interior products.

$$\eta[\mathrm{m\_}] := \sum_{\lambda=0}^{m-2}\left(1+(-1)^{(m-\lambda)}\right)\overset{m,\lambda,m}{\sigma}\;\underset{m}{\alpha_1}\underset{\lambda}{\triangle}\underset{m}{\alpha_2}\ //\ \mathbf{GeneralizedProductSimplify}$$

With this function we can make a palette of souls for various values of m. We do this below for m equals 1 to 6 to cover spaces of dimension from 1 to 6.

```
PaletteTemplate[Table[{m, η[m]}, {m, 1, 6}]]
```

1	0
2	$2\,\overset{2,0,2}{\sigma}\;\underset{2}{\alpha_1}\wedge\underset{2}{\alpha_2}$
3	$2\left(\underset{3}{\alpha_1}\underset{1}{\triangle}\underset{3}{\alpha_2}\right)\overset{3,1,3}{\sigma}$
4	$2\left(\underset{4}{\alpha_1}\underset{2}{\triangle}\underset{4}{\alpha_2}\right)\overset{4,2,4}{\sigma}+2\,\overset{4,0,4}{\sigma}\;\underset{4}{\alpha_1}\wedge\underset{4}{\alpha_2}$
5	$2\left(\underset{5}{\alpha_1}\underset{1}{\triangle}\underset{5}{\alpha_2}\right)\overset{5,1,5}{\sigma}+2\left(\underset{5}{\alpha_1}\underset{3}{\triangle}\underset{5}{\alpha_2}\right)\overset{5,3,5}{\sigma}$
6	$2\left(\underset{6}{\alpha_1}\underset{2}{\triangle}\underset{6}{\alpha_2}\right)\overset{6,2,6}{\sigma}+2\left(\underset{6}{\alpha_1}\underset{4}{\triangle}\underset{6}{\alpha_2}\right)\overset{6,4,6}{\sigma}+2\,\overset{6,0,6}{\sigma}\;\underset{6}{\alpha_1}\wedge\underset{6}{\alpha_2}$

Of particular note is the soul of the symmetrized sum of two different simple 2-elements.

$$2\ {}^{2,0,2}\sigma\ \underset{2}{\alpha_1} \wedge \underset{2}{\alpha_2}$$

This 4-element is the critical potentially non-scalar residue in a norm calculation. It is always zero in spaces of dimension 2, or 3. *Hence the norm of a Grassmann number in these spaces under the hypercomplex product is guaranteed to be scalar.*

In a space of 4 dimensions, on the other hand, this element may not be zero. *Hence the space of dimension 3 is the highest-dimensional space in which the norm of a Grassmann number is guaranteed to be scalar.*

We have already seen that Grassmann numbers in a 2-space under the hypercomplex product generate the quaternions. In the next section we shall see that Grassmann numbers in a 3-space under the hypercomplex product generate the Octonions. The Octonions therefore will be the last hypercomplex system with a scalar norm.

Summary of results of this section

Consider a general Grassmann number **X**.

$$X == a + \underset{m}{\alpha} + \underset{k}{\beta} + \underset{p}{\gamma} + \ldots$$

Here, we suppose **a** to be scalar, and the rest of the terms to be nonscalar simple or non-simple elements.

The generalized hypercomplex norm $n[X]$ of **X** may be written as:

$$n[X] == a^2 - \underset{m}{\alpha} \circ \underset{m}{\alpha} - \underset{k}{\beta} \circ \underset{k}{\beta} - \underset{p}{\gamma} \circ \underset{p}{\gamma} - \ldots$$

The hypercomplex square $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ of an element, has in general, both a body and a soul.

The body of $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ is the negative of the inner product.

$$\text{Body}\left[\underset{m}{\alpha} \circ \underset{m}{\alpha}\right] == -\underset{m}{\alpha} \ominus \underset{m}{\alpha}$$

The soul of $\underset{m}{\alpha} \circ \underset{m}{\alpha}$ depends on the terms of which it is composed. If $\underset{m}{\alpha} == \underset{m}{\alpha_1} + \underset{m}{\alpha_2} + \underset{m}{\alpha_3} + \ldots$ then

$$\text{Soul}\left[\underset{m}{\alpha} \circ \underset{m}{\alpha}\right] ==$$

$$\sum_{\lambda=0}^{m-2} \left(1 + (-1)^{(m-\lambda)}\right)\ {}^{m,\lambda,m}\sigma \left(\underset{m}{\alpha_1} \underset{\lambda}{\triangle} \underset{m}{\alpha_2} + \underset{m}{\alpha_1} \underset{\lambda}{\triangle} \underset{m}{\alpha_3} + \underset{m}{\alpha_2} \underset{\lambda}{\triangle} \underset{m}{\alpha_3} + \ldots\right)$$

If the component elements of different grade of **X** are simple, then its soul is zero, and its norm becomes the scalar:

$$n[X] == a^2 + \underset{m}{\alpha} \ominus \underset{m}{\alpha} + \underset{k}{\beta} \ominus \underset{k}{\beta} + \underset{p}{\gamma} \ominus \underset{p}{\gamma} + \ldots$$

It is only in 3-space that we are guaranteed that all the components of a Grassmann number are simple. *Therefore it is only in 3-space that we are guaranteed that the norm of a Grassmann number under the hypercomplex product is scalar.*

11.8 Octonions

To be completed

12 Exploring Clifford Algebra

12.17 Rotations

To be completed

12.1 Introduction

In this chapter we define a new type of algebra: the Clifford algebra. Clifford algebra was originally proposed by William Kingdon Clifford (Clifford) based on Grassmann's ideas. Clifford algebras have now found an important place as mathematical systems for describing some of the more fundamental theories of mathematical physics. Clifford algebra is based on a new kind of product called the *Clifford product*.

In this chapter we will show how the Clifford product can be defined straightforwardly in terms of the exterior and interior products that we have already developed, without introducing any new axioms. This approach has the dual advantage of ensuring consistency and enabling all the results we have developed previously to be applied to Clifford numbers.

In the previous chapter we have gone to some lengths to define and explore the generalized Grassmann product. In this chapter we will use the generalized Grassmann product to define the Clifford product in its most general form as a simple sum of generalized products. Computations in general Clifford algebras can be complicated, and this approach at least permits us to render the complexities in a straightforward manner.

From the general case, we then show how Clifford products of intersecting and orthogonal elements simplify. This is the normal case treated in introductory discussions of Clifford algebra.

Clifford algebras occur throughout mathematical physics, the most well known being the real numbers, complex numbers, quaternions, and complex quaternions. In this book we show how Clifford algebras can be firmly based on the Grassmann algebra as a sum of generalized Grassmann products.

Historical Note

The seminal work on Clifford algebra is by Clifford in his paper *Applications of Grassmann's Extensive Algebra* that he published in the *American Journal of Mathematics Pure and Applied* in 1878 (Clifford). Clifford became a great admirer of Grassmann and one of those rare contemporaries who appears to have understood his work. The first paragraph of this paper contains the following passage.

> Until recently I was unacquainted with the *Ausdehnungslehre*, and knew only so much of it as is contained in the author's geometrical papers in *Crelle's Journal* and in *Hankel's Lectures on Complex Numbers*. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work, and my conviction that its principles will exercise a vast influence on the future of mathematical science.

12.2 The Clifford Product

Definition of the Clifford product

The Clifford product of elements $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is denoted by $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ and defined to be

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.1$$

Here, $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}$ is the generalized Grassmann product of order λ.

The most surprising property of the Clifford product is its associativity, even though it is defined in terms of products which are non-associative. The associativity of the Clifford products is not directly evident from the definition.

Tabulating Clifford products

In order to see what this formula gives we can tabulate the Clifford products in terms of generalized products. Here we maintain the grade of the first factor general, and vary the grade of the second.

```
Table[
  {α_m ⋄ β_k, Sum[CreateSignedGeneralizedProducts[λ][α_m, β_k], {λ, 0, k}]},
  {k, 0, 4}] // SimplifySigns // TableForm
```

$\underset{m}{\alpha} \diamond \underset{0}{\beta} \qquad \underset{m}{\alpha} \underset{0}{\triangle} \underset{0}{\beta}$

$\underset{m}{\alpha} \diamond \beta \qquad \underset{m}{\alpha} \underset{0}{\triangle} \beta - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \beta\right)$

$\underset{m}{\alpha} \diamond \underset{2}{\beta} \qquad \underset{m}{\alpha} \underset{0}{\triangle} \underset{2}{\beta} - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}\right) - \underset{m}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}$

$\underset{m}{\alpha} \diamond \underset{3}{\beta} \qquad \underset{m}{\alpha} \underset{0}{\triangle} \underset{3}{\beta} - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \underset{3}{\beta}\right) - \underset{m}{\alpha} \underset{2}{\triangle} \underset{3}{\beta} + (-1)^m \left(\underset{m}{\alpha} \underset{3}{\triangle} \underset{3}{\beta}\right)$

$\underset{m}{\alpha} \diamond \underset{4}{\beta} \qquad \underset{m}{\alpha} \underset{0}{\triangle} \underset{4}{\beta} - (-1)^m \left(\underset{m}{\alpha} \underset{1}{\triangle} \underset{4}{\beta}\right) - \underset{m}{\alpha} \underset{2}{\triangle} \underset{4}{\beta} + (-1)^m \left(\underset{m}{\alpha} \underset{3}{\triangle} \underset{4}{\beta}\right) + \underset{m}{\alpha} \underset{4}{\triangle} \underset{4}{\beta}$

The grade of a Clifford product

It can be seen that, from its definition 12.1 in terms of generalized products, the Clifford product of $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ is a sum of elements with grades ranging from $m + k$ to $| m - k |$ in steps of 2. All the elements of a given Clifford product are either of even grade (if $m + k$ is even), or of odd grade (if $m + k$ is odd). To calculate the full range of grades in a space of unspecified dimension we can use the *GrassmannAlgebra* function `RawGrade`. For example:

RawGrade$\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$

{1, 3, 5}

Given a space of a certain dimension, some of these elements may necessarily be zero (and thus result in a grade of `Grade0`), because their grade is larger than the dimension of the space. For example, in a 3-space:

$\mathbb{V}_3$**; Grade**$\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$

{1, 3, Grade0}

In the general discussion of the Clifford product that follows, we will assume that the dimension of the space is high enough to avoid any terms of the product becoming zero because their grade exceeds the dimension of the space. In later more specific examples, however, the dimension of the space becomes an important factor in determining the structure of the particular Clifford algebra under consideration.

Clifford products in terms of generalized products

As can be seen from the definition, the Clifford product of a simple *m*-element and a simple *k*-element can be expressed as the sum of signed generalized products.

For example, $\underset{3}{\alpha} \diamond \underset{2}{\beta}$ may be expressed as the sum of three signed generalized products of grades 1, 3 and 5. In *GrassmannAlgebra* we can effect this conversion by applying the `ToGeneralizedProducts` function.

ToGeneralizedProducts$\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$

$$\underset{3}{\alpha} \underset{0}{\triangle} \underset{2}{\beta} + \underset{3}{\alpha} \underset{1}{\triangle} \underset{2}{\beta} - \underset{3}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}$$

Multiple Clifford products can be expanded in the same way. For example:

ToGeneralizedProducts$\left[\underset{3}{\alpha} \diamond \underset{2}{\beta} \diamond \gamma\right]$

$$\left(\underset{3}{\alpha} \underset{0}{\triangle} \underset{2}{\beta}\right) \underset{0}{\triangle} \gamma + \left(\underset{3}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}\right) \underset{0}{\triangle} \gamma + -\left(\underset{3}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}\right) \underset{0}{\triangle} \gamma + \left(\underset{3}{\alpha} \underset{0}{\triangle} \underset{2}{\beta}\right) \underset{1}{\triangle} \gamma + \left(\underset{3}{\alpha} \underset{1}{\triangle} \underset{2}{\beta}\right) \underset{1}{\triangle} \gamma + -\left(\underset{3}{\alpha} \underset{2}{\triangle} \underset{2}{\beta}\right) \underset{1}{\triangle} \gamma$$

`ToGeneralizedProducts` can also be used to expand Clifford products in any Grassmann expression, or list of Grassmann expressions. For example we can express the Clifford product of two general Grassmann numbers in 2-space in terms of generalized products.

```
𝕍2; X = CreateGrassmannNumber[ξ] ◇ CreateGrassmannNumber[ψ]
```

$$(\xi_0 + e_1\,\xi_1 + e_2\,\xi_2 + \xi_3\,e_1\wedge e_2)\diamond(\psi_0 + e_1\,\psi_1 + e_2\,\psi_2 + \psi_3\,e_1\wedge e_2)$$

```
X1 = ToGeneralizedProducts[X]
```

$$\begin{aligned}&\xi_0 \underset{0}{\triangle} \psi_0 + \xi_0 \underset{0}{\triangle} (e_1\,\psi_1) + \xi_0 \underset{0}{\triangle} (e_2\,\psi_2) + \xi_0 \underset{0}{\triangle} (\psi_3\,e_1\wedge e_2) + (e_1\,\xi_1) \underset{0}{\triangle} \psi_0 +\\ &(e_1\,\xi_1) \underset{0}{\triangle} (e_1\,\psi_1) + (e_1\,\xi_1) \underset{0}{\triangle} (e_2\,\psi_2) + (e_1\,\xi_1) \underset{0}{\triangle} (\psi_3\,e_1\wedge e_2) + (e_2\,\xi_2) \underset{0}{\triangle} \psi_0 +\\ &(e_2\,\xi_2) \underset{0}{\triangle} (e_1\,\psi_1) + (e_2\,\xi_2) \underset{0}{\triangle} (e_2\,\psi_2) + (e_2\,\xi_2) \underset{0}{\triangle} (\psi_3\,e_1\wedge e_2) +\\ &(\xi_3\,e_1\wedge e_2) \underset{0}{\triangle} \psi_0 + (\xi_3\,e_1\wedge e_2) \underset{0}{\triangle} (e_1\,\psi_1) + (\xi_3\,e_1\wedge e_2) \underset{0}{\triangle} (e_2\,\psi_2) +\\ &(\xi_3\,e_1\wedge e_2) \underset{0}{\triangle} (\psi_3\,e_1\wedge e_2) + (e_1\,\xi_1) \underset{1}{\triangle} (e_1\,\psi_1) + (e_1\,\xi_1) \underset{1}{\triangle} (e_2\,\psi_2) +\\ &(e_1\,\xi_1) \underset{1}{\triangle} (\psi_3\,e_1\wedge e_2) + (e_2\,\xi_2) \underset{1}{\triangle} (e_1\,\psi_1) + (e_2\,\xi_2) \underset{1}{\triangle} (e_2\,\psi_2) +\\ &(e_2\,\xi_2) \underset{1}{\triangle} (\psi_3\,e_1\wedge e_2) - (\xi_3\,e_1\wedge e_2) \underset{1}{\triangle} (e_1\,\psi_1) - (\xi_3\,e_1\wedge e_2) \underset{1}{\triangle} (e_2\,\psi_2) -\\ &(\xi_3\,e_1\wedge e_2) \underset{1}{\triangle} (\psi_3\,e_1\wedge e_2) - (\xi_3\,e_1\wedge e_2) \underset{2}{\triangle} (\psi_3\,e_1\wedge e_2)\end{aligned}$$

Clifford products in terms of interior products

The primary definitions of Clifford and generalized products are in terms of exterior and interior products. To transform a Clifford product to exterior and interior products, we can either apply the *GrassmannAlgebra* function `ToInteriorProducts` to the results obtained above, or directly to the expression involving the Clifford product.

$$\texttt{ToInteriorProducts}\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$$

$$-(\alpha_1\wedge\alpha_2\wedge\alpha_3\ominus\beta_1\wedge\beta_2) + (\alpha_1\wedge\alpha_2\wedge\alpha_3\ominus\beta_1)\wedge\beta_2 - (\alpha_1\wedge\alpha_2\wedge\alpha_3\ominus\beta_2)\wedge\beta_1 + \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\beta_1\wedge\beta_2$$

Note that `ToInteriorProducts` works with explicit elements, but also as above, automatically creates a simple exterior product from an underscripted element.

`ToInteriorProducts` expands the Clifford product by using the A form of the generalized product. A second function `ToInteriorProductsB` expands the Clifford product by using the `B` form of the expansion to give a different but equivalent expression.

$$\texttt{ToInteriorProductsB}\left[\underset{3}{\alpha} \diamond \underset{2}{\beta}\right]$$

$$-(\alpha_1\wedge\alpha_2\wedge\alpha_3\ominus\beta_1\wedge\beta_2) - \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\beta_1\ominus\beta_2 + \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\beta_2\ominus\beta_1 + \alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\beta_1\wedge\beta_2$$

In this example, the difference is evidenced in the second and third terms.

Clifford products in terms of inner products

Clifford products may also be expressed in terms of inner products, some of which may be scalar products. From this form it is easy to read the grade of the terms.

```
ToInnerProducts[α₃ ⋄ β₂]
```

$$-(\alpha_2\wedge\alpha_3\ominus\beta_1\wedge\beta_2)\,\alpha_1+(\alpha_1\wedge\alpha_3\ominus\beta_1\wedge\beta_2)\,\alpha_2-(\alpha_1\wedge\alpha_2\ominus\beta_1\wedge\beta_2)\,\alpha_3-(\alpha_3\ominus\beta_2)\,\alpha_1\wedge\alpha_2\wedge\beta_1+(\alpha_3\ominus\beta_1)\,\alpha_1\wedge\alpha_2\wedge\beta_2+(\alpha_2\ominus\beta_2)\,\alpha_1\wedge\alpha_3\wedge\beta_1-(\alpha_2\ominus\beta_1)\,\alpha_1\wedge\alpha_3\wedge\beta_2-(\alpha_1\ominus\beta_2)\,\alpha_2\wedge\alpha_3\wedge\beta_1+(\alpha_1\ominus\beta_1)\,\alpha_2\wedge\alpha_3\wedge\beta_2+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\beta_1\wedge\beta_2$$

Clifford products in terms of scalar products

Finally, Clifford products may be expressed in terms of scalar and exterior products only.

```
ToScalarProducts[α₃ ⋄ β₂]
```

$$(\alpha_2\ominus\beta_2)\,(\alpha_3\ominus\beta_1)\,\alpha_1-(\alpha_2\ominus\beta_1)\,(\alpha_3\ominus\beta_2)\,\alpha_1-(\alpha_1\ominus\beta_2)\,(\alpha_3\ominus\beta_1)\,\alpha_2+(\alpha_1\ominus\beta_1)\,(\alpha_3\ominus\beta_2)\,\alpha_2+(\alpha_1\ominus\beta_2)\,(\alpha_2\ominus\beta_1)\,\alpha_3-(\alpha_1\ominus\beta_1)\,(\alpha_2\ominus\beta_2)\,\alpha_3-(\alpha_3\ominus\beta_2)\,\alpha_1\wedge\alpha_2\wedge\beta_1+(\alpha_3\ominus\beta_1)\,\alpha_1\wedge\alpha_2\wedge\beta_2+(\alpha_2\ominus\beta_2)\,\alpha_1\wedge\alpha_3\wedge\beta_1-(\alpha_2\ominus\beta_1)\,\alpha_1\wedge\alpha_3\wedge\beta_2-(\alpha_1\ominus\beta_2)\,\alpha_2\wedge\alpha_3\wedge\beta_1+(\alpha_1\ominus\beta_1)\,\alpha_2\wedge\alpha_3\wedge\beta_2+\alpha_1\wedge\alpha_2\wedge\alpha_3\wedge\beta_1\wedge\beta_2$$

12.3 The Reverse of an Exterior Product

Defining the reverse

We will find in our discussion of Clifford products that many operations and formulae are simplified by expressing some of the exterior products in a form which reverses the order of the 1-element factors in the products.

We denote the *reverse* of a simple m-element $\underset{m}{\alpha}$ by $\underset{m}{\alpha}^{\dagger}$, and define it to be:

$$(\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_m)^{\dagger} == \alpha_m\wedge\alpha_{m-1}\wedge\cdots\wedge\alpha_1 \qquad 12.2$$

We can easily work out the number of permutations to achieve this rearrangement as $\frac{1}{2}\,m\,(m-1)$. *Mathematica* automatically simplifies $(-1)^{\frac{1}{2}m\,(m-1)}$ to $\mathbb{i}^{m\,(m-1)}$.

$$\underset{m}{\alpha}^{\dagger} == (-1)^{\frac{1}{2} m (m-1)} \underset{m}{\alpha} == \mathbb{i}^{m (m-1)} \underset{m}{\alpha} \qquad 12.3$$

The operation of taking the reverse of an element is called *reversion*.

In *Mathematica*, the superscript dagger is represented by `SuperDagger`. Thus **`SuperDagger`**$[\underset{m}{\alpha}]$ returns $\underset{m}{\alpha}^{\dagger}$.

The pattern of signs, as m increases from zero, alternates in pairs:

1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, …

Computing the reverse

In *GrassmannAlgebra* you can take the reverse of the exterior products in a Grassmann expression or list of Grassmann expressions by using the function `GrassmannReverse`. `GrassmannReverse` expands products and factors scalars before reversing the factors. It will operate with graded variables of either numeric or symbolic grade. For example:

`GrassmannReverse`$\left[\left\{1, x, x\wedge y, \underset{3}{\alpha}\wedge 3, \underset{k}{\beta}, \underset{m}{\alpha}\ominus\left(\underset{k}{\beta}-\underset{p}{\gamma}\right)\right\}\right]$

$\{1, x, y\wedge x, 3\,\alpha_3\wedge\alpha_2\wedge\alpha_1, \beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1,$
$\alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1 - \alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\gamma_p\wedge\cdots\wedge\gamma_2\wedge\gamma_1\}$

On the other hand, if we wish just to put exterior products into reverse *form* (that is, reversing the factors, and changing the sign of the product so that the result remains equal to the original), we can use the *GrassmannAlgebra* function `ReverseForm`.

`ReverseForm`$\left[\left\{1, x, x\wedge y, \underset{3}{\alpha}, \underset{k}{\beta}, \underset{m}{\alpha}\ominus\left(\underset{k}{\beta}-\underset{p}{\gamma}\right)\right\}\right]$

$\{1, x, -(y\wedge x), -(\alpha_3\wedge\alpha_2\wedge\alpha_1), \mathbb{i}^{(-1+k)\,k}\,\beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1,$
$\mathbb{i}^{(-1+k)\,k+(-1+m)\,m}\,(\alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\beta_k\wedge\cdots\wedge\beta_2\wedge\beta_1) -$
$\mathbb{i}^{(-1+m)\,m+(-1+p)\,p}\,(\alpha_m\wedge\cdots\wedge\alpha_2\wedge\alpha_1\ominus\gamma_p\wedge\cdots\wedge\gamma_2\wedge\gamma_1)\}$

12.4 Special Cases of Clifford Products

The Clifford product with scalars

The Clifford product of a scalar with any element simplifies to the usual (field) product.

```
ToScalarProducts[a ⋄ α_m]
```

$a\ \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$

$$a \diamond \underset{m}{\alpha} = a\ \underset{m}{\alpha} \qquad 12.4$$

Hence the Clifford product of any number of scalars is just their underlying field product.

```
ToScalarProducts[a ⋄ b ⋄ c]
```

$a\ b\ c$

The Clifford product of 1-elements

The Clifford product of two 1-elements is just the sum of their interior (here scalar) and exterior products. Hence it is of grade {0, 2}.

```
ToScalarProducts[x ⋄ y]
```

$x \ominus y + x \wedge y$

$$x \diamond y = x \wedge y + x \ominus y \qquad 12.5$$

The Clifford product of any number of 1-elements can be computed.

```
ToScalarProducts[x ⋄ y ⋄ z]
```

$z\ (x \ominus y) - y\ (x \ominus z) + x\ (y \ominus z) + x \wedge y \wedge z$

```
ToScalarProducts[w ⋄ x ⋄ y ⋄ z]
```

$(w \ominus z)\ (x \ominus y) - (w \ominus y)\ (x \ominus z) + (w \ominus x)\ (y \ominus z) + (y \ominus z)\ w \wedge x - (x \ominus z)\ w \wedge y +$
$(x \ominus y)\ w \wedge z + (w \ominus z)\ x \wedge y - (w \ominus y)\ x \wedge z + (w \ominus x)\ y \wedge z + w \wedge x \wedge y \wedge z$

The Clifford product of an *m*-element and a 1-element

The Clifford product of an arbitrary simple *m*-element and a 1-element results in just two terms: their exterior and interior products.

$$x \diamond \underset{m}{\alpha} = x \wedge \underset{m}{\alpha} + \underset{m}{\alpha} \ominus x \qquad 12.6$$

$$\underset{m}{\alpha} \diamond x == \underset{m}{\alpha} \wedge x - (-1)^m \underset{m}{\alpha} \ominus x \qquad 12.7$$

By rewriting equation 12.7 as:

$$(-1)^m \underset{m}{\alpha} \diamond x == x \wedge \underset{m}{\alpha} - \underset{m}{\alpha} \ominus x$$

we can add and subtract equations 12.6 and 12.7 to express the exterior product and interior products of a 1-element and an *m*-element in terms of Clifford products

$$x \wedge \underset{m}{\alpha} == \frac{1}{2}\left(x \diamond \underset{m}{\alpha} + (-1)^m \underset{m}{\alpha} \diamond x\right) \qquad 12.8$$

$$\underset{m}{\alpha} \ominus x == \frac{1}{2}\left(x \diamond \underset{m}{\alpha} - (-1)^m \underset{m}{\alpha} \diamond x\right) \qquad 12.9$$

The Clifford product of an *m*-element and a 2-element

The Clifford product of an arbitrary *m*-element and a 2-element is given by just three terms, an exterior product, a generalized product of order 1, and an interior product.

$$\underset{m}{\alpha} \diamond \underset{2}{\beta} == \underset{m}{\alpha} \wedge \underset{2}{\beta} - (-1)^m \underset{m}{\alpha} \underset{1}{\triangle} \underset{2}{\beta} - \underset{m}{\alpha} \ominus \underset{2}{\beta} \qquad 12.10$$

The Clifford product of two 2-elements

By way of example we explore the various forms into which we can cast a Clifford product of two 2-elements. The highest level is into a sum of generalized products. From this we can expand the terms into interior, inner or scalar products where appropriate.

We expect the grade of the Clifford product to be a composite one.

```
RawGrade[(x ∧ y) ⋄ (u ∧ v)]
```

{0, 2, 4}

```
ToGeneralizedProducts[(x ∧ y) ⋄ (u ∧ v)]
```

$$x \wedge y \underset{0}{\triangle} u \wedge v - x \wedge y \underset{1}{\triangle} u \wedge v - x \wedge y \underset{2}{\triangle} u \wedge v$$

The first of these generalized products is equivalent to an exterior product of grade 4. The second is a generalized product of grade 2. The third reduces to an interior product of grade 0. We can see this more explicitly by converting to interior products.

```
ToInteriorProducts[(x ∧ y) ⋄ (u ∧ v)]
```

$$-(u \wedge v \ominus x \wedge y) - (x \wedge y \ominus u) \wedge v + (x \wedge y \ominus v) \wedge u + u \wedge v \wedge x \wedge y$$

We can convert the middle terms to inner (in this case scalar) products.

```
ToInnerProducts[(x ∧ y) ⋄ (u ∧ v)]
```

$$-(u \wedge v \ominus x \wedge y) + (v \ominus y)\, u \wedge x - (v \ominus x)\, u \wedge y - (u \ominus y)\, v \wedge x + (u \ominus x)\, v \wedge y + u \wedge v \wedge x \wedge y$$

Finally we can express the Clifford number in terms only of exterior and scalar products.

```
ToScalarProducts[(x ∧ y) ⋄ (u ∧ v)]
```

$$(u \ominus y)\,(v \ominus x) - (u \ominus x)\,(v \ominus y) + (v \ominus y)\, u \wedge x - (v \ominus x)\, u \wedge y - (u \ominus y)\, v \wedge x + (u \ominus x)\, v \wedge y + u \wedge v \wedge x \wedge y$$

The Clifford product of two identical elements

The Clifford product of two identical elements $\underset{p}{\gamma}$ is, by definition

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} == \sum_{\lambda=0}^{p} (-1)^{\lambda(p-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \left(\underset{p}{\gamma} \underset{\lambda}{\Delta} \underset{p}{\gamma}\right)$$

Since the only non-zero generalized product of the form $\underset{p}{\gamma} \underset{\lambda}{\Delta} \underset{p}{\gamma}$ is that for which $\lambda = p$, that is $\underset{p}{\gamma} \ominus \underset{p}{\gamma}$, we have immediately that:

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} == (-1)^{\frac{1}{2}p(p-1)} \underset{p}{\gamma} \ominus \underset{p}{\gamma}$$

Or, alternatively:

$$\boxed{\underset{p}{\gamma} \ominus \underset{p}{\gamma} == \underset{p}{\gamma}^{\dagger} \diamond \underset{p}{\gamma} == \underset{p}{\gamma} \diamond \underset{p}{\gamma}^{\dagger}} \qquad \mathbf{12.11}$$

12.5 Alternate Forms for the Clifford Product

Alternate expansions of the Clifford product

The Clifford product has been defined in Section 12.2 as:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda(m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

Alternate forms for the generalized product have been discussed in Chapter 10. The generalized product, and hence the Clifford product, may be expanded by decomposition of $\underset{m}{\alpha}$, by decomposition of $\underset{k}{\beta}$, or by decomposition of both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$.

The Clifford product expressed by decomposition of the first factor

The generalized product, expressed by decomposition of $\underset{m}{\alpha}$ is:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == \sum_{i=1}^{\binom{m}{\lambda}} \underset{m-\lambda}{\alpha^i} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^i}\right) \qquad \underset{m}{\alpha} == \underset{\lambda}{\alpha^1} \wedge \underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2} \wedge \underset{m-\lambda}{\alpha^2} == \cdots$$

Substituting this into the expression for the Clifford product gives:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} (-1)^{\lambda(m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \underset{m-\lambda}{\alpha^i} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^i}\right)$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1} \wedge \underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2} \wedge \underset{m-\lambda}{\alpha^2} == \cdots$$

Our objective is to rearrange the formula for the Clifford product so that the signs are absorbed into the formula, thus making the form of the formula independent of the values of *m* and λ. We can do this by writing $(-1)^{\frac{1}{2}\lambda(\lambda-1)} \underset{\lambda}{\alpha^i} == \underset{\lambda}{\alpha^i}^\dagger$ (where $\underset{\lambda}{\alpha^i}^\dagger$ is the reverse of $\underset{\lambda}{\alpha^i}$) and interchanging the order of the decomposition of $\underset{m}{\alpha}$ into a λ-element and a (m-λ)-element to absorb the $(-1)^{\lambda(m-\lambda)}$ factor: $(-1)^{\lambda(m-\lambda)} \underset{m}{\alpha} == \underset{m-\lambda}{\alpha^1} \wedge \underset{\lambda}{\alpha^1} == \underset{m-\lambda}{\alpha^2} \wedge \underset{\lambda}{\alpha^2} == \cdots$.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \underset{m-\lambda}{\alpha^i} \wedge \left(\underset{k}{\beta} \ominus \underset{\lambda}{\alpha^i}^\dagger\right)$$

$$\underset{m}{\alpha} == \underset{m-\lambda}{\alpha^1} \wedge \underset{\lambda}{\alpha^1} == \underset{m-\lambda}{\alpha^2} \wedge \underset{\lambda}{\alpha^2} == \cdots$$

Because of the symmetry of the expression with respect to λ and (m-λ), we can write μ equal to (m-λ) and then λ becomes (m-μ). This enables us to write the formula a little simpler by arranging for the factors of grade μ to come before those of (m-μ). Finally, because of the inherent arbitrariness of the symbol μ, we can change it back to λ to get the formula in a more accustomed form.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} \underset{\lambda}{\alpha^{i}} \wedge \left(\underset{k}{\beta} \ominus \underset{m-\lambda}{\alpha^{i}}{}^{\dagger}\right) \qquad \mathbf{12.12}$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^{1}} \wedge \underset{m-\lambda}{\alpha^{1}} == \underset{\lambda}{\alpha^{2}} \wedge \underset{m-\lambda}{\alpha^{2}} == \cdots$$

The right hand side of this expression is a sum of interior products. In *GrassmannAlgebra* we can develop the Clifford product $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ as a sum of interior products in this particular form by using `ToInteriorProductsD` (note the final '`D`'). Since $\underset{m}{\alpha}$ is to be decomposed, it must have an explicit numerical grade.

Example 1

We can expand any explicit elements.

```
ToInteriorProductsD[(x∧y)◇(u∧v)]
```

$x \wedge y \ominus v \wedge u + x \wedge (u \wedge v \ominus y) - y \wedge (u \wedge v \ominus x) + u \wedge v \wedge x \wedge y$

Note that this is a different (although equivalent) result from that obtained using `ToInteriorProducts`.

```
ToInteriorProducts[(x∧y)◇(u∧v)]
```

$-(u \wedge v \ominus x \wedge y) - (x \wedge y \ominus u) \wedge v + (x \wedge y \ominus v) \wedge u + u \wedge v \wedge x \wedge y$

Example 2

We can also enter the elements as graded variables and have *GrassmannAlgebra* create the requisite products.

$\texttt{ToInteriorProductsD}\left[\underset{2}{\alpha} \diamond \underset{2}{\beta}\right]$

$\beta_1 \wedge \beta_2 \ominus \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge (\beta_1 \wedge \beta_2 \ominus \alpha_2) - \alpha_2 \wedge (\beta_1 \wedge \beta_2 \ominus \alpha_1) + \alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2$

Example 3

The formula shows that it does not depend on the grade of $\underset{k}{\beta}$ for its form. Thus we can still obtain an expansion for general $\underset{k}{\beta}$. For example we can take:

```
A = ToInteriorProductsD[α_2 ⋄ β_k]
```

$$\underset{k}{\beta}\ominus\alpha_2\wedge\alpha_1+\alpha_1\wedge\left(\underset{k}{\beta}\ominus\alpha_2\right)-\alpha_2\wedge\left(\underset{k}{\beta}\ominus\alpha_1\right)+\alpha_1\wedge\alpha_2\wedge\underset{k}{\beta}$$

Example 4

Note that if *k* is less than the grade of the first factor, some of the interior product terms may be zero, thus simplifying the expression.

```
B = ToInteriorProductsD[α_2 ⋄ β]
```

$$-(\alpha_1\wedge\alpha_2\ominus\beta)+\beta\wedge\alpha_1\wedge\alpha_2$$

If we put *k* equal to 1 in the expression derived for general *k* in Example 3 we get:

```
A1 = A /. k → 1
```

$$\beta\ominus\alpha_2\wedge\alpha_1+\alpha_1\wedge(\beta\ominus\alpha_2)-\alpha_2\wedge(\beta\ominus\alpha_1)+\alpha_1\wedge\alpha_2\wedge\beta$$

Although this does not immediately look like the expression **B** above, we can see that it is the same by noting that the first term of **A1** is zero, and expanding their difference to scalar products.

```
ToScalarProducts[B - A1]
```

0

Alternative expression by decomposition of the first factor

An alternative expression for the Clifford product expressed by a decomposition of $\underset{m}{\alpha}$ is obtained by reversing the order of the factors in the generalized products, and then expanding the generalized products in their B form expansion.

$$\underset{m}{\alpha}\diamond\underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]}(-1)^{\lambda\,(m-\lambda)+\frac{1}{2}\lambda\,(\lambda-1)+(m-\lambda)\,(k-\lambda)}\left(\underset{k}{\beta}\underset{\lambda}{\Delta}\underset{m}{\alpha}\right)$$

$$== \sum_{\lambda=0}^{\mathrm{Min}[m,k]}\sum_{i=1}^{\binom{m}{\lambda}}(-1)^{k\,(m-\lambda)+\frac{1}{2}\lambda\,(\lambda-1)}\left(\underset{k}{\beta}\wedge\underset{m-\lambda}{\alpha^i}\right)\ominus\underset{\lambda}{\alpha^i}$$

As before we write $(-1)^{\frac{1}{2}\lambda\,(\lambda-1)}\underset{\lambda}{\alpha^i} == \underset{\lambda}{\alpha^i}^{\dagger}$ and interchange the order of $\underset{k}{\beta}$ and $\underset{m-\lambda}{\alpha^i}$ to absorb the sign $(-1)^{k\,(m-\lambda)}$ to get:

$$\underset{m}{\alpha}\diamond\underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]}\sum_{i=1}^{\binom{m}{\lambda}}\left(\underset{m-\lambda}{\alpha^i}\wedge\underset{k}{\beta}\right)\ominus\underset{\lambda}{\alpha^i}^{\dagger}$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1}\wedge\underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2}\wedge\underset{m-\lambda}{\alpha^2} == \cdots$$

In order to get this into a form for direct comparison of our previously derived result in formula 12.12, we write $(-1)^{\lambda(m-\lambda)} \underset{m}{\alpha} == \underset{m-\lambda}{\alpha^1} \wedge \underset{\lambda}{\alpha^1} == \underset{m-\lambda}{\alpha^2} \wedge \underset{\lambda}{\alpha^2} == \cdots$, then interchange λ and $(m-\lambda)$ as before to get finally:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{i=1}^{\binom{m}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{\lambda}{\alpha^i} \wedge \underset{k}{\beta}\right) \ominus \underset{m-\lambda}{\alpha^i}{}^{\dagger} \qquad \mathbf{12.13}$$
$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1} \wedge \underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2} \wedge \underset{m-\lambda}{\alpha^2} == \cdots$$

As before, the right hand side of this expression is a sum of interior products. In *GrassmannAlgebra* we can develop the Clifford product $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ in this form by using `ToInteriorProductsC`. This is done in the Section 12.6 below.

The Clifford product expressed by decomposition of the second factor

If we wish to expand a Clifford product in terms of the second factor $\underset{k}{\beta}$, we can use formulas A and B of the generalized product theorem (Section 10.5) and substitute directly to get either of:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{j=1}^{\binom{k}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{m}{\alpha} \ominus \underset{\lambda}{\beta^j}{}^{\dagger}\right) \wedge \underset{k-\lambda}{\beta^j} \qquad \mathbf{12.14}$$
$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots$$

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} \sum_{j=1}^{\binom{k}{\lambda}} (-1)^{\lambda(m-\lambda)} \left(\underset{m}{\alpha} \wedge \underset{k-\lambda}{\beta^j}\right) \ominus \underset{\lambda}{\beta^j}{}^{\dagger} \qquad \mathbf{12.15}$$
$$\underset{k}{\beta} == \underset{\lambda}{\beta^1} \wedge \underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2} \wedge \underset{k-\lambda}{\beta^2} == \cdots$$

The Clifford product expressed by decomposition of both factors

We can similarly decompose the Clifford product in terms of both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$. The sign $(-1)^{\frac{1}{2}\lambda(\lambda-1)}$ can be absorbed by taking the reverse of either $\underset{\lambda}{\alpha^i}$ or $\underset{\lambda}{\beta^j}$.

$$\underset{m}{\alpha}\diamond\underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]}\sum_{i=1}^{\binom{m}{\lambda}}\sum_{j=1}^{\binom{k}{\lambda}}(-1)^{\lambda(m-\lambda)}\left(\underset{\lambda}{\alpha^i}^{\dagger}\ominus\underset{\lambda}{\beta^j}\right)\underset{m-\lambda}{\alpha^i}\wedge\underset{k-\lambda}{\beta^j}$$

$$\underset{m}{\alpha}\diamond\underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]}\sum_{i=1}^{\binom{m}{\lambda}}\sum_{j=1}^{\binom{k}{\lambda}}(-1)^{\lambda(m-\lambda)}\left(\underset{\lambda}{\alpha^i}\ominus\underset{\lambda}{\beta^j}^{\dagger}\right)\underset{m-\lambda}{\alpha^i}\wedge\underset{k-\lambda}{\beta^j} \qquad 12.16$$

$$\underset{m}{\alpha} == \underset{\lambda}{\alpha^1}\wedge\underset{m-\lambda}{\alpha^1} == \underset{\lambda}{\alpha^2}\wedge\underset{m-\lambda}{\alpha^2} == \cdots \qquad \underset{k}{\beta} == \underset{\lambda}{\beta^1}\wedge\underset{k-\lambda}{\beta^1} == \underset{\lambda}{\beta^2}\wedge\underset{k-\lambda}{\beta^2} == \cdots$$

12.6 Writing Down a General Clifford Product

The form of a Clifford product expansion

We take as example the Clifford product $\underset{4}{\alpha}\diamond\underset{k}{\beta}$ and expand it in terms of the factors of its first factor in both the D form and the C form. We see that all terms except the interior and exterior products at the ends of the expression appear to be different. There are of course equal numbers of terms in either form. Note that if the parentheses were not there, *the expressions would be identical* except for (possibly) the signs of the terms. Although these central terms differ between the two forms, we can show by reducing them to scalar products that their *sums* are the same.

The D form

```
X = ToInteriorProductsD[α ⋄ β]
```
$$X = \text{ToInteriorProductsD}\left[\underset{4}{\alpha} \diamond \underset{k}{\beta}\right]$$

$$\begin{aligned}
&\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2\right) - \\
&\alpha_2 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_1\right) + \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2 \wedge \alpha_1\right) - \\
&\alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2 \wedge \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3\right) - \alpha_1 \wedge \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2\right) + \\
&\alpha_1 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2\right) + \alpha_2 \wedge \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_1\right) - \alpha_2 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_1\right) + \\
&\alpha_3 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_2 \wedge \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4\right) - \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3\right) + \\
&\alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_2\right) - \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \underset{k}{\beta}
\end{aligned}$$

The C form

Note that since the exterior product has higher precedence in *Mathematica* than the interior product, $\alpha_1 \wedge \underset{k}{\beta} \ominus \alpha_2$ is equivalent to $\left(\alpha_1 \wedge \underset{k}{\beta}\right) \ominus \alpha_2$, and no parentheses are necessary in the terms of the C form.

$$\text{ToInteriorProductsC}\left[\underset{4}{\alpha} \diamond \underset{k}{\beta}\right]$$

$$\begin{aligned}
&\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2 \wedge \alpha_1 - \alpha_1 \wedge \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2 + \alpha_2 \wedge \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_1 - \\
&\alpha_3 \wedge \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2 \wedge \alpha_1 + \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge \alpha_2 \wedge \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 - \\
&\alpha_1 \wedge \alpha_3 \wedge \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2 + \alpha_1 \wedge \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2 + \alpha_2 \wedge \alpha_3 \wedge \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_1 - \\
&\alpha_2 \wedge \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_1 + \alpha_3 \wedge \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_2 \wedge \alpha_1 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \underset{k}{\beta} \ominus \alpha_4 + \\
&\alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_3 - \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_2 + \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \underset{k}{\beta} \ominus \alpha_1 + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \underset{k}{\beta}
\end{aligned}$$

Either form is easy to write down directly by observing simple mnemonic rules.

A mnemonic way to write down a general Clifford product

We can begin developing the D form mnemonically by listing the $\underset{\lambda}{\alpha^{i}}$. This list has the same form as the basis of the Grassmann algebra of a 4-space.

◆ 1. Calculate the $\underset{\lambda}{\alpha^{i}}$

These are all the essentially different combinations of factors of all grades. The list has the same form as the basis of the Grassmann algebra whose n-element is $\underset{m}{\alpha}$.

```
DeclareBasis[4, α]; B = BasisΛ[]
```

$\{1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 \wedge \alpha_2, \alpha_1 \wedge \alpha_3, \alpha_1 \wedge \alpha_4, \alpha_2 \wedge \alpha_3, \alpha_2 \wedge \alpha_4, \alpha_3 \wedge \alpha_4, \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \alpha_1 \wedge \alpha_2 \wedge \alpha_4, \alpha_1 \wedge \alpha_3 \wedge \alpha_4, \alpha_2 \wedge \alpha_3 \wedge \alpha_4, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4\}$

◆ 2. Calculate the cobasis with respect to $\underset{m}{\alpha}$

```
A = CobasisΛ[]
```

$\{\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4, \alpha_2 \wedge \alpha_3 \wedge \alpha_4, -(\alpha_1 \wedge \alpha_3 \wedge \alpha_4), \alpha_1 \wedge \alpha_2 \wedge \alpha_4, -(\alpha_1 \wedge \alpha_2 \wedge \alpha_3), \alpha_3 \wedge \alpha_4, -(\alpha_2 \wedge \alpha_4), \alpha_2 \wedge \alpha_3, \alpha_1 \wedge \alpha_4, -(\alpha_1 \wedge \alpha_3), \alpha_1 \wedge \alpha_2, \alpha_4, -\alpha_3, \alpha_2, -\alpha_1, 1\}$

◆ 3. Take the reverse of the cobasis

```
R = GrassmannReverse[A]
```

$\{\alpha_4 \wedge \alpha_3 \wedge \alpha_2 \wedge \alpha_1, \alpha_4 \wedge \alpha_3 \wedge \alpha_2, -(\alpha_4 \wedge \alpha_3 \wedge \alpha_1), \alpha_4 \wedge \alpha_2 \wedge \alpha_1, -(\alpha_3 \wedge \alpha_2 \wedge \alpha_1), \alpha_4 \wedge \alpha_3, -(\alpha_4 \wedge \alpha_2), \alpha_3 \wedge \alpha_2, \alpha_4 \wedge \alpha_1, -(\alpha_3 \wedge \alpha_1), \alpha_2 \wedge \alpha_1, \alpha_4, -\alpha_3, \alpha_2, -\alpha_1, 1\}$

◆ 4. Take the interior product of each of these elements with $\underset{k}{\beta}$

$\mathbf{S = Thread}[\underset{k}{\beta} \ominus \mathbf{R}]$

$\{\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2 \wedge \alpha_1, \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2, \underset{k}{\beta} \ominus -(\alpha_4 \wedge \alpha_3 \wedge \alpha_1), \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2 \wedge \alpha_1, \underset{k}{\beta} \ominus -(\alpha_3 \wedge \alpha_2 \wedge \alpha_1), \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3, \underset{k}{\beta} \ominus -(\alpha_4 \wedge \alpha_2), \underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2, \underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_1, \underset{k}{\beta} \ominus -(\alpha_3 \wedge \alpha_1), \underset{k}{\beta} \ominus \alpha_2 \wedge \alpha_1, \underset{k}{\beta} \ominus \alpha_4, \underset{k}{\beta} \ominus -\alpha_3, \underset{k}{\beta} \ominus \alpha_2, \underset{k}{\beta} \ominus -\alpha_1, \underset{k}{\beta} \ominus 1\}$

◆ 5. Take the exterior product of the elements of the two lists B and S

```
T = Thread[B ∧ S]
```

$\{1 \wedge (\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2 \wedge \alpha_1), \alpha_1 \wedge (\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2), \alpha_2 \wedge (\underset{k}{\beta} \ominus -(\alpha_4 \wedge \alpha_3 \wedge \alpha_1)), \alpha_3 \wedge (\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2 \wedge \alpha_1), \alpha_4 \wedge (\underset{k}{\beta} \ominus -(\alpha_3 \wedge \alpha_2 \wedge \alpha_1)), \alpha_1 \wedge \alpha_2 \wedge (\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3), \alpha_1 \wedge \alpha_3 \wedge (\underset{k}{\beta} \ominus -(\alpha_4 \wedge \alpha_2)), \alpha_1 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2), \alpha_2 \wedge \alpha_3 \wedge (\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_1), \alpha_2 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus -(\alpha_3 \wedge \alpha_1)), \alpha_3 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus \alpha_2 \wedge \alpha_1), \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge (\underset{k}{\beta} \ominus \alpha_4), \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus -\alpha_3), \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus \alpha_2), \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus -\alpha_1), \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge (\underset{k}{\beta} \ominus 1)\}$

◆ 6. Add the terms and simplify the result by factoring out the scalars

```
U = FactorScalars[Plus @@ T]
```

$$\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2 \wedge \alpha_1 + \alpha_1 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_2\right) - \alpha_2 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3 \wedge \alpha_1\right) + \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2 \wedge \alpha_1\right) - \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2 \wedge \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_3\right) - \alpha_1 \wedge \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_2\right) + \alpha_1 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_2\right) + \alpha_2 \wedge \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4 \wedge \alpha_1\right) - \alpha_2 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3 \wedge \alpha_1\right) + \alpha_3 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_2 \wedge \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \left(\underset{k}{\beta} \ominus \alpha_4\right) - \alpha_1 \wedge \alpha_2 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_3\right) + \alpha_1 \wedge \alpha_3 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_2\right) - \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \left(\underset{k}{\beta} \ominus \alpha_1\right) + \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \wedge \underset{k}{\beta}$$

◆ 7. Compare to the original expression

```
X == U
```

```
True
```

12.7 The Clifford Product of Intersecting Elements

General formulae for intersecting elements

Suppose two simple elements $\underset{p}{\gamma} \wedge \underset{m}{\alpha}$ and $\underset{p}{\gamma} \wedge \underset{k}{\beta}$ which have a simple element $\underset{p}{\gamma}$ in common. Then by definition their Clifford product may be written as a sum of generalized products.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) = \sum_{\lambda=0}^{\mathrm{Min}[m,k]+p} (-1)^{\lambda (p+m-\lambda) + \frac{1}{2}\lambda(\lambda-1)} \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\Delta} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)$$

But it has been shown in Section 10.12 that for $\lambda \geq p$ that:

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\Delta} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) = (-1)^{p (\lambda-p)} \left(\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{-p+\lambda}{\Delta} \underset{k}{\beta}\right) \ominus \underset{p}{\gamma}$$

Substituting in the formula above gives:

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) = \sum_{\lambda=0}^{\mathrm{Min}[m,k]+p} (-1)^{p+\lambda (m-\lambda) + \frac{1}{2}\lambda(\lambda-1)} \left(\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda-p}{\Delta} \underset{k}{\beta}\right) \ominus \underset{p}{\gamma}$$

Since the terms on the right hand side are zero for $\lambda < p$, we can define $\mu = \lambda - p$ and rewrite the right hand side as:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{\omega}\sum_{\mu=0}^{\mathrm{Min}[m,k]}(-1)^{\mu\,(p+m-\mu)+\frac{1}{2}\mu\,(\mu-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\underset{\mu}{\Delta}\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}$$

where $\omega = m\,p + \frac{1}{2}\,p\,(p-1)$. Hence we can finally cast the right hand side as a Clifford product by taking out the second $\underset{p}{\gamma}$ factor.

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p+\frac{1}{2}\,p\,(p-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.17$$

In a similar fashion we can show that the right hand side can be written in the alternative form:

$$\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{\frac{1}{2}\,p\,(p-1)}\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} \qquad 12.18$$

By reversing the order of $\underset{m}{\alpha}$ and $\underset{p}{\gamma}$ in the first factor, formula 12.17 can be written as:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{\frac{1}{2}\,p\,(p-1)}\left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.19$$

And by noting that the reverse, $\underset{p}{\gamma}^{\dagger}$ of $\underset{p}{\gamma}$ is $(-1)^{\frac{1}{2}\,p\,(p-1)}\,\underset{p}{\gamma}$, we can absorb the sign into the formulae by changing one of the $\underset{p}{\gamma}$ to $\underset{p}{\gamma}^{\dagger}$. For example:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == \left(\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.20$$

In sum: we can write:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p}\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma} \qquad 12.21$$

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} \qquad 12.22$$

$$\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}==\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma} \qquad 12.23$$

The relations derived up to this point are completely general and apply to Clifford products in an arbitrary space with an arbitrary metric. For example they do not require any of the elements to be orthogonal.

Special cases of intersecting elements

We can derive special cases of the formulae derived above by putting $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ equal to unity. First, put $\underset{k}{\beta}$ equal to unity. Remembering that the Clifford product with a scalar reduces to the ordinary product we obtain:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\underset{p}{\gamma}==(-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma}==(-1)^{m\,p}\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma}$$

Next, put $\underset{m}{\alpha}$ equal to unity, and then, to enable comparison with the previous formulae, replace $\underset{k}{\beta}$ by $\underset{m}{\alpha}$.

$$\underset{p}{\gamma}^{\dagger}\diamond\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)==\left(\underset{p}{\gamma}\diamond\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma}==\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma}$$

Finally we note that, since the far right hand sides of these two equations are equal, all the expressions are equal.

$$\begin{aligned}&\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\underset{p}{\gamma}==\underset{p}{\gamma}^{\dagger}\diamond\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\\ &==\left(\underset{p}{\gamma}\diamond\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma}==\left(\underset{p}{\gamma}\wedge\underset{m}{\alpha}\right)\ominus\underset{p}{\gamma}==(-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\underset{p}{\gamma}\right)\ominus\underset{p}{\gamma}\end{aligned} \qquad 12.24$$

By putting $\underset{m}{\alpha}$ to unity in these equations we can recover the relation 12.11 between the Clifford product of identical elements and their interior product.

12.8 The Clifford Product of Orthogonal Elements

The Clifford product of totally orthogonal elements

In Chapter 10 on generalized products we showed that if $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal (that is, $\alpha_i \ominus \beta_j == 0$ for each α_i belonging to $\underset{m}{\alpha}$ and β_j belonging to $\underset{k}{\beta}$, then $\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta} == 0$, *except* when λ = 0.

Thus we see immediately from the definition of the Clifford product that the Clifford product of two totally orthogonal elements is equal to their exterior product.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.25$$

The Clifford product of partially orthogonal elements

Suppose now we introduce an arbitrary element $\underset{p}{\gamma}$ into $\underset{m}{\alpha} \diamond \underset{k}{\beta}$, and expand the expression in terms of generalized products.

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \sum_{\lambda=0}^{\mathrm{Min}[m,k+p]} (-1)^{\lambda (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \underset{m}{\alpha} \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)$$

But from formula 10.35 we have that:

$$\underset{m}{\alpha} \underset{\lambda}{\triangle} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{p}{\gamma}\right) \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad \lambda \leq \mathrm{Min}[m,\ p]$$

Hence:

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \qquad 12.26$$

Similarly, expressing $\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta}$ in terms of generalized products and substituting from equation 10.37 gives:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[p,k]} (-1)^{\lambda (m+p-\lambda)+\frac{1}{2}\lambda(\lambda-1)+m\lambda} \underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

But $(-1)^{\lambda(m+p-\lambda)+\frac{1}{2}\lambda(\lambda-1)+m\lambda} == (-1)^{\lambda(p-\lambda)+\frac{1}{2}\lambda(\lambda-1)}$, hence we can write:

$$\boxed{\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta} == \underset{m}{\alpha}\wedge\left(\underset{p}{\gamma}\diamond\underset{k}{\beta}\right) \qquad \alpha_i\ominus\beta_j == 0} \quad 12.27$$

Testing the formulae

To test any of these formulae we may always tabulate specific cases. Here we convert the difference between the sides of equation 12.27 to scalar products and then put to zero any products whose factors are orthogonal. To do this we use the *GrassmannAlgebra* function `OrthogonalSimplificationRules`. We verify the formula for the first 50 cases.

```
? OrthogonalSimplificationRules
```

```
OrthogonalSimplificationRules[{{X1,Y1},{X2,Y2},…}] develops a list of
  rules which put to zero all the scalar products of a 1-element from
  Xi and a 1-element from Yi. Xi and Yi may be either variables,
  basis elements, exterior products of these, or graded variables.
```

$$\text{Flatten}\left[\text{Table}\left[\text{ToScalarProducts}\left[\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}-\underset{m}{\alpha}\wedge\left(\underset{p}{\gamma}\diamond\underset{k}{\beta}\right)\right]\text{ /. } \text{OrthogonalSimplificationRules}\left[\left\{\left\{\underset{m}{\alpha},\underset{k}{\beta}\right\}\right\}\right],\ \{m,0,3\},\{k,0,m\},\{p,0,m+2\}\right]\right]$$

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

12.9 The Clifford Product of Intersecting Orthogonal Elements

Orthogonal union

We have shown in Section 12.7 above that the Clifford product of arbitrary intersecting elements may be expressed by:

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p}\left(\left(\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}\right)\diamond\underset{k}{\beta}\right)\ominus\underset{p}{\gamma}\right)$$

$$\left(\underset{m}{\alpha}\wedge\underset{p}{\gamma}^{\dagger}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right) == (-1)^{m\,p}\left(\underset{m}{\alpha}\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\right)\ominus\underset{p}{\gamma}$$

But we have also shown in Section 12.8 that if $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal, then:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right)$$

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta}$$

Hence the right hand sides of the first equations may be written, in the orthogonal case, as:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right)\right) \ominus \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == 0 \tag{12.28}$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta}\right) \ominus \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == 0 \tag{12.29}$$

Orthogonal intersection

Consider the case of three simple elements $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$ where $\underset{p}{\gamma}$ is *totally orthogonal* to both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ (and hence to $\underset{m}{\alpha} \wedge \underset{k}{\beta}$). A simple element $\underset{p}{\gamma}$ is totally orthogonal to an element $\underset{m}{\alpha}$ if and only if $\underset{m}{\alpha} \ominus \gamma_i == 0$ for all γ_i belonging to $\underset{p}{\gamma}$. Here, we do *not* assume that $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are orthogonal.

As before we consider the Clifford product of intersecting elements, but because of the orthogonality relationships, we can replace the generalized products on the right hand side by the right hand side of equation 10.39.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \sum_{\lambda=0}^{\mathrm{Min}[m,k]+p} (-1)^{\lambda\,(p+m-\lambda)+\frac{1}{2}\lambda\,(\lambda-1)} \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \underset{\lambda}{\Delta} \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)$$

$$== \sum_{\lambda=p}^{\mathrm{Min}[m,k]+p} (-1)^{\lambda\,(p+m-\lambda)+\frac{1}{2}\lambda\,(\lambda-1)} \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \underset{\lambda-p}{\Delta} \underset{k}{\beta}\right)$$

Let $\mu = \lambda - p$, then this equals:

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \sum_{\mu=0}^{\mathrm{Min}[m,k]} (-1)^{(\mu+p)\,(m-\mu)+\frac{1}{2}(\mu+p)\,(\mu+p-1)} \left(\underset{m}{\alpha} \underset{\mu}{\Delta} \underset{k}{\beta}\right)$$

$$== (-1)^{m p+\frac{1}{2} p (p-1)} \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \sum_{\mu=0}^{\mathrm{Min}[m,k]} (-1)^{\mu (m-\mu)+\frac{1}{2} \mu (\mu-1)} \left(\underset{m}{\alpha} \underset{\mu}{\triangle} \underset{k}{\beta}\right)$$

Hence we can cast the right hand side as a Clifford product.

$$\left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m p+\frac{1}{2} p (p-1)} \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)$$

Or finally, by absorbing the signs into the left-hand side as:

$$\boxed{\begin{gathered}\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \\ \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \qquad \underset{m}{\alpha} \ominus \gamma_i == \underset{k}{\beta} \ominus \gamma_i == 0\end{gathered}} \quad 12.30$$

Note carefully, that for this formula to be true, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are not necessarily orthogonal. However, if they are orthogonal we can express the Clifford product on the right hand side as an exterior product.

$$\boxed{\begin{gathered}\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \\ \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \qquad \alpha_i \ominus \beta_j == \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0\end{gathered}} \quad 12.31$$

12.10 Summary of Special Cases of Clifford Products

Arbitrary elements

The main results of the preceding sections concerning Clifford Products of intersecting and orthogonal elements are summarized below.

◆ $\underset{p}{\gamma}$ is an arbitrary repeated factor

$$\boxed{\underset{p}{\gamma}^{\dagger} \diamond \underset{p}{\gamma} == \underset{p}{\gamma} \diamond \underset{p}{\gamma}^{\dagger} == \underset{p}{\gamma} \ominus \underset{p}{\gamma}} \quad 12.32$$

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} == \underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma} == \underset{p}{\gamma} \ominus \underset{p}{\gamma}^{\dagger} \qquad 12.33$$

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} \diamond \underset{p}{\gamma} == \left(\underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma}\right) \underset{p}{\gamma} == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}^{\dagger}\right) \underset{p}{\gamma} \qquad 12.34$$

$$\underset{p}{\gamma} \diamond \underset{p}{\gamma} \diamond \underset{p}{\gamma} \diamond \underset{p}{\gamma} == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right)^{2} \qquad 12.35$$

- **$\underset{p}{\gamma}$ is an arbitrary repeated factor, $\underset{m}{\alpha}$ is arbitrary**

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \underset{p}{\gamma} == \underset{p}{\gamma}^{\dagger} \diamond \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right)$$
$$== \left(\underset{p}{\gamma} \diamond \underset{m}{\alpha}\right) \ominus \underset{p}{\gamma} == \left(\underset{p}{\gamma} \wedge \underset{m}{\alpha}\right) \ominus \underset{p}{\gamma} == (-1)^{m\,p} \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \ominus \underset{p}{\gamma} \qquad 12.36$$

- **$\underset{p}{\gamma}$ is an arbitrary repeated factor, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are arbitrary**

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta}\right) \ominus \underset{p}{\gamma} \qquad 12.37$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right)\right) \ominus \underset{p}{\gamma} \qquad 12.38$$

Arbitrary and orthogonal elements

- **$\underset{p}{\gamma}$ is arbitrary, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal**

$$\underset{m}{\alpha} \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta} \qquad \alpha_{i} \ominus \beta_{j} == 0 \qquad 12.39$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right) \qquad \alpha_i \ominus \beta_j == 0 \tag{12.40}$$

◆ $\underset{p}{\gamma}$ is an arbitrary repeated factor, $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\underset{m}{\alpha} \wedge \left(\underset{p}{\gamma} \diamond \underset{k}{\beta}\right)\right) \ominus \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == 0 \tag{12.41}$$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == (-1)^{m\,p} \left(\left(\underset{m}{\alpha} \diamond \underset{p}{\gamma}\right) \wedge \underset{k}{\beta}\right) \ominus \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j == 0 \tag{12.42}$$

◆ $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are arbitrary, $\underset{p}{\gamma}$ is a repeated factor totally orthogonal to both $\underset{m}{\alpha}$ and $\underset{k}{\beta}$

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}^{\dagger}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \qquad \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0 \tag{12.43}$$

Orthogonal elements

◆ $\underset{m}{\alpha}$ and $\underset{k}{\beta}$ are totally orthogonal

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \underset{m}{\alpha} \wedge \underset{k}{\beta} \qquad \alpha_i \ominus \beta_j == 0 \tag{12.44}$$

◆ $\underset{m}{\alpha}$, $\underset{k}{\beta}$, and $\underset{p}{\gamma}$ are totally orthogonal

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) = \left(\underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \qquad \alpha_i \ominus \beta_j = \alpha_i \ominus \gamma_s = \beta_j \ominus \gamma_s = 0 \tag{12.45}$$

Calculating with Clifford products

The previous section summarizes some alternative formulations for a Clifford product when its factors contain a common element, and when they have an orthogonality relationship between them. In this section we discuss how these relations can make it easy to calculate with these type of Clifford products.

◆ The Clifford product of totally orthogonal elements reduces to the exterior product

If we know that *all* the factors of a Clifford product are totally orthogonal, then we can interchange the Clifford product and the exterior product at will. Hence, *for totally orthogonal elements, the Clifford and exterior products are associative*, and we do not need to include parentheses.

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} \diamond \underset{p}{\gamma} = \underset{m}{\alpha} \diamond \underset{k}{\beta} \wedge \underset{p}{\gamma} = \underset{m}{\alpha} \wedge \underset{k}{\beta} \diamond \underset{p}{\gamma} = \underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{p}{\gamma} \qquad \alpha_i \ominus \beta_j = \alpha_i \ominus \gamma_s = \beta_j \ominus \gamma_s = 0 \tag{12.46}$$

Note carefully however, that this associativity does not extend to the *factors* of the *m*-, *k*-, or *p*-elements unless the factors of the *m*-, *k*-, or *p*-element concerned are mutually orthogonal. In which case we could for example then write:

$$\alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_m = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \qquad \alpha_i \ominus \alpha_j = 0 \quad i \neq j \tag{12.47}$$

■ Example

For example if $x \wedge y$ and z are totally orthogonal, that is $x \ominus z = 0$ and $y \ominus z = 0$, then we can write

$$(x \wedge y) \diamond z = x \wedge y \wedge z = x \wedge (y \diamond z) = -y \wedge (x \diamond z)$$

But since $\mathbf{x}$ *is not necessarily orthogonal to* $\mathbf{y}$, these are *not* the same as

$$x \diamond y \diamond z == (x \diamond y) \wedge z == x \diamond (y \wedge z)$$

We can see the difference by reducing the Clifford products to scalar products.

```
ToScalarProducts[{(x∧y)⋄z, x∧y∧z, x∧(y⋄z), -y∧(x⋄z)}] /.
 OrthogonalSimplificationRules[{{x∧y, z}}]
```

```
{x∧y∧z, x∧y∧z, x∧y∧z, x∧y∧z}
```

```
ToScalarProducts[{x⋄y⋄z, (x⋄y)∧z, x⋄(y∧z)}] /.
 OrthogonalSimplificationRules[{{x∧y, z}}]
```

```
{z (x⊖y) + x∧y∧z, z (x⊖y) + x∧y∧z, z (x⊖y) + x∧y∧z}
```

12.11 Associativity of the Clifford Product

Associativity of orthogonal elements

In what follows we use formula 12.45 to show that the Clifford product as defined at the beginning of this chapter is associative for totally orthogonal elements. This result will then enable us to show that by re-expressing the elements in terms in an orthogonal basis, the Clifford product is, *in general*, associative. Note that the approach we have adopted in this book has not required us to adopt *ab initio* the associativity of the Clifford product as an axiom, but rather show that associativity is a natural consequence of its definition.

Formula 12.45 tells us that the Clifford product of (possibly) intersecting and totally orthogonal elements is given by:

$$\left(\underset{m}{\alpha} \wedge \underset{p}{\gamma}\right) \diamond \left(\underset{p}{\gamma} \wedge \underset{k}{\beta}\right) == \left(\underset{p}{\gamma}^{\dagger} \ominus \underset{p}{\gamma}\right) \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right)$$
$$\alpha_i \ominus \beta_j == \alpha_i \ominus \gamma_s == \beta_j \ominus \gamma_s == 0$$

Note that it is not necessary that the factors of $\underset{m}{\alpha}$ be orthogonal to each other. This is true also for the factors of $\underset{k}{\beta}$ or of $\underset{p}{\gamma}$.

We begin by writing a Clifford product of three elements, and associating the first pair. The elements contain factors which are specific to the element ($\underset{m}{\alpha}$, $\underset{k}{\beta}$, $\underset{r}{\omega}$), pairwise common ($\underset{p}{\gamma}$, $\underset{q}{\delta}$, $\underset{s}{\epsilon}$), and common to all three ($\underset{z}{\theta}$). This product will therefore represent the most general case since other cases may be obtained by letting one or more factors reduce to unity.

$$\left(\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{z}{\theta}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

$$=\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{p}{\gamma^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

$$=\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{p}{\gamma^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)(-1)^{sk}\left(\left(\underset{s}{\epsilon^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\right)\wedge\left(\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

$$=(-1)^{sk}\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{p}{\gamma^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)\left(\left(\underset{s}{\epsilon^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

On the other hand we can associate the second pair to obtain the same result.

$$\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\left(\underset{p}{\gamma}\wedge\underset{z}{\theta}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)\right)$$

$$=\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond(-1)^{zk+z\,(s+r)}\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\right)\wedge\left(\underset{s}{\epsilon}\wedge\underset{r}{\omega}\right)$$

$$=(-1)^{zk+z\,(s+r)}\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{k}{\beta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\right)$$

$$=(-1)^{zk+z\,(s+r)+sz+ks}\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{z}{\theta}\wedge\underset{s}{\epsilon}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{s}{\epsilon}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\right)$$

$$=(-1)^{zk+zr+ks}\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{q}{\delta}\right)\right)\left(\left(\underset{s}{\epsilon^{\dagger}}\wedge\underset{p}{\gamma^{\dagger}}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{p}{\gamma}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{z}{\theta}\right)\wedge\left(\underset{k}{\beta}\wedge\underset{r}{\omega}\right)$$

$$=(-1)^{sk}\left(\left(\underset{z}{\theta^{\dagger}}\wedge\underset{p}{\gamma^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\right)\left(\left(\underset{s}{\epsilon^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$

In sum: We have shown that the Clifford product of possibly intersecting but otherwise totally orthogonal elements is associative. Any factors which make up a given individual element are not specifically involved, and hence it is of no consequence to the associativity of the element with another whether or not these factors are mutually orthogonal.

A mnemonic formula for products of orthogonal elements

Because $\underset{z}{\theta}$ and $\underset{p}{\gamma}$ are orthogonal to each other, and $\underset{s}{\epsilon}$ and $\underset{q}{\delta}$ are orthogonal to each other we can rewrite the inner products in the alternative forms:

$$\left(\underset{z}{\theta^{\dagger}}\wedge\underset{p}{\gamma^{\dagger}}\right)\ominus\left(\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)=\left(\underset{z}{\theta^{\dagger}}\ominus\underset{z}{\theta}\right)\left(\underset{p}{\gamma^{\dagger}}\ominus\underset{p}{\gamma}\right)$$

$$\left(\underset{s}{\epsilon^{\dagger}}\wedge\underset{q}{\delta^{\dagger}}\right)\ominus\left(\underset{s}{\epsilon}\wedge\underset{q}{\delta}\right)=\left(\underset{s}{\epsilon^{\dagger}}\ominus\underset{s}{\epsilon}\right)\left(\underset{q}{\delta^{\dagger}}\ominus\underset{q}{\delta}\right)$$

The results of the previous section may then summarized as:

$$\left(\underset{m}{\alpha}\wedge\underset{s}{\epsilon}\wedge\underset{z}{\theta}\wedge\underset{p}{\gamma}\right)\diamond\left(\underset{p}{\gamma}\wedge\underset{z}{\theta}\wedge\underset{k}{\beta}\wedge\underset{q}{\delta}\right)\diamond\left(\underset{q}{\delta}\wedge\underset{s}{\epsilon}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right)$$
$$=(-1)^{sk}\left(\underset{p}{\gamma}^{\dagger}\ominus\underset{p}{\gamma}\right)\left(\underset{z}{\theta}^{\dagger}\ominus\underset{z}{\theta}\right)$$
$$\left(\underset{s}{\epsilon}^{\dagger}\ominus\underset{s}{\epsilon}\right)\left(\underset{q}{\delta}^{\dagger}\ominus\underset{q}{\delta}\right)\left(\underset{m}{\alpha}\wedge\underset{k}{\beta}\wedge\underset{r}{\omega}\wedge\underset{z}{\theta}\right) \qquad \mathbf{12.48}$$

A mnemonic for making this transformation is then

1. Rearrange the factors in a Clifford product to get common factors adjacent to the Clifford product symbol, taking care to include any change of sign due to the quasi-commutativity of the exterior product.

2. Replace the common factors by their inner product, *but with one copy being reversed*.

3. If there are no common factors in a Clifford product, the Clifford product can be replaced by the exterior product.

Remember that for these relations to hold all the elements must be totally orthogonal to each other.

Note that if, in addition, the 1-element factors of any of these elements, $\underset{p}{\gamma}$ say, are orthogonal to each other, then:

$$\underset{p}{\gamma}\ominus\underset{p}{\gamma}=(\gamma_1\ominus\gamma_1)\,(\gamma_2\ominus\gamma_2)\,\cdots\,(\gamma_p\ominus\gamma_p)$$

Associativity of non-orthogonal elements

Consider the Clifford product $\left(\underset{m}{\alpha}\diamond\underset{k}{\beta}\right)\diamond\underset{p}{\gamma}$ where there are no restrictions on the factors $\underset{m}{\alpha}$, $\underset{k}{\beta}$ and $\underset{p}{\gamma}$. It has been shown in Section 6.3 that an arbitrary simple *m*-element may be expressed in terms of *m* orthogonal 1-element factors (the Gram-Schmidt orthogonalization process). Suppose that *ν* such orthogonal 1-elements $\mathbf{e}_1,\ \mathbf{e}_2,\ \cdots,\ \mathbf{e}_\nu$ have been found in terms of which $\underset{m}{\alpha}$, $\underset{k}{\beta}$, and $\underset{p}{\gamma}$ can be expressed. Writing the *m*-elements, *k*-elements and *p*-elements formed from the $\mathbf{e}_i$ as $\underset{m}{\mathbf{e}_j}$, $\underset{k}{\mathbf{e}_r}$, and $\underset{p}{\mathbf{e}_s}$ respectively, we can write:

$$\underset{m}{\alpha}=\Sigma\,\mathbf{a}^j\,\underset{m}{\mathbf{e}_j}\qquad \underset{k}{\beta}=\Sigma\,\mathbf{b}^r\,\underset{k}{\mathbf{e}_r}\qquad \underset{p}{\gamma}=\Sigma\,\mathbf{c}^s\,\underset{p}{\mathbf{e}_s}$$

Thus we can write the Clifford product as:

$$\left(\underset{m}{\alpha}\diamond\underset{k}{\beta}\right)\diamond\underset{p}{\gamma}=\left(\Sigma\,\mathbf{a}^j\,\underset{m}{\mathbf{e}_j}\diamond\Sigma\,\mathbf{b}^r\,\underset{k}{\mathbf{e}_r}\right)\diamond\Sigma\,\mathbf{c}^s\,\underset{p}{\mathbf{e}_s}=\Sigma\,\Sigma\,\Sigma\,\mathbf{a}^j\,\mathbf{b}^r\,\mathbf{c}^s\left(\underset{m}{\mathbf{e}_j}\diamond\underset{k}{\mathbf{e}_r}\right)\diamond\underset{p}{\mathbf{e}_s}$$

But we have already shown in the previous section that the Clifford product of orthogonal elements is associative. That is:

$$\left(\underset{m}{e_j} \diamond \underset{k}{e_r}\right) \diamond \underset{p}{e_s} == \underset{m}{e_j} \diamond \left(\underset{k}{e_r} \diamond \underset{p}{e_s}\right) == \underset{m}{e_j} \diamond \underset{k}{e_r} \diamond \underset{p}{e_s}$$

Hence we can write:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \diamond \underset{p}{\gamma} == \Sigma\,\Sigma\,\Sigma\, a^j\, b^r\, c^s\, \underset{m}{e_j} \diamond \left(\underset{k}{e_r} \diamond \underset{p}{e_s}\right) == \underset{m}{\alpha} \diamond \left(\underset{k}{\beta} \diamond \underset{p}{\gamma}\right)$$

We thus see that *the Clifford product of general elements is associative*.

$$\boxed{\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right) \diamond \underset{p}{\gamma} == \underset{m}{\alpha} \diamond \left(\underset{k}{\beta} \diamond \underset{p}{\gamma}\right) == \underset{m}{\alpha} \diamond \underset{k}{\beta} \diamond \underset{p}{\gamma}} \qquad \mathbf{12.49}$$

The associativity of the Clifford product is usually taken as an axiom. However, in this book we have chosen to show that associativity is a consequence of our definition of the Clifford product in terms of exterior and interior products. In this way we can ensure that the Grassmann and Clifford algebras are consistent.

Testing the general associativity of the Clifford product

We can easily create a function in *Mathematica* to test the associativity of Clifford products of elements of different grades in spaces of different dimensions. Below we define a function called `CliffordAssociativityTest` which takes the dimension of the space and the grades of three elements as arguments. The steps are as follows:

- Declare a space of the given dimension. It does not matter what the metric is since we do not use it.
- Create general elements of the given grades in a space of the given dimension.
- To make the code more readable, define a function which converts a product to scalar product form.
- Compute the products associated in the two different ways, and subtract them.
- The associativity test is successful if a result of 0 is returned.

```
CliffordAssociativityTest[n_][m_, k_, p_] :=
 Module[{X, Y, Z, S}, 𝕍n; X = CreateElement[ξ_m];
  Y = CreateElement[ψ_k]; Z = CreateElement[ζ_p];
  S[x_] := ToScalarProducts[x]; S[S[X ⋄ Y] ⋄ Z] - S[X ⋄ S[Y ⋄ Z]]]
```

We can either test individual cases, for example 2-elements in a 4-space:

```
CliffordAssociativityTest[4][2, 2, 2]
```

```
0
```

Or, we can tabulate a number of results together. For example, elements of all grades in all spaces of dimension 0, 1, 2, and 3.

```
Table[CliffordAssociativityTest[n][m, k, p],
 {n, 0, 3}, {m, 0, n}, {k, 0, n}, {p, 0, m}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

12.12 Decomposing a Clifford Product

Even and odd generalized products

A Clifford product $\underset{m}{\alpha} \diamond \underset{k}{\beta}$ can be decomposed into two sums, those whose terms are generalized products of even order $\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e$, and those whose terms are generalized products of odd order $\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o$. That is:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e + \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.50$$

where, from the definition of the Clifford product:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \sum_{\lambda=0}^{\mathrm{Min}[m,k]} (-1)^{\lambda (m-\lambda)+\frac{1}{2}\lambda(\lambda-1)} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right)$$

we can take just the even generalized products to get:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e == \sum_{\lambda=0,2,4,\cdots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.51$$

(Here, the limit of the sum is understood to mean the greatest even integer less than or equal to **Min[m,k]**.)

The odd generalized products are:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o == (-1)^m \sum_{\lambda=1,3,5,\cdots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda+1}{2}} \left(\underset{m}{\alpha} \underset{\lambda}{\triangle} \underset{k}{\beta}\right) \qquad 12.52$$

(In this case, the limit of the sum is understood to mean the greatest odd integer less than or equal to **Min[m,k]**.)

Note carefully that the evenness and oddness to which we refer is to the *order of the generalized product* not to the grade of the Clifford product.

The Clifford product in reverse order

The expression for the even and odd components of the Clifford product taken in the reverse order is simply obtained from the formulae above by using the quasi-commutativity of the generalized product.

$$\left(\underset{k}{\beta}\diamond\underset{m}{\alpha}\right)_e == \sum_{\lambda=0,2,4,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}} \left(\underset{k}{\beta}\underset{\lambda}{\triangle}\underset{m}{\alpha}\right) == \sum_{\lambda=0,2,4,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}+(m-\lambda)(k-\lambda)} \left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right)$$

But, since λ is even, this simplifies to:

$$\left(\underset{k}{\beta}\diamond\underset{m}{\alpha}\right)_e == (-1)^{m\,k} \sum_{\lambda=0,2,4,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda}{2}} \left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right) \qquad 12.53$$

Hence, the even terms of both products are equal, except when both factors of the product are of odd grade.

$$\left(\underset{k}{\beta}\diamond\underset{m}{\alpha}\right)_e == (-1)^{m\,k} \left(\underset{m}{\alpha}\diamond\underset{k}{\beta}\right)_e \qquad 12.54$$

In a like manner we can show that for λ odd:

$$\left(\underset{k}{\beta}\diamond\underset{m}{\alpha}\right)_o == (-1)^{k} \sum_{\lambda=1,3,5,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda+1}{2}+(m-\lambda)(k-\lambda)} \left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right)$$

$$\left(\underset{k}{\beta}\diamond\underset{m}{\alpha}\right)_o == -(-1)^{m\,k}(-1)^{m} \sum_{\lambda=1,3,5,\ldots}^{\mathrm{Min}[m,k]} (-1)^{\frac{\lambda+1}{2}} \left(\underset{m}{\alpha}\underset{\lambda}{\triangle}\underset{k}{\beta}\right) \qquad 12.55$$

$$\left(\underset{k}{\beta}\diamond\underset{m}{\alpha}\right)_o == -(-1)^{m\,k} \left(\underset{m}{\alpha}\diamond\underset{k}{\beta}\right)_o \qquad 12.56$$

The decomposition of a Clifford product

Finally, therefore, we can write:

$$\underset{m}{\alpha} \diamond \underset{k}{\beta} == \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e + \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.57$$

$$(-1)^{m\,k}\, \underset{k}{\beta} \diamond \underset{m}{\alpha} == \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e - \left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o \qquad 12.58$$

which we can add and subtract to give:

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_e == \frac{1}{2}\left(\underset{m}{\alpha} \diamond \underset{k}{\beta} + (-1)^{m\,k}\, \underset{k}{\beta} \diamond \underset{m}{\alpha}\right) \qquad 12.59$$

$$\left(\underset{m}{\alpha} \diamond \underset{k}{\beta}\right)_o == \frac{1}{2}\left(\underset{m}{\alpha} \diamond \underset{k}{\beta} - (-1)^{m\,k}\, \underset{k}{\beta} \diamond \underset{m}{\alpha}\right) \qquad 12.60$$

■ Example: An *m*-element and a 1-element

Putting *k* equal to 1 gives:

$$\underset{m}{\alpha} \wedge \beta == \frac{1}{2}\left(\underset{m}{\alpha} \diamond \beta + (-1)^{m}\, \beta \diamond \underset{m}{\alpha}\right) \qquad 12.61$$

$$\frac{1}{2}\left(\underset{m}{\alpha} \diamond \beta - (-1)^{m}\, \beta \diamond \underset{m}{\alpha}\right) == -(-1)^{m}\, \underset{m}{\alpha} \ominus \beta \qquad 12.62$$

■ Example: An *m*-element and a 2-element

Putting *k* equal to 2 gives:

$$\frac{1}{2}\left(\underset{m}{\alpha} \diamond \underset{2}{\beta} + \underset{2}{\beta} \diamond \underset{m}{\alpha}\right) == \underset{m}{\alpha} \wedge \underset{2}{\beta} - \underset{m}{\alpha} \ominus \underset{2}{\beta} \qquad 12.63$$

$$\frac{1}{2}\left(\underset{m}{\alpha}\diamond\underset{2}{\beta}-\underset{2}{\beta}\diamond\underset{m}{\alpha}\right)==-(-1)^{m}\underset{m}{\alpha}\underset{1}{\triangle}\underset{2}{\beta} \qquad 12.64$$

12.13 Clifford Algebra

Generating Clifford algebras

Up to this point we have concentrated on the definition and properties of the Clifford product. We are now able to turn our attention to the algebras that such a product is capable of generating in different spaces.

Broadly speaking, an algebra can be constructed from a set of elements of a linear space which has a product operation, *provided* all products of the elements are again elements of the set.

In what follows we shall discuss Clifford algebras. The generating elements will be selected subsets of the basis elements of the full Grassmann algebra on the space. The product operation will be the Clifford product which we have defined in the first part of this chapter in terms of the generalized Grassmann product. Thus, Clifford algebras may viewed as living very much within the Grassmann algebra, relying on it for both its elements and its operations.

In this development therefore, a particular Clifford algebra is ultimately defined by the values of the scalar products of basis vectors of the underlying Grassmann algebra, and thus by the metric on the space.

In many cases we will be defining the specific Clifford algebras in terms of orthogonal (not necessarily orthonormal) basis elements.

Clifford algebras include the real numbers, complex numbers, quaternions, biquaternions, and the Pauli and Dirac algebras.

Real algebra

All the products that we have developed up to this stage, the exterior, interior, generalized Grassmann and Clifford products possess the valuable and consistent property that when applied to scalars yield results equivalent to the usual (underlying field) product. The field has been identified with a space of zero dimensions, and the scalars identified with its elements.

Thus, if **a** and **b** are scalars, then:

$$\mathbf{a} \diamond \mathbf{b} == \mathbf{a} \wedge \mathbf{b} == \mathbf{a} \ominus \mathbf{b} == \mathbf{a} \underset{0}{\triangle} \mathbf{b} == \mathbf{a}\,\mathbf{b} \qquad 12.65$$

Thus the real algebra is isomorphic with the Clifford algebra of a space of zero dimensions.

Clifford algebras of a 1-space

We begin our discussion of Clifford algebras with the simplest case: the Clifford algebras of 1-space. Suppose the basis for the 1-space is e_1, then the basis for the associated Grassmann algebra is $\{1, e_1\}$.

```
𝕍1; Basis∧[]
```

```
{1, e1}
```

There are only four possible Clifford products of these basis elements. We can construct a table of these products by using the *GrassmannAlgebra* function **CliffordProductTable**.

```
CliffordProductTable[]
```

```
{{1 ⋄ 1, 1 ⋄ e1}, {e1 ⋄ 1, e1 ⋄ e1}}
```

Usually however, to make the products easier to read and use, we will display them in the form of a palette using the *GrassmannAlgebra* function **PaletteForm**. (We can click on the palette to enter any of its expressions into the notebook).

```
C1 = CliffordProductTable[]; PaletteForm[C1]
```

$1 \diamond 1$	$1 \diamond e_1$
$e_1 \diamond 1$	$e_1 \diamond e_1$

In the general case any Clifford product may be expressed in terms of exterior and interior products. We can see this by applying **ToInteriorProducts** to the table (although only interior (here scalar) products result from this simple case),

```
C2 = ToInteriorProducts[C1]; PaletteForm[C2]
```

1	e_1
e_1	$e_1 \ominus e_1$

Different Clifford algebras may be generated depending on the metric chosen for the space. In this example we can see that the types of Clifford algebra which we can generate in a 1-space are dependent only on the choice of a single scalar value for the scalar product $e_1 \ominus e_1$. The Clifford product of two general elements of the algebra is

```
(a + b e1) ⋄ (c + d e1) // ToScalarProducts
```

$a\,c + b\,d\,(e_1 \ominus e_1) + b\,c\,e_1 + a\,d\,e_1$

It is clear to see that if we choose $e_1 \ominus e_1 == -1$, we have an algebra isomorphic to the complex algebra. The basis 1-element e_1 then plays the role of the imaginary unit i. We can generate this particular algebra immediately by declaring the metric, and then generating the product table.

```
DeclareBasis[{i}]; DeclareMetric[{{-1}}];
CliffordProductTable[] // ToScalarProducts // ToMetricForm //
 PaletteForm
```

1	i
i	-1

However, our main purpose in discussing this very simple example in so much detail is to emphasize that even in this case, there are an *infinite* number of Clifford algebras on a 1-space depending on the choice of the scalar value for $e_1 \ominus e_1$. The complex algebra, although it has surely proven itself to be the most useful, is just one among many.

Finally, we note that all Clifford algebras possess the real algebra as their simplest even subalgebra.

12.14 Clifford Algebras of a 2-Space

The Clifford product table in 2-space

In this section we explore the Clifford algebras of 2-space. As might be expected, the Clifford algebras of 2-space are significantly richer than those of 1-space. First we declare a (not necessarily orthogonal) basis for the 2-space, and generate the associated Clifford product table.

```
𝕍2; C1 = CliffordProductTable[]; PaletteForm[C1]
```

$1 \diamond 1$	$1 \diamond e_1$	$1 \diamond e_2$	$1 \diamond (e_1 \wedge e_2)$
$e_1 \diamond 1$	$e_1 \diamond e_1$	$e_1 \diamond e_2$	$e_1 \diamond (e_1 \wedge e_2)$
$e_2 \diamond 1$	$e_2 \diamond e_1$	$e_2 \diamond e_2$	$e_2 \diamond (e_1 \wedge e_2)$
$(e_1 \wedge e_2) \diamond 1$	$(e_1 \wedge e_2) \diamond e_1$	$(e_1 \wedge e_2) \diamond e_2$	$(e_1 \wedge e_2) \diamond (e_1 \wedge e_2)$

To see the way in which these Clifford products reduce to generalized Grassmann products we can apply the *GrassmannAlgebra* function `ToGeneralizedProducts`.

C_2 = ToGeneralizedProducts[C_1]; PaletteForm[C_2]

$1 \underset{0}{\triangle} 1$	$1 \underset{0}{\triangle} e_1$	$1 \underset{0}{\triangle} e_2$	
$e_1 \underset{0}{\triangle} 1$	$e_1 \underset{0}{\triangle} e_1 + e_1 \underset{1}{\triangle} e_1$	$e_1 \underset{0}{\triangle} e_2 + e_1 \underset{1}{\triangle} e_2$	
$e_2 \underset{0}{\triangle} 1$	$e_2 \underset{0}{\triangle} e_1 + e_2 \underset{1}{\triangle} e_1$	$e_2 \underset{0}{\triangle} e_2 + e_2 \underset{1}{\triangle} e_2$	
$(e_1 \wedge e_2) \underset{0}{\triangle} 1$	$(e_1 \wedge e_2) \underset{0}{\triangle} e_1 - (e_1 \wedge e_2) \underset{1}{\triangle} e_1$	$(e_1 \wedge e_2) \underset{0}{\triangle} e_2 - (e_1 \wedge e_2) \underset{1}{\triangle} e_2$	$(e_1$

The next level of expression is obtained by applying `ToInteriorProducts` to reduce the generalized products to exterior and interior products.

C_3 = ToInteriorProducts[C_2]; PaletteForm[C_3]

1	e_1	e_2	$e_1 \wedge e_2$
e_1	$e_1 \ominus e_1$	$e_1 \ominus e_2 + e_1 \wedge e_2$	$e_1 \wedge e_2 \ominus e_1$
e_2	$e_1 \ominus e_2 - e_1 \wedge e_2$	$e_2 \ominus e_2$	$e_1 \wedge e_2 \ominus e_2$
$e_1 \wedge e_2$	$-(e_1 \wedge e_2 \ominus e_1)$	$-(e_1 \wedge e_2 \ominus e_2)$	$-(e_1 \wedge e_2 \ominus e_1 \wedge e_2) - (e_1 \wedge e_2 \ominus e_1) \wedge e_2$

Finally, we can expand the interior products to scalar products.

C_4 = ToScalarProducts[C_3]; PaletteForm[C_4]

1	e_1	e_2	$e_1 \wedge e$
e_1	$e_1 \ominus e_1$	$e_1 \ominus e_2 + e_1 \wedge e_2$	$-(e_1 \ominus e_2)\, e_1 +$
e_2	$e_1 \ominus e_2 - e_1 \wedge e_2$	$e_2 \ominus e_2$	$-(e_2 \ominus e_2)\, e_1 +$
$e_1 \wedge e_2$	$(e_1 \ominus e_2)\, e_1 - (e_1 \ominus e_1)\, e_2$	$(e_2 \ominus e_2)\, e_1 - (e_1 \ominus e_2)\, e_2$	$(e_1 \ominus e_2)^2 - (e_1 \ominus$

It should be noted that the *GrassmannAlgebra* operation `ToScalarProducts` also applies the function `ToStandardOrdering`. Hence scalar products are reordered to present the basis element with lower index first. For example the scalar product $\mathbf{e_2 \ominus e_1}$ does not appear in the table above.

Product tables in a 2-space with an orthogonal basis

This is the Clifford product table for a general basis. If however,we choose an orthogonal basis in which $\mathbf{e_1 \ominus e_2}$ is zero, the table simplifies to:

C_5 = C_4 /. $e_1 \ominus e_2 \to 0$; PaletteForm[C_5]

1	e_1	e_2	$e_1 \wedge e_2$
e_1	$e_1 \ominus e_1$	$e_1 \wedge e_2$	$(e_1 \ominus e_1)\, e_2$
e_2	$-(e_1 \wedge e_2)$	$e_2 \ominus e_2$	$-(e_2 \ominus e_2)\, e_1$
$e_1 \wedge e_2$	$-(e_1 \ominus e_1)\, e_2$	$(e_2 \ominus e_2)\, e_1$	$-(e_1 \ominus e_1)\,(e_2 \ominus e_2)$

This table defines *all* the Clifford algebras on the 2-space. Different Clifford algebras may be generated by choosing different metrics for the space, that is, by choosing the two scalar values

$e_1 \ominus e_1$ and $e_2 \ominus e_2$.
Note however that allocation of scalar values **a** to $e_1 \ominus e_1$ and **b** to $e_2 \ominus e_2$ would lead to essentially the same structure as allocating **b** to $e_1 \ominus e_1$ and **a** to $e_2 \ominus e_2$.

In the rest of what follows however, we will restrict ourselves to metrics in which $e_1 \ominus e_1 == \pm 1$ and $e_2 \ominus e_2 == \pm 1$, whence there are three cases of interest.

$$\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to +1\}$$
$$\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to -1\}$$
$$\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to -1\}$$

As observed previously the case $\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to +1\}$ is isomorphic to the case $\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to -1\}$, so we do not need to consider it.

Case 1: $\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to +1\}$

This is the standard case of a 2-space with an orthonormal basis. Making the replacements in the table gives:

```
C6 = C5 /. {e1 ⊖ e1 → +1, e2 ⊖ e2 → +1}; PaletteForm[C6]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	1	$e_1 \wedge e_2$	e_2
e_2	$-(e_1 \wedge e_2)$	1	$-e_1$
$e_1 \wedge e_2$	$-e_2$	e_1	-1

Inspecting this table for interesting structures or substructures, we note first that the even subalgebra (that is, the algebra based on products of the even basis elements) is isomorphic to the complex algebra. For our own explorations we can use the palette to construct a product table for the subalgebra, or we can create a table using the *GrassmannAlgebra* function `TableTemplate`, and edit it by deleting the middle rows and columns.

```
TableTemplate[C6]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	1	$e_1 \wedge e_2$	e_2
e_2	$-(e_1 \wedge e_2)$	1	$-e_1$
$e_1 \wedge e_2$	$-e_2$	e_1	-1

1	$e_1 \wedge e_2$
$e_1 \wedge e_2$	-1

If we want to create a palette for the subalgebra, we have to edit the normal list (matrix) form and then apply `PaletteForm`. For even subalgebras we can also apply the *GrassmannAlgebra* function `EvenCliffordProductTable` which creates a Clifford product table from just the basis elements of even grade. We then set the metric we want, convert the Clifford product to scalar products, evaluate the scalar products according to the chosen metric, and display the resulting table as a palette.

```
C7 = EvenCliffordProductTable[]; DeclareMetric[{1, 0}, {0, 1}];
PaletteForm[ToMetricForm[ToScalarProducts[C7]]]
```

1	$e_1 \wedge e_2$
$e_1 \wedge e_2$	-1

Here we see that the basis element $\mathbf{e_1 \wedge e_2}$ has the property that its (Clifford) square is **-1**. We can see how this arises by carrying out the elementary operations on the product. Note that $\mathbf{e_1 \ominus e_1 == e_2 \ominus e_2 == 1}$ since we have assumed the 2-space under consideration is Euclidean.

$$(\mathbf{e_1 \wedge e_2}) \diamond (\mathbf{e_1 \wedge e_2}) == -(\mathbf{e_2 \wedge e_1}) \diamond (\mathbf{e_1 \wedge e_2}) == -(\mathbf{e_1 \ominus e_1})\ (\mathbf{e_2 \ominus e_2}) == -1$$

Thus the pair $\{1,\ \mathbf{e_1 \wedge e_2}\}$ generates an algebra under the Clifford product operation, isomorphic to the complex algebra. It is also the basis of the even Grassmann algebra of $\underset{2}{\Lambda}$.

Case 2: $\{e_1 \ominus e_1 \to +1,\ e_2 \ominus e_2 \to -1\}$

Here is an example of a Clifford algebra which does not have any popular applications of which the author is aware.

```
C7 = C5 /. {e1⊖e1 → +1, e2⊖e2 → -1}; PaletteForm[C7]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	1	$e_1 \wedge e_2$	e_2
e_2	$-(e_1 \wedge e_2)$	-1	e_1
$e_1 \wedge e_2$	$-e_2$	$-e_1$	1

Case 3: $\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to -1\}$

Our third case in which the metric is **{{-1,0},{0,-1}}**, is isomorphic to the quaternions.

```
C8 = C5 /. {e1⊖e1 → -1, e2⊖e2 → -1}; PaletteForm[C8]
```

1	e_1	e_2	$e_1 \wedge e_2$
e_1	-1	$e_1 \wedge e_2$	$-e_2$
e_2	$-(e_1 \wedge e_2)$	-1	e_1
$e_1 \wedge e_2$	e_2	$-e_1$	-1

We can see this isomorphism more clearly by substituting the usual quaternion symbols (here we *Mathematica*'s double-struck symbols and choose the correspondence $\{\mathbf{e_1} \to \mathbb{i},\ \mathbf{e_2} \to \mathbb{j},\ \mathbf{e_1 \wedge e_2} \to \mathbb{k}\}$.

```
C₉ = C₈ /. {e₁ → 𝕚, e₂ → 𝕛, e₁ ∧ e₂ → 𝕜}; PaletteForm[C₉]
```

1	𝕚	𝕛	𝕜
𝕚	-1	𝕜	-𝕛
𝕛	-𝕜	-1	𝕚
𝕜	𝕛	-𝕚	-1

Having verified that the structure is indeed quaternionic, let us return to the original specification in terms of the basis of the Grassmann algebra. A quaternion can be written in terms of these basis elements as:

$$Q == a + b\, e_1 + c\, e_2 + d\, e_1 \wedge e_2 == (a + b\, e_1) + (c + d\, e_1) \wedge e_2$$

Now, because e_1 and e_2 are orthogonal, $e_1 \wedge e_2$ is equal to $e_1 \diamond e_2$. But for any further calculations we will need to use the Clifford product form. Hence we write

$$\boxed{\begin{array}{c} Q == a + b\, e_1 + c\, e_2 + d\, e_1 \diamond e_2 == \\ (a + b\, e_1) + (c + d\, e_1) \diamond e_2 \end{array}} \quad 12.66$$

Hence under one interpretation, each of e_1 and e_2 *and* their Clifford product $e_1 \diamond e_2$ behaves as a different imaginary unit. Under the second interpretation, a quaternion is a complex number with imaginary unit e_2, whose components are complex numbers based on e_1 as the imaginary unit.

12.15 Clifford Algebras of a 3-Space

The Clifford product table in 3-space

In this section we explore the Clifford algebras of 3-space. First we declare a (not necessarily orthogonal) basis for the 3-space, and generate the associated Clifford product table. Because of the size of the table, only the first few columns are shown in the print version.

```
𝕍₃; C₁ = CliffordProductTable[]; PaletteForm[C₁]
```

$1 \diamond 1$	$1 \diamond e_1$	$1 \diamond e_2$	$1 \diamond e_3$	
$e_1 \diamond 1$	$e_1 \diamond e_1$	$e_1 \diamond e_2$	$e_1 \diamond e_3$	
$e_2 \diamond 1$	$e_2 \diamond e_1$	$e_2 \diamond e_2$	$e_2 \diamond e_3$	
$e_3 \diamond 1$	$e_3 \diamond e_1$	$e_3 \diamond e_2$	$e_3 \diamond e_3$	
$(e_1 \wedge e_2) \diamond 1$	$(e_1 \wedge e_2) \diamond e_1$	$(e_1 \wedge e_2) \diamond e_2$	$(e_1 \wedge e_2) \diamond e_3$	$(e_1$
$(e_1 \wedge e_3) \diamond 1$	$(e_1 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_3) \diamond e_2$	$(e_1 \wedge e_3) \diamond e_3$	$(e_1$
$(e_2 \wedge e_3) \diamond 1$	$(e_2 \wedge e_3) \diamond e_1$	$(e_2 \wedge e_3) \diamond e_2$	$(e_2 \wedge e_3) \diamond e_3$	$(e_2$
$(e_1 \wedge e_2 \wedge e_3) \diamond 1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_2$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_3$	$(e_1 \wedge$

$\mathcal{C}\ell_3$: The Pauli algebra

Our first exploration is to Clifford algebras in Euclidean space, hence we accept the default metric which was automatically declared when we declared the basis.

```
MatrixForm[Metric]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the basis elements are orthogonal, we can use the *GrassmannAlgebra* function `CliffordToOrthogonalScalarProducts` for computing the Clifford products. This function is faster for larger calculations than the more general `ToScalarProducts` used in the previous examples.

```
C2 = ToMetricForm[CliffordToOrthogonalScalarProducts[C1]];
PaletteForm[C2]
```

1	e_1	e_2	e_3	$e_1 \wedge e_2$	$e_1 \wedge$
e_1	1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	e_2	e
e_2	$-(e_1 \wedge e_2)$	1	$e_2 \wedge e_3$	$-e_1$	$-(e_1 \wedge e$
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	1	$e_1 \wedge e_2 \wedge e_3$	$-e$
$e_1 \wedge e_2$	$-e_2$	e_1	$e_1 \wedge e_2 \wedge e_3$	-1	$-(e_2$
$e_1 \wedge e_3$	$-e_3$	$-(e_1 \wedge e_2 \wedge e_3)$	e_1	$e_2 \wedge e_3$	$-$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	$-e_3$	e_2	$-(e_1 \wedge e_3)$	$e_1 \wedge$
$e_1 \wedge e_2 \wedge e_3$	$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$	$-e_3$	e

We can show that this Clifford algebra of Euclidean 3-space is isomorphic to the Pauli algebra. Pauli's representation of the algebra was by means of 2×2 matrices over the complex field:

$$\sigma_1 == \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 == \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 == \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The isomorphism is direct and straightforward:

$$\{I \leftrightarrow 1,\ \sigma_1 \leftrightarrow e_1,\ \sigma_2 \leftrightarrow e_2,\ \sigma_3 \leftrightarrow e_3,\ \sigma_1\,\sigma_2 \leftrightarrow e_1 \wedge e_2,$$
$$\sigma_1\,\sigma_3 \leftrightarrow e_1 \wedge e_3,\ \sigma_2\,\sigma_3 \leftrightarrow e_2 \wedge e_3,\ \sigma_1\,\sigma_2\,\sigma_3 \leftrightarrow e_1 \wedge e_2 \wedge e_3\}$$

To show this we can:
1) Replace the symbols for the basis elements of the Grassmann algebra in the table above by symbols for the Pauli matrices. Replace the exterior product operation by the matrix product operation.
2) Replace the symbols by the actual matrices, allowing *Mathematica* to perform the matrix products.
3) Replace the resulting Pauli matrices with the corresponding basis elements of the Grassmann

algebra.
4) Verify that the resulting table is the same as the original table.

We perform these steps in sequence. Because of the size of the output, only the electronic version will show the complete tables.

Step 1: Replace symbols for entities and operations

```
C₃ = (C₂ // ReplaceNegativeUnit) /. {1 → I, e₁ → σ₁, e₂ → σ₂, e₃ → σ₃} /.
  Wedge → Dot; PaletteForm[C₃]
```

I	σ_1	σ_2	σ_3	$\sigma_1 . \sigma_2$	$\sigma_1 . \sigma_3$	σ_2 .
σ_1	I	$\sigma_1 . \sigma_2$	$\sigma_1 . \sigma_3$	σ_2	σ_3	$\sigma_1 . \sigma$
σ_2	$-\sigma_1 . \sigma_2$	I	$\sigma_2 . \sigma_3$	$-\sigma_1$	$-\sigma_1 . \sigma_2 . \sigma_3$	σ
σ_3	$-\sigma_1 . \sigma_3$	$-\sigma_2 . \sigma_3$	I	$\sigma_1 . \sigma_2 . \sigma_3$	$-\sigma_1$	$-$
$\sigma_1 . \sigma_2$	$-\sigma_2$	σ_1	$\sigma_1 . \sigma_2 . \sigma_3$	$-I$	$-\sigma_2 . \sigma_3$	σ_1 .
$\sigma_1 . \sigma_3$	$-\sigma_3$	$-\sigma_1 . \sigma_2 . \sigma_3$	σ_1	$\sigma_2 . \sigma_3$	$-I$	$-\sigma_1$
$\sigma_2 . \sigma_3$	$\sigma_1 . \sigma_2 . \sigma_3$	$-\sigma_3$	σ_2	$-\sigma_1 . \sigma_3$	$\sigma_1 . \sigma_2$	$-$
$\sigma_1 . \sigma_2 . \sigma_3$	$\sigma_2 . \sigma_3$	$-\sigma_1 . \sigma_3$	$\sigma_1 . \sigma_2$	$-\sigma_3$	σ_2	$-$

Step 2: Substitute matrices and calculate

$$C_4 = C_3 \;/.\; \left\{I \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 \to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\};$$

```
MatrixForm[C₄]
```

$$\begin{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\end{pmatrix}$$

Step 3:

Here we let the first row (column) of the product table correspond back to the basis elements of the Grassmann representation, and make the substitution: throughout the whole table.

```
C5 = C4 /. Thread[First[C4] → Basis∧[]] /.
  Thread[-First[C4] → -Basis∧[]]
```

```
{{1, e1, e2, e3, e1∧e2, e1∧e3, e2∧e3, e1∧e2∧e3},
 {e1, 1, e1∧e2, e1∧e3, e2, e3, e1∧e2∧e3, e2∧e3},
 {e2, -(e1∧e2), 1, e2∧e3, -e1, -(e1∧e2∧e3), e3, -(e1∧e3)},
 {e3, -(e1∧e3), -(e2∧e3), 1, e1∧e2∧e3, -e1, -e2, e1∧e2},
 {e1∧e2, -e2, e1, e1∧e2∧e3, -1, -(e2∧e3), e1∧e3, -e3},
 {e1∧e3, -e3, -(e1∧e2∧e3), e1, e2∧e3, -1, -(e1∧e2), e2},
 {e2∧e3, e1∧e2∧e3, -e3, e2, -(e1∧e3), e1∧e2, -1, -e1},
 {e1∧e2∧e3, e2∧e3, -(e1∧e3), e1∧e2, -e3, e2, -e1, -1}}
```

Step 4: Verification

Verify that this is the table with which we began.

```
C5 == C2
```

```
True
```

$C\ell_3{}^{+}$: The Quaternions

Multiplication of two even elements always generates an even element, hence the even elements form a subalgebra. In this case the basis for the subalgebra is composed of the unit **1** and the bivectors $e_i \wedge e_j$.

```
C4 = EvenCliffordProductTable[] //
   CliffordToOrthogonalScalarProducts //
  ToMetricForm; PaletteForm[C4]
```

1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_2 \wedge e_3$
$e_1 \wedge e_2$	-1	$-(e_2 \wedge e_3)$	$e_1 \wedge e_3$
$e_1 \wedge e_3$	$e_2 \wedge e_3$	-1	$-(e_1 \wedge e_2)$
$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$	-1

From this multiplication table we can see that the even subalgebra of the Clifford algebra of 3-space is isomorphic to the quaternions. To see the isomorphism more clearly, replace the bivectors by 𝕚, 𝕛, and 𝕜.

```
C5 = C4 /. {e2∧e3 → 𝕚, e1∧e3 → 𝕛, e1∧e2 → 𝕜}; PaletteForm[C5]
```

1	𝕜	𝕛	𝕚
𝕜	−1	−𝕚	𝕛
𝕛	𝕚	−1	−𝕜
𝕚	−𝕛	𝕜	−1

The Complex subalgebra

The subalgebra generated by the pair of basis elements $\{1, e_1 \wedge e_2 \wedge e_3\}$ is isomorphic to the complex algebra. Although, under the Clifford product, each of the 2-elements behaves *like* an imaginary unit, it is only the 3-element $e_1 \wedge e_2 \wedge e_3$ that also commutes with each of the other basis elements.

Biquaternions

We now explore the metric in which $\{e_1 \ominus e_1 \to -1,\ e_2 \ominus e_2 \to -1,\ e_3 \ominus e_3 \to -1\}$.

To generate the Clifford product table for this metric we enter:

```
DeclareMetric[DiagonalMatrix[{-1, -1, -1}]];
C6 = CliffordProductTable[] // ToScalarProducts // ToMetricForm;
PaletteForm[C6]
```

1	e_1	e_2	e_3	$e_1 \wedge e_2$	$e_1 \wedge e_3$
e_1	-1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$-e_2$	$-e_3$
e_2	$-(e_1 \wedge e_2)$	-1	$e_2 \wedge e_3$	e_1	$-(e_1 \wedge e_2 \wedge e_3)$
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	-1	$e_1 \wedge e_2 \wedge e_3$	e_1
$e_1 \wedge e_2$	e_2	$-e_1$	$e_1 \wedge e_2 \wedge e_3$	-1	$e_2 \wedge e_3$
$e_1 \wedge e_3$	e_3	$-(e_1 \wedge e_2 \wedge e_3)$	$-e_1$	$-(e_2 \wedge e_3)$	-1
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e_3	$-e_2$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$
$e_1 \wedge e_2 \wedge e_3$	$-(e_2 \wedge e_3)$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$	$-e_3$	e_2

Note that every one of the basis elements of the Grassmann algebra (except 1 and $e_1 \wedge e_2 \wedge e_3$) acts as an imaginary unit under the Clifford product. This enables us to build up a general element of the algebra as a sum of nested complex numbers, or of nested quaternions. To show this, we begin by writing a general element in terms of the basis elements of the Grassmann algebra:

$$QQ == a + b\, e_1 + c\, e_2 + d\, e_3 + e\, e_1 \wedge e_2 + f\, e_1 \wedge e_3 + g\, e_2 \wedge e_3 + h\, e_1 \wedge e_2 \wedge e_3$$

Or, as previously argued, since the basis 1-elements are orthogonal, we can replace the exterior product by the Clifford product and rearrange the terms in the sum to give

$$QQ == (a + b\, e_1 + c\, e_2 + e\, e_1 \diamond e_2) + (d + f\, e_1 + g\, e_2 + h\, e_1 \diamond e_2) \diamond e_3$$

This sum may be viewed as the complex sum of two quaternions

$$Q_1 == a + b\, e_1 + c\, e_2 + e\, e_1 \diamond e_2 == (a + b\, e_1) + (c + e\, e_1) \diamond e_2$$
$$Q_2 == d + f\, e_1 + g\, e_2 + h\, e_1 \diamond e_2 == (d + f\, e_1) + (g + h\, e_1) \diamond e_2$$

$$QQ == Q_1 + Q_2 \diamond e_3$$

Historical Note

This complex sum of two quaternions was called a biquaternion by Hamilton [Hamilton, Elements of Quaternions, p133] but Clifford in a footnote to his *Preliminary Sketch of Biquaternions* [Clifford, Mathematical Papers, Chelsea] says 'Hamilton's biquaternion is a quaternion with complex coefficients; but it is convenient (as Prof. Pierce remarks) to suppose from the beginning that all scalars may be complex. As the word is thus no longer wanted in its old meaning, I have made bold to use it in a new one."

Hamilton uses the word *biscalar* for a *complex number* and *bivector* [p 225 Elements of Quaternions] for a *complex vector*, that is, for a vector $\mathbf{x} + i\,\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are vectors; and the word *biquaternion* for a complex quaternion $\mathbf{q_0} + i\,\mathbf{q_1}$, where $\mathbf{q_0}$ and $\mathbf{q_1}$ are quaternions. He emphasizes here that "… i is the (scalar) *imaginary of algebra*, and *not* a symbol for a *geometrically real right versor* …"

Hamilton introduces his biquaternion as the quotient of a bivector (his usage) by a (real) vector.

12.16 Clifford Algebras of a 4-Space

The Clifford product table in 4-space

In this section we explore the Clifford algebras of 4-space. First we declare a (not necessarily orthogonal) basis for the 4-space, and generate the associated Clifford product table. Because of the size of the table, only the first few columns are shown in the print version.

$\mathbb{V}_4$; C_1 = CliffordProductTable[]; PaletteForm[C_1]

$1 \diamond 1$	$1 \diamond e_1$	$1 \diamond e_2$	$1 \diamond$
$e_1 \diamond 1$	$e_1 \diamond e_1$	$e_1 \diamond e_2$	$e_1 \diamond$
$e_2 \diamond 1$	$e_2 \diamond e_1$	$e_2 \diamond e_2$	$e_2 \diamond$
$e_3 \diamond 1$	$e_3 \diamond e_1$	$e_3 \diamond e_2$	$e_3 \diamond$
$e_4 \diamond 1$	$e_4 \diamond e_1$	$e_4 \diamond e_2$	$e_4 \diamond$
$(e_1 \wedge e_2) \diamond 1$	$(e_1 \wedge e_2) \diamond e_1$	$(e_1 \wedge e_2) \diamond e_2$	$(e_1 \wedge e$
$(e_1 \wedge e_3) \diamond 1$	$(e_1 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_3) \diamond e_2$	$(e_1 \wedge e$
$(e_1 \wedge e_4) \diamond 1$	$(e_1 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_4) \diamond e_2$	$(e_1 \wedge e$
$(e_2 \wedge e_3) \diamond 1$	$(e_2 \wedge e_3) \diamond e_1$	$(e_2 \wedge e_3) \diamond e_2$	$(e_2 \wedge e$
$(e_2 \wedge e_4) \diamond 1$	$(e_2 \wedge e_4) \diamond e_1$	$(e_2 \wedge e_4) \diamond e_2$	$(e_2 \wedge e$
$(e_3 \wedge e_4) \diamond 1$	$(e_3 \wedge e_4) \diamond e_1$	$(e_3 \wedge e_4) \diamond e_2$	$(e_3 \wedge e$
$(e_1 \wedge e_2 \wedge e_3) \diamond 1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_3) \diamond e_2$	$(e_1 \wedge e_2 \wedge$
$(e_1 \wedge e_2 \wedge e_4) \diamond 1$	$(e_1 \wedge e_2 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_4) \diamond e_2$	$(e_1 \wedge e_2 \wedge$
$(e_1 \wedge e_3 \wedge e_4) \diamond 1$	$(e_1 \wedge e_3 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_3 \wedge e_4) \diamond e_2$	$(e_1 \wedge e_3 \wedge$
$(e_2 \wedge e_3 \wedge e_4) \diamond 1$	$(e_2 \wedge e_3 \wedge e_4) \diamond e_1$	$(e_2 \wedge e_3 \wedge e_4) \diamond e_2$	$(e_2 \wedge e_3 \wedge$
$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \diamond 1$	$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \diamond e_1$	$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \diamond e_2$	$(e_1 \wedge e_2 \wedge e$

$\mathcal{C}\ell_4$: The Clifford algebra of Euclidean 4-space

Our first exploration is to the Clifford algebra of Euclidean space, hence we accept the default metric which was automatically declared when we declared the basis.

Metric

{{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

Since the basis elements are orthogonal, we can use the faster *GrassmannAlgebra* function `CliffordToOrthogonalScalarProducts` for computing the Clifford products.

```
C₂ = CliffordToOrthogonalScalarProducts[C₁] // ToMetricForm;
PaletteForm[C₂]
```

1	e_1	e_2	e_3
e_1	1	$e_1 \wedge e_2$	$e_1 \wedge e_3$
e_2	$-(e_1 \wedge e_2)$	1	$e_2 \wedge e_3$
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	1
e_4	$-(e_1 \wedge e_4)$	$-(e_2 \wedge e_4)$	$-(e_3 \wedge e_4)$
$e_1 \wedge e_2$	$-e_2$	e_1	$e_1 \wedge e_2 \wedge e_3$
$e_1 \wedge e_3$	$-e_3$	$-(e_1 \wedge e_2 \wedge e_3)$	e_1
$e_1 \wedge e_4$	$-e_4$	$-(e_1 \wedge e_2 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	$-e_3$	e_2
$e_2 \wedge e_4$	$e_1 \wedge e_2 \wedge e_4$	$-e_4$	$-(e_2 \wedge e_3 \wedge e_4)$
$e_3 \wedge e_4$	$e_1 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	$-e_4$
$e_1 \wedge e_2 \wedge e_3$	$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$
$e_1 \wedge e_2 \wedge e_4$	$e_2 \wedge e_4$	$-(e_1 \wedge e_4)$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$
$e_1 \wedge e_3 \wedge e_4$	$e_3 \wedge e_4$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_4)$
$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$e_3 \wedge e_4$	$-(e_2 \wedge e_4)$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_2 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_4)$

This table is well off the page in the printed version, but we can condense the notation for the basis elements of Λ by replacing $\mathbf{e_i \wedge \cdots \wedge e_j}$ by $\mathbf{e_{i,\ldots,j}}$. To do this we can use the *GrassmannAlgebra* function `ToBasisIndexedForm`. For example

```
1 - (e₁ ∧ e₂ ∧ e₃) // ToBasisIndexedForm
```

$1 - e_{1,2,3}$

To display the table in condensed notation we make up a rule for each of the basis elements.

```
C₃ = C₂ /. Reverse[Thread[Basis∧[] → ToBasisIndexedForm[Basis∧[]]]];
PaletteForm[C₃]
```

1	e_1	e_2	e_3	e_4	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$
e_1	1	$e_{1,2}$	$e_{1,3}$	$e_{1,4}$	e_2	e_3	e_4
e_2	$-e_{1,2}$	1	$e_{2,3}$	$e_{2,4}$	$-e_1$	$-e_{1,2,3}$	$-e_{1,2,}$
e_3	$-e_{1,3}$	$-e_{2,3}$	1	$e_{3,4}$	$e_{1,2,3}$	$-e_1$	$-e_{1,3,}$
e_4	$-e_{1,4}$	$-e_{2,4}$	$-e_{3,4}$	1	$e_{1,2,4}$	$e_{1,3,4}$	$-e_1$
$e_{1,2}$	$-e_2$	e_1	$e_{1,2,3}$	$e_{1,2,4}$	-1	$-e_{2,3}$	$-e_{2,4}$
$e_{1,3}$	$-e_3$	$-e_{1,2,3}$	e_1	$e_{1,3,4}$	$e_{2,3}$	-1	$-e_{3,4}$
$e_{1,4}$	$-e_4$	$-e_{1,2,4}$	$-e_{1,3,4}$	e_1	$e_{2,4}$	$e_{3,4}$	-1
$e_{2,3}$	$e_{1,2,3}$	$-e_3$	e_2	$e_{2,3,4}$	$-e_{1,3}$	$e_{1,2}$	$e_{1,2,3,}$
$e_{2,4}$	$e_{1,2,4}$	$-e_4$	$-e_{2,3,4}$	e_2	$-e_{1,4}$	$-e_{1,2,3,4}$	$e_{1,2}$
$e_{3,4}$	$e_{1,3,4}$	$e_{2,3,4}$	$-e_4$	e_3	$e_{1,2,3,4}$	$-e_{1,4}$	$e_{1,3}$
$e_{1,2,3}$	$e_{2,3}$	$-e_{1,3}$	$e_{1,2}$	$e_{1,2,3,4}$	$-e_3$	e_2	$e_{2,3,4}$
$e_{1,2,4}$	$e_{2,4}$	$-e_{1,4}$	$-e_{1,2,3,4}$	$e_{1,2}$	$-e_4$	$-e_{2,3,4}$	e_2
$e_{1,3,4}$	$e_{3,4}$	$e_{1,2,3,4}$	$-e_{1,4}$	$e_{1,3}$	$e_{2,3,4}$	$-e_4$	e_3
$e_{2,3,4}$	$-e_{1,2,3,4}$	$e_{3,4}$	$-e_{2,4}$	$e_{2,3}$	$-e_{1,3,4}$	$e_{1,2,4}$	$-e_{1,2,}$
$e_{1,2,3,4}$	$-e_{2,3,4}$	$e_{1,3,4}$	$-e_{1,2,4}$	$e_{1,2,3}$	$-e_{3,4}$	$e_{2,4}$	$-e_{2,3}$

$\mathcal{C}\ell_4{}^+$: The even Clifford algebra of Euclidean 4-space

The even subalgebra is composed of the unit **1**, the bivectors $\mathbf{e_i \wedge e_j}$, and the single basis 4-element $\mathbf{e_1 \wedge e_2 \wedge e_3 \wedge e_4}$.

```
C₃ = EvenCliffordProductTable[] //
   CliffordToOrthogonalScalarProducts //
  ToMetricForm; PaletteForm[C₃]
```

1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	$e_1 \wedge e_4$	$e_2 \wedge e_3$
$e_1 \wedge e_2$	-1	$-(e_2 \wedge e_3)$	$-(e_2 \wedge e_4)$	$e_1 \wedge e_3$
$e_1 \wedge e_3$	$e_2 \wedge e_3$	-1	$-(e_3 \wedge e_4)$	$-(e_1 \wedge e$
$e_1 \wedge e_4$	$e_2 \wedge e_4$	$e_3 \wedge e_4$	-1	$e_1 \wedge e_2 \wedge e_3$
$e_2 \wedge e_3$	$-(e_1 \wedge e_3)$	$e_1 \wedge e_2$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	-1
$e_2 \wedge e_4$	$-(e_1 \wedge e_4)$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_2$	$e_3 \wedge e_4$
$e_3 \wedge e_4$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_4)$	$e_1 \wedge e_3$	$-(e_2 \wedge e$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_3 \wedge e_4)$	$e_2 \wedge e_4$	$-(e_2 \wedge e_3)$	$-(e_1 \wedge e$

$\mathcal{C}\ell_{1,3}$: The Dirac algebra

To generate the Dirac algebra we need the Minkowski metric in which there is one time-like basis element $\mathbf{e_1} \ominus \mathbf{e_1} == +1$, and three space-like basis elements $\mathbf{e_i} \ominus \mathbf{e_i} == -1$.

```
{e₁ ⊖ e₁ == 1, e₂ ⊖ e₂ == -1, e₃ ⊖ e₃ == -1, e₄ ⊖ e₄ == -1}
```

To generate the Clifford product table for this metric we enter:

```
𝕍₄; DeclareMetric[DiagonalMatrix[{1, -1, -1, -1}]]
```

```
{{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
```

```
C₆ = CliffordToOrthogonalScalarProducts[C₁] // ToMetricForm;
PaletteForm[C₆]
```

1	e_1	e_2	e_3	
e_1	1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	e_1
e_2	$-(e_1 \wedge e_2)$	-1	$e_2 \wedge e_3$	e_2
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	-1	e_3
e_4	$-(e_1 \wedge e_4)$	$-(e_2 \wedge e_4)$	$-(e_3 \wedge e_4)$	
$e_1 \wedge e_2$	$-e_2$	$-e_1$	$e_1 \wedge e_2 \wedge e_3$	$e_1 \wedge$
$e_1 \wedge e_3$	$-e_3$	$-(e_1 \wedge e_2 \wedge e_3)$	$-e_1$	$e_1 \wedge$
$e_1 \wedge e_4$	$-e_4$	$-(e_1 \wedge e_2 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$	$-$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e_3	$-e_2$	$e_2 \wedge$
$e_2 \wedge e_4$	$e_1 \wedge e_2 \wedge e_4$	e_4	$-(e_2 \wedge e_3 \wedge e_4)$	$-$
$e_3 \wedge e_4$	$e_1 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	e_4	$-$
$e_1 \wedge e_2 \wedge e_3$	$e_2 \wedge e_3$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$	$e_1 \wedge e_2$
$e_1 \wedge e_2 \wedge e_4$	$e_2 \wedge e_4$	$e_1 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e$
$e_1 \wedge e_3 \wedge e_4$	$e_3 \wedge e_4$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$e_1 \wedge e_4$	$-(e$
$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e_3 \wedge e_4)$	$e_2 \wedge e_4$	$-(e$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$-(e_2 \wedge e_3 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_2 \wedge e_4$	$-(e_1 \wedge$

To confirm that this structure is isomorphic to the Dirac algebra we can go through the same procedure that we followed in the case of the Pauli algebra.

Step 1: Replace symbols for entities and operations

$$C_7 = (C_6\ //\ \mathtt{ReplaceNegativeUnit})\ /.\ \{1 \to \mathrm{I},\ e_1 \to \gamma_0,\ e_2 \to \gamma_1,\ e_3 \to \gamma_2,\ e_4 \to \gamma_3\}\ /.\ \mathtt{Wedge} \to \mathtt{Dot};\ \mathtt{PaletteForm}[C_7]$$

I	γ_0	γ_1	γ_2	γ_3
γ_0	I	$\gamma_0\cdot\gamma_1$	$\gamma_0\cdot\gamma_2$	$\gamma_0\cdot\gamma_3$
γ_1	$-\gamma_0\cdot\gamma_1$	$-\mathrm{I}$	$\gamma_1\cdot\gamma_2$	$\gamma_1\cdot\gamma_3$
γ_2	$-\gamma_0\cdot\gamma_2$	$-\gamma_1\cdot\gamma_2$	$-\mathrm{I}$	$\gamma_2\cdot\gamma_3$
γ_3	$-\gamma_0\cdot\gamma_3$	$-\gamma_1\cdot\gamma_3$	$-\gamma_2\cdot\gamma_3$	$-\mathrm{I}$
$\gamma_0\cdot\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\cdot\gamma_1\cdot\gamma_2$	$\gamma_0\cdot\gamma_1\cdot\gamma_3$
$\gamma_0\cdot\gamma_2$	$-\gamma_2$	$-\gamma_0\cdot\gamma_1\cdot\gamma_2$	$-\gamma_0$	$\gamma_0\cdot\gamma_2\cdot\gamma_3$
$\gamma_0\cdot\gamma_3$	$-\gamma_3$	$-\gamma_0\cdot\gamma_1\cdot\gamma_3$	$-\gamma_0\cdot\gamma_2\cdot\gamma_3$	$-\gamma_0$
$\gamma_1\cdot\gamma_2$	$\gamma_0\cdot\gamma_1\cdot\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\cdot\gamma_2\cdot\gamma_3$
$\gamma_1\cdot\gamma_3$	$\gamma_0\cdot\gamma_1\cdot\gamma_3$	γ_3	$-\gamma_1\cdot\gamma_2\cdot\gamma_3$	$-\gamma_1$
$\gamma_2\cdot\gamma_3$	$\gamma_0\cdot\gamma_2\cdot\gamma_3$	$\gamma_1\cdot\gamma_2\cdot\gamma_3$	γ_3	$-\gamma_2$
$\gamma_0\cdot\gamma_1\cdot\gamma_2$	$\gamma_1\cdot\gamma_2$	$\gamma_0\cdot\gamma_2$	$-\gamma_0\cdot\gamma_1$	$\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3$
$\gamma_0\cdot\gamma_1\cdot\gamma_3$	$\gamma_1\cdot\gamma_3$	$\gamma_0\cdot\gamma_3$	$-\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3$	$-\gamma_0\cdot\gamma_1$
$\gamma_0\cdot\gamma_2\cdot\gamma_3$	$\gamma_2\cdot\gamma_3$	$\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3$	$\gamma_0\cdot\gamma_3$	$-\gamma_0\cdot\gamma_2$
$\gamma_1\cdot\gamma_2\cdot\gamma_3$	$-\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3$	$-\gamma_2\cdot\gamma_3$	$\gamma_1\cdot\gamma_3$	$-\gamma_1\cdot\gamma_2$
$\gamma_0\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3$	$-\gamma_1\cdot\gamma_2\cdot\gamma_3$	$-\gamma_0\cdot\gamma_2\cdot\gamma_3$	$\gamma_0\cdot\gamma_1\cdot\gamma_3$	$-\gamma_0\cdot\gamma_1\cdot\gamma_2$

Step 2: Substitute matrices and calculate

$$C_8 = C_7\ /.\ \left\{\mathrm{I} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \gamma_0 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \gamma_1 \to \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\right.$$
$$\left.\gamma_2 \to \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},\ \gamma_3 \to \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right\};\ \mathtt{MatrixForm}[C_8]$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \quad \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

Step 3:

```
C9 = C8 /. Thread[First[C8] → BasisΛ[]] /.
   Thread[-First[C8] → -BasisΛ[]];
```

Step 4: Verification

```
C9 == C6
```

```
True
```

$\mathcal{C}\ell_{0,4}$: The Clifford algebra of complex quaternions

To generate the Clifford algebra of complex quaternions we need a metric in which all the scalar products of orthogonal basis elements are of the form $e_i \ominus e_j$ (*i* not equal to *j*) are equal to **-1**. That is

$$\{e_1 \ominus e_1 == -1,\ e_2 \ominus e_2 == -1,\ e_3 \ominus e_3 == -1,\ e_4 \ominus e_4 == -1\}$$

To generate the Clifford product table for this metric we enter:

```
V4; DeclareMetric[DiagonalMatrix[{-1, -1, -1, -1}]]
```

```
{{-1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}}
```

```
C6 = CliffordToOrthogonalScalarProducts[C1] // ToMetricForm;
PaletteForm[C6]
```

1	e_1	e_2	e_3	
e_1	-1	$e_1 \wedge e_2$	$e_1 \wedge e_3$	e_1
e_2	$-(e_1 \wedge e_2)$	-1	$e_2 \wedge e_3$	e_2
e_3	$-(e_1 \wedge e_3)$	$-(e_2 \wedge e_3)$	-1	e_3
e_4	$-(e_1 \wedge e_4)$	$-(e_2 \wedge e_4)$	$-(e_3 \wedge e_4)$	
$e_1 \wedge e_2$	e_2	$-e_1$	$e_1 \wedge e_2 \wedge e_3$	$e_1 \wedge$
$e_1 \wedge e_3$	e_3	$-(e_1 \wedge e_2 \wedge e_3)$	$-e_1$	$e_1 \wedge$
$e_1 \wedge e_4$	e_4	$-(e_1 \wedge e_2 \wedge e_4)$	$-(e_1 \wedge e_3 \wedge e_4)$	$-$
$e_2 \wedge e_3$	$e_1 \wedge e_2 \wedge e_3$	e_3	$-e_2$	$e_2 \wedge$
$e_2 \wedge e_4$	$e_1 \wedge e_2 \wedge e_4$	e_4	$-(e_2 \wedge e_3 \wedge e_4)$	$-$
$e_3 \wedge e_4$	$e_1 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	e_4	$-$
$e_1 \wedge e_2 \wedge e_3$	$-(e_2 \wedge e_3)$	$e_1 \wedge e_3$	$-(e_1 \wedge e_2)$	$e_1 \wedge e_2$
$e_1 \wedge e_2 \wedge e_4$	$-(e_2 \wedge e_4)$	$e_1 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e$
$e_1 \wedge e_3 \wedge e_4$	$-(e_3 \wedge e_4)$	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$e_1 \wedge e_4$	$-(e$
$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$	$-(e_3 \wedge e_4)$	$e_2 \wedge e_4$	$-(e$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$e_2 \wedge e_3 \wedge e_4$	$-(e_1 \wedge e_3 \wedge e_4)$	$e_1 \wedge e_2 \wedge e_4$	$-(e_1 \wedge$

Note that both the vectors and bivectors act as an imaginary units under the Clifford product. This enables us to build up a general element of the algebra as a sum of nested complex numbers, quaternions, or complex quaternions. To show this, we begin by writing a general element in terms of the basis elements of the Grassmann algebra:

X = CreateGrassmannNumber[a]

$$a^0 + e_1 a^1 + e_2 a^2 + e_3 a^3 + e_4 a^4 + a^5 e_1 \wedge e_2 + a^6 e_1 \wedge e_3 + a^7 e_1 \wedge e_4 + a^8 e_2 \wedge e_3 + a^9 e_2 \wedge e_4 + a^{10} e_3 \wedge e_4 + a^{11} e_1 \wedge e_2 \wedge e_3 + a^{12} e_1 \wedge e_2 \wedge e_4 + a^{13} e_1 \wedge e_3 \wedge e_4 + a^{14} e_2 \wedge e_3 \wedge e_4 + a^{15} e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

We can then factor this expression to display it as a nested sum of numbers of the form $C_k == a^i + a^j e_1$. (Of course any basis element will do upon which to base the factorization.)

$$X == (a^0 + a^1 e_1) + (a^2 + a^5 e_1) \wedge e_2 + ((a^3 + a^6 e_1) + (a^8 + a^{11} e_1) \wedge e_2) \wedge e_3 + ((a^4 + a^7 e_1) + (a^9 + a^{12} e_1) \wedge e_2 + ((a^8 + a^{13} e_1) + (a^{14} + a^{15} e_1) \wedge e_2) \wedge e_3) \wedge e_4$$

Which can be rewritten in terms of the C_i as

$$X == C_1 + C_2 \wedge e_2 + (C_3 + C_4 \wedge e_2) \wedge e_3 + (C_5 + C_6 \wedge e_2 + (C_7 + C_8 \wedge e_2) \wedge e_3) \wedge e_4$$

Again, we can rewrite each of these elements as $Q_k == C_i + C_j \wedge e_2$ to get

$$X == Q_1 + Q_2 \wedge e_3 + (Q_3 + Q_4 \wedge e_3) \wedge e_4$$

Finally, we write $QQ_k == Q_i + Q_j \wedge e_3$ to get

$$X == QQ_1 + QQ_2 \wedge e_4$$

Since the basis 1-elements are orthogonal, we can replace the exterior product by the Clifford product to give

$$X == (a^0 + a^1 e_1) + (a^2 + a^5 e_1) \diamond e_2 + ((a^3 + a^6 e_1) + (a^8 + a^{11} e_1) \diamond e_2) \diamond e_3 + ((a^4 + a^7 e_1) + (a^9 + a^{12} e_1) \diamond e_2 + ((a^8 + a^{13} e_1) + (a^{14} + a^{15} e_1) \diamond e_2) \diamond e_3) \diamond e_4$$

$$== C_1 + C_2 \diamond e_2 + (C_3 + C_4 \diamond e_2) \diamond e_3 + (C_5 + C_6 \diamond e_2 + (C_7 + C_8 \diamond e_2) \diamond e_3) \diamond e_4$$

$$== Q_1 + Q_2 \diamond e_3 + (Q_3 + Q_4 \diamond e_3) \diamond e_4$$

$$== QQ_1 + QQ_2 \diamond e_4$$

12.17 Rotations

To be completed

13 Exploring Grassmann Matrix Algebra

13.1 Introduction

This chapter introduces the concept of a matrix of Grassmann numbers, which we will call a *Grassmann matrix*. Wherever it makes sense, the operations discussed will work also for listed collections of the components of tensors of any order as per the *Mathematica* representation: a set of lists nested to a certain number of levels. Thus, for example, an operation may also work for vectors or a list containing only one element.

We begin by discussing some quick methods for generating matrices of Grassmann numbers, particularly matrices of symbolic Grassmann numbers, where it can become tedious to define all the scalar factors involved. We then discuss the common algebraic operations like multiplication by scalars, addition, multiplication, taking the complement, finding the grade of elements, simplification, taking components or determining the type of elements involved.

Separate sections are devoted to discussing the notions of transpose, determinant and adjoint of a Grassmann matrix. These notions do not carry over directly into the algebra of Grassmann matrices due to the non-commutative nature of Grassmann numbers.

Matrix powers are then discussed, and the inverse of a Grassmann matrix defined in terms of its positive integer powers. Non-integer powers are defined for matrices whose bodies have distinct eigenvalues. The determination of the eigensystem of a Grassmann matrix is discussed for this class of matrix.

Finally, functions of Grassmann matrices are discussed based on the determination of their eigensystems. We show that relationships that we expect of scalars, for example Log[Exp[A]] = A or `Sin[A]`^{2} `+ Cos[A]`^{2} `= 1,` can still apply to Grassmann matrices.

13.2 Creating Matrices

Creating general symbolic matrices

Unless a Grassmann matrix is particularly simple, and particularly when a large symbolic matrix is required, it is useful to be able to generate the initial form of a the matrix automatically. We do this with the *GrassmannAlgebra* `CreateMatrixForm` function.

```
? CreateMatrixForm
```

```
CreateMatrixForm[D][X,S] constructs an array of the specified
  dimensions with copies of the expression X formed by indexing
  its scalars and variables with subscripts. S is an optional
  list of excluded symbols. D is a list of dimensions of
  the array (which may be symbolic for lists and matrices).
  Note that an indexed  scalar is not recognised as a scalar
  unless it has an underscripted 0, or is declared as a scalar.
```

Suppose we require a 2×2 matrix of general Grassmann numbers of the form **X** (given below) in a 2-space.

```
X = a + b e1 + c e2 + d e1 ∧ e2
```

$a + b\, e_1 + c\, e_2 + d\, e_1 \wedge e_2$

We declare the 2-space, and then create a 2×2 matrix of the *form* **X**.

```
𝕍2; M = CreateMatrixForm[{2, 2}][X]; MatrixForm[M]
```

$$\begin{pmatrix} a_{1,1} + e_1\, b_{1,1} + e_2\, c_{1,1} + d_{1,1}\, e_1 \wedge e_2 & a_{1,2} + e_1\, b_{1,2} + e_2\, c_{1,2} + d_{1,2}\, e_1 \wedge e_2 \\ a_{2,1} + e_1\, b_{2,1} + e_2\, c_{2,1} + d_{2,1}\, e_1 \wedge e_2 & a_{2,2} + e_1\, b_{2,2} + e_2\, c_{2,2} + d_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

We need to declare the extra scalars that we have formed. This can be done with just one pattern (provided we wish all symbols with this pattern to be scalar).

```
DeclareExtraScalars[{Subscript[_, _, _]}]
```

$\{a, b, c, d, e, f, g, h, \Bbbk, (\_ \ominus \_)?\text{InnerProductQ}, \_{}_{\_,\_}, \underset{0}{\_}\}$

Creating matrices with specific coefficients

To enter a Grassmann matrix with specific coefficients, enter a placeholder as the kernel symbol. An output will be returned which will be copied to an input form as soon as you attempt to edit it. You can select a placeholder and enter the value you want. The tab key can be used to select the next placeholder. The steps are:

- Generate the form you want.

```
Y = CreateGrassmannNumber[□]
```

$\square + \square\, e_1 + \square\, e_2 + \square\, e_1 \wedge e_2$

```
CreateMatrixForm[{2, 2}][Y]
```

$\{\{\square + \square\, e_1 + \square\, e_2 + \square\, e_1 \wedge e_2,\ \square + \square\, e_1 + \square\, e_2 + \square\, e_1 \wedge e_2\},$
$\ \{\square + \square\, e_1 + \square\, e_2 + \square\, e_1 \wedge e_2,\ \square + \square\, e_1 + \square\, e_2 + \square\, e_1 \wedge e_2\}\}$

- Start editing the form above.

```
{{2 + □ e1 + □ e2 + □ e1 ∧ e2, □ + □ e1 + □ e2 + □ e1 ∧ e2},
 {□ + □ e1 + □ e2 + □ e1 ∧ e2, □ + □ e1 + □ e2 + □ e1 ∧ e2}}
```

- Finish off your editing and give the matrix a name.

```
A = {{2 + 3 e1, e1 + 6 e2}, {5 - 9 e1 ∧ e2, e2 + e1 ∧ e2}}; MatrixForm[A]
```

$$\begin{pmatrix} 2+3 e_1 & e_1+6 e_2 \\ 5-9 e_1 \wedge e_2 & e_2+e_1 \wedge e_2 \end{pmatrix}$$

Creating matrices with variable elements

To create a matrix in which the Grassmann numbers are expressed in terms of **x**, **y**, and **z**, say, you can temporarily declare the basis as **{x,y,z}** before generating the form with `CreateGrassmannNumber`. For example:

```
B = Basis; DeclareBasis[{x, y, z}]
```

{x, y, z}

```
Z = CreateGrassmannNumber[□]
```

$\square + x\,\square + y\,\square + z\,\square + \square\, x \wedge y + \square\, x \wedge z + \square\, y \wedge z + \square\, x \wedge y \wedge z$

Here we create a vector of elements of the form **Z**.

```
CreateMatrixForm[{3}][Z] // MatrixForm
```

$$\begin{pmatrix} \square + x\,\square + y\,\square + z\,\square + \square\, x \wedge y + \square\, x \wedge z + \square\, y \wedge z + \square\, x \wedge y \wedge z \\ \square + x\,\square + y\,\square + z\,\square + \square\, x \wedge y + \square\, x \wedge z + \square\, y \wedge z + \square\, x \wedge y \wedge z \\ \square + x\,\square + y\,\square + z\,\square + \square\, x \wedge y + \square\, x \wedge z + \square\, y \wedge z + \square\, x \wedge y \wedge z \end{pmatrix}$$

Remember to return to your original basis.

```
DeclareBasis[B]
```

$\{e_1, e_2, e_3\}$

13.3 Manipulating Matrices

Matrix simplification

Simplification of the elements of a matrix may be effected by using `GrassmannSimplify`. For example, suppose we are working in a Euclidean 3-space and have a matrix **B**:

```
𝕍3; B = {{1 + x∧x, 2 + e1⊖e2}, {\overline{e1∧e2}, 3 + x∧y∧z∧w}};
MatrixForm[B]
```

$$\begin{pmatrix} 1 + x \wedge x & 2 + e_1 \ominus e_2 \\ \overline{e_1 \wedge e_2} & 3 + x \wedge y \wedge z \wedge w \end{pmatrix}$$

To simplify the elements of the matrix we enter:

```
𝒢[B]
```

$\{\{1, 2 + e_1 \ominus e_2\}, \{e_3, 3\}\}$

If we wish to simplify any orthogonal elements, we can use `ToMetricForm`. Since we are in a Euclidean space the scalar product term is zero.

```
ToMetricForm[𝒢[B]]
```

$\{\{1, 2\}, \{e_3, 3\}\}$

The grades of the elements of a matrix

The `GrassmannAlgebra` function `Grade` returns a matrix with each element replaced by a list of the *different* grades of its components. Thus, if an element is composed of several components of the same grade, only that one grade will be returned for that element. For example, take the matrix **A** created in the previous section:

```
A // MatrixForm
```

$$\begin{pmatrix} 2 + 3 e_1 & e_1 + 6 e_2 \\ 5 - 9 e_1 \wedge e_2 & e_2 + e_1 \wedge e_2 \end{pmatrix}$$

```
Grade[A] // MatrixForm
```

$$\begin{pmatrix} \{0, 1\} & 1 \\ \{0, 2\} & \{1, 2\} \end{pmatrix}$$

Taking components of a matrix

We take the matrix **M** created in Section 13.2.

```
M // MatrixForm
```

$$\begin{pmatrix} a_{1,1} + e_1\, b_{1,1} + e_2\, c_{1,1} + d_{1,1}\, e_1 \wedge e_2 & a_{1,2} + e_1\, b_{1,2} + e_2\, c_{1,2} + d_{1,2}\, e_1 \wedge e_2 \\ a_{2,1} + e_1\, b_{2,1} + e_2\, c_{2,1} + d_{2,1}\, e_1 \wedge e_2 & a_{2,2} + e_1\, b_{2,2} + e_2\, c_{2,2} + d_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

The body of **M** is given by:

```
Body[M] // MatrixForm
```

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

and its soul by:

```
Soul[M] // MatrixForm
```

$$\begin{pmatrix} e_1\, b_{1,1} + e_2\, c_{1,1} + d_{1,1}\, e_1 \wedge e_2 & e_1\, b_{1,2} + e_2\, c_{1,2} + d_{1,2}\, e_1 \wedge e_2 \\ e_1\, b_{2,1} + e_2\, c_{2,1} + d_{2,1}\, e_1 \wedge e_2 & e_1\, b_{2,2} + e_2\, c_{2,2} + d_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

The even components are extracted by using `EvenGrade`.

```
EvenGrade[M] // MatrixForm
```

$$\begin{pmatrix} a_{1,1} + d_{1,1}\, e_1 \wedge e_2 & a_{1,2} + d_{1,2}\, e_1 \wedge e_2 \\ a_{2,1} + d_{2,1}\, e_1 \wedge e_2 & a_{2,2} + d_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

The odd components are extracted by using `OddGrade`.

```
OddGrade[M] // MatrixForm
```

$$\begin{pmatrix} e_1\, b_{1,1} + e_2\, c_{1,1} & e_1\, b_{1,2} + e_2\, c_{1,2} \\ e_1\, b_{2,1} + e_2\, c_{2,1} & e_1\, b_{2,2} + e_2\, c_{2,2} \end{pmatrix}$$

Finally, we can extract the elements of any specific grade using `ExtractGrade`.

```
ExtractGrade[2][M] // MatrixForm
```

$$\begin{pmatrix} d_{1,1}\, e_1 \wedge e_2 & d_{1,2}\, e_1 \wedge e_2 \\ d_{2,1}\, e_1 \wedge e_2 & d_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

Determining the type of elements

We take the matrix **A** created in Section 13.2.

```
A // MatrixForm
```

$$\begin{pmatrix} 2+3\,e_1 & e_1+6\,e_2 \\ 5-9\,e_1\wedge e_2 & e_2+e_1\wedge e_2 \end{pmatrix}$$

`EvenGradeQ`, `OddGradeQ`, `EvenSoulQ` and `GradeQ` all interrogate the individual elements of a matrix.

```
EvenGradeQ[A] // MatrixForm
```

$$\begin{pmatrix} \text{False} & \text{False} \\ \text{True} & \text{False} \end{pmatrix}$$

```
OddGradeQ[A] // MatrixForm
```

$$\begin{pmatrix} \text{False} & \text{True} \\ \text{False} & \text{False} \end{pmatrix}$$

```
EvenSoulQ[A] // MatrixForm
```

$$\begin{pmatrix} \text{False} & \text{False} \\ \text{False} & \text{False} \end{pmatrix}$$

```
GradeQ[1][A] // MatrixForm
```

$$\begin{pmatrix} \text{False} & \text{True} \\ \text{False} & \text{False} \end{pmatrix}$$

To check the type of a complete matrix we can ask whether it is free of either `True` or `False` values.

```
FreeQ[GradeQ[1][A], False]
```

```
False
```

```
FreeQ[EvenSoulQ[A], True]
```

```
True
```

13.4 Matrix Algebra

Multiplication by scalars

Due to the `Listable` attribute of the `Times` operation in *Mathematica*, multiplication by scalars behaves as expected. For example, multiplying the Grassmann matrix **A** created in Section 13.2 by the scalar **a** gives:

```
A // MatrixForm
```

$$\begin{pmatrix} 2+3\,e_1 & e_1+6\,e_2 \\ 5-9\,e_1\wedge e_2 & e_2+e_1\wedge e_2 \end{pmatrix}$$

```
a A // MatrixForm
```

$$\begin{pmatrix} a\,(2+3\,e_1) & a\,(e_1+6\,e_2) \\ a\,(5-9\,e_1\wedge e_2) & a\,(e_2+e_1\wedge e_2) \end{pmatrix}$$

Matrix addition

Due to the `Listable` attribute of the `Plus` operation in *Mathematica*, matrices of the same size add automatically.

```
A + M
```

$$\{\{2+3\,e_1+a_{1,1}+e_1\,b_{1,1}+e_2\,c_{1,1}+d_{1,1}\,e_1\wedge e_2,\\ e_1+6\,e_2+a_{1,2}+e_1\,b_{1,2}+e_2\,c_{1,2}+d_{1,2}\,e_1\wedge e_2\},\\ \{5+a_{2,1}+e_1\,b_{2,1}+e_2\,c_{2,1}-9\,e_1\wedge e_2+d_{2,1}\,e_1\wedge e_2,\\ e_2+a_{2,2}+e_1\,b_{2,2}+e_2\,c_{2,2}+e_1\wedge e_2+d_{2,2}\,e_1\wedge e_2\}\}$$

We can collect like terms by using `GrassmannSimplify`.

```
𝒢[A + M]
```

$$\{\{2+a_{1,1}+e_1\,(3+b_{1,1})+e_2\,c_{1,1}+d_{1,1}\,e_1\wedge e_2,\\ a_{1,2}+e_1\,(1+b_{1,2})+e_2\,(6+c_{1,2})+d_{1,2}\,e_1\wedge e_2\},\\ \{5+a_{2,1}+e_1\,b_{2,1}+e_2\,c_{2,1}+(-9+d_{2,1})\,e_1\wedge e_2,\\ a_{2,2}+e_1\,b_{2,2}+e_2\,(1+c_{2,2})+(1+d_{2,2})\,e_1\wedge e_2\}\}$$

The complement of a matrix

Since the complement is a linear operation, the complement of a matrix is just the matrix of the complements of the elements.

```
M̄ // MatrixForm
```

$$\begin{pmatrix} \overline{a_{1,1} + e_1\, b_{1,1} + e_2\, c_{1,1} + d_{1,1}\, e_1 \wedge e_2} & \overline{a_{1,2} + e_1\, b_{1,2} + e_2\, c_{1,2} + d_{1,2}\, e_1 \wedge e_2} \\ \overline{a_{2,1} + e_1\, b_{2,1} + e_2\, c_{2,1} + d_{2,1}\, e_1 \wedge e_2} & \overline{a_{2,2} + e_1\, b_{2,2} + e_2\, c_{2,2} + d_{2,2}\, e_1 \wedge e_2} \end{pmatrix}$$

This can be simplified by using GrassmannSimplify. In a Euclidean space this gives:

```
𝒢[M̄] // MatrixForm
```

$$\begin{pmatrix} e_2\, b_{1,1} - e_1\, c_{1,1} + d_{1,1} + a_{1,1}\, e_1 \wedge e_2 & e_2\, b_{1,2} - e_1\, c_{1,2} + d_{1,2} + a_{1,2}\, e_1 \wedge e_2 \\ e_2\, b_{2,1} - e_1\, c_{2,1} + d_{2,1} + a_{2,1}\, e_1 \wedge e_2 & e_2\, b_{2,2} - e_1\, c_{2,2} + d_{2,2} + a_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

Matrix products

As with the product of two Grassmann numbers, entering the product of two matrices of Grassmann numbers, does not effect any multiplication. For example, take the matrix **A** in a 2-space created in the previous section.

```
MatrixForm[A] ∧ MatrixForm[A]
```

$$\begin{pmatrix} 2 + 3\, e_1 & e_1 + 6\, e_2 \\ 5 - 9\, e_1 \wedge e_2 & e_2 + e_1 \wedge e_2 \end{pmatrix} \wedge \begin{pmatrix} 2 + 3\, e_1 & e_1 + 6\, e_2 \\ 5 - 9\, e_1 \wedge e_2 & e_2 + e_1 \wedge e_2 \end{pmatrix}$$

To multiply out the product **A∧A**, we use the *GrassmannAlgebra* `MatrixProduct` function which performs the multiplication. The short-hand for MatrixProduct is a script capital M $\mathcal{M}$, obtained by typing ⎋scM⎋.

```
ℳ[A ∧ A]
```

$$\{\{(2 + 3\, e_1) \wedge (2 + 3\, e_1) + (e_1 + 6\, e_2) \wedge (5 - 9\, e_1 \wedge e_2),$$
$$(2 + 3\, e_1) \wedge (e_1 + 6\, e_2) + (e_1 + 6\, e_2) \wedge (e_2 + e_1 \wedge e_2)\},$$
$$\{(5 - 9\, e_1 \wedge e_2) \wedge (2 + 3\, e_1) + (e_2 + e_1 \wedge e_2) \wedge (5 - 9\, e_1 \wedge e_2),$$
$$(5 - 9\, e_1 \wedge e_2) \wedge (e_1 + 6\, e_2) + (e_2 + e_1 \wedge e_2) \wedge (e_2 + e_1 \wedge e_2)\}\}$$

To take the product and simplify at the same time, apply the `GrassmannSimplify` function.

```
𝒢[ℳ[A ∧ A]] // MatrixForm
```

$$\begin{pmatrix} 4+17 e_1+30 e_2 & 2 e_1+12 e_2+19 e_1 \wedge e_2 \\ 10+15 e_1+5 e_2-13 e_1 \wedge e_2 & 5 e_1+30 e_2 \end{pmatrix}$$

Regressive products

We can also take the regressive product.

```
𝒢[ℳ[A ∨ A]] // MatrixForm
```

$$\begin{pmatrix} e_1 \wedge e_2 \vee (-9 e_1-54 e_2) & e_1 \wedge e_2 \vee (19+e_1+6 e_2) \\ e_1 \wedge e_2 \vee (-13-27 e_1-9 e_2-9 e_1 \wedge e_2) & e_1 \wedge e_2 \vee (-9 e_1-52 e_2+e_1 \wedge e_2) \end{pmatrix}$$

Interior products

We calculate an interior product of two matrices in the same way.

```
P = 𝒢[ℳ[A ⊖ A]]
```

$$\{\{4+9 (e_1 \ominus e_1)+11 e_1+30 e_2,\ 3 (e_1 \ominus e_1)+19 (e_1 \ominus e_2)+6 (e_2 \ominus e_2)\},$$
$$\{10-27 (e_1 \wedge e_2 \ominus e_1)-9 (e_1 \wedge e_2 \ominus e_1 \wedge e_2)+5 e_2-13 e_1 \wedge e_2,$$
$$e_2 \ominus e_2-9 (e_1 \wedge e_2 \ominus e_1)-53 (e_1 \wedge e_2 \ominus e_2)+e_1 \wedge e_2 \ominus e_1 \wedge e_2\}\}$$

To convert the interior products to scalar products, use the *GrassmannAlgebra* function `ToScalarProducts` on the matrix.

```
P1 = ToScalarProducts[P]
```

$$\{\{4+9 (e_1 \ominus e_1)+11 e_1+30 e_2,\ 3 (e_1 \ominus e_1)+19 (e_1 \ominus e_2)+6 (e_2 \ominus e_2)\},$$
$$\{10+9 (e_1 \ominus e_2)^2-9 (e_1 \ominus e_1) (e_2 \ominus e_2)+27 (e_1 \ominus e_2) e_1+5 e_2-$$
$$27 (e_1 \ominus e_1) e_2-13 e_1 \wedge e_2,\ -(e_1 \ominus e_2)^2+e_2 \ominus e_2+(e_1 \ominus e_1) (e_2 \ominus e_2)+$$
$$9 (e_1 \ominus e_2) e_1+53 (e_2 \ominus e_2) e_1-9 (e_1 \ominus e_1) e_2-53 (e_1 \ominus e_2) e_2\}\}$$

This product can be simplified further if the values of the scalar products of the basis elements are known. In a Euclidean space, we obtain:

```
ToMetricForm[P1] // MatrixForm
```

$$\begin{pmatrix} 13+11 e_1+30 e_2 & 9 \\ 1-22 e_2-13 e_1 \wedge e_2 & 2+53 e_1-9 e_2 \end{pmatrix}$$

Clifford products

The Clifford product of two matrices can be calculated in stages. First we multiply the elements and expand and simplify the terms with `GrassmannSimplify`.

```
C1 = 𝒢[ℳ[A ◇ A]]
```

$$\{\{2\diamond 2+3\;2\diamond e_1+3\;e_1\diamond 2+e_1\diamond 5+9\;e_1\diamond e_1-9\;e_1\diamond(e_1\wedge e_2)+6\;e_2\diamond 5-54\;e_2\diamond(e_1\wedge e_2),\;2\diamond e_1+6\;2\diamond e_2+3\;e_1\diamond e_1+19\;e_1\diamond e_2+e_1\diamond(e_1\wedge e_2)+6\;e_2\diamond e_2+6\;e_2\diamond(e_1\wedge e_2)\},$$
$$\{5\diamond 2+3\;5\diamond e_1+e_2\diamond 5-9\;e_2\diamond(e_1\wedge e_2)-9\;(e_1\wedge e_2)\diamond 2+(e_1\wedge e_2)\diamond 5-27\;(e_1\wedge e_2)\diamond e_1-9\;(e_1\wedge e_2)\diamond(e_1\wedge e_2),$$
$$5\diamond e_1+6\;5\diamond e_2+e_2\diamond e_2+e_2\diamond(e_1\wedge e_2)-9\;(e_1\wedge e_2)\diamond e_1-53\;(e_1\wedge e_2)\diamond e_2+(e_1\wedge e_2)\diamond(e_1\wedge e_2)\}\}$$

Next we can convert the Clifford products into exterior and scalar products with `ToScalarProducts`.

```
C2 = ToScalarProducts[C1]
```

$$\{\{4+9\;(e_1\ominus e_1)+17\;e_1+9\;(e_1\ominus e_2)\;e_1+54\;(e_2\ominus e_2)\;e_1+30\;e_2-9\;(e_1\ominus e_1)\;e_2-54\;(e_1\ominus e_2)\;e_2,$$
$$3\;(e_1\ominus e_1)+19\;(e_1\ominus e_2)+6\;(e_2\ominus e_2)+2\;e_1-(e_1\ominus e_2)\;e_1-6\;(e_2\ominus e_2)\;e_1+12\;e_2+(e_1\ominus e_1)\;e_2+6\;(e_1\ominus e_2)\;e_2+19\;e_1\wedge e_2\},$$
$$\{10-9\;(e_1\ominus e_2)^2+9\;(e_1\ominus e_1)\;(e_2\ominus e_2)+15\;e_1-27\;(e_1\ominus e_2)\;e_1+9\;(e_2\ominus e_2)\;e_1+5\;e_2+27\;(e_1\ominus e_1)\;e_2-9\;(e_1\ominus e_2)\;e_2-13\;e_1\wedge e_2,$$
$$(e_1\ominus e_2)^2+e_2\ominus e_2-(e_1\ominus e_1)\;(e_2\ominus e_2)+5\;e_1-9\;(e_1\ominus e_2)\;e_1-54\;(e_2\ominus e_2)\;e_1+30\;e_2+9\;(e_1\ominus e_1)\;e_2+54\;(e_1\ominus e_2)\;e_2\}\}$$

Finally, if we are in a Euclidean space, this simplifies significantly.

```
ToMetricForm[C2] // MatrixForm
```

$$\begin{pmatrix} 13+71\;e_1+21\;e_2 & 9-4\;e_1+13\;e_2+19\;e_1\wedge e_2 \\ 19+24\;e_1+32\;e_2-13\;e_1\wedge e_2 & -49\;e_1+39\;e_2 \end{pmatrix}$$

13.5 The Transpose

Conditions on the transpose of an exterior product

If we adopt the usual definition of the transpose of a matrix as being the matrix where rows become columns and columns become rows, we find that, in general, we lose the relation for the transpose of a product being the product of the transposes in reverse order. To see under what conditions this does indeed remain true we break the matrices up into their even and odd components. Let $A == A_e + A_o$ and $B == B_e + B_o$, where the subscript *e* refers to the even component and *o* refers to the odd component. Then

$$(A \wedge B)^T == ((A_e + A_o) \wedge (B_e + B_o))^T == (A_e \wedge B_e + A_e \wedge B_o + A_o \wedge B_e + A_o \wedge B_o)^T ==$$
$$(A_e \wedge B_e)^T + (A_e \wedge B_o)^T + (A_o \wedge B_e)^T + (A_o \wedge B_o)^T$$

$$B^T \wedge A^T == (B_e + B_o)^T \wedge (A_e + A_o)^T ==$$
$$(B_e^T + B_o^T) \wedge (A_e^T + A_o^T) == B_e^T \wedge A_e^T + B_e^T \wedge A_o^T + B_o^T \wedge A_e^T + B_o^T \wedge A_o^T$$

If the elements of two Grassmann matrices commute, then just as for the usual case, the transpose of a product is the product of the transposes in reverse order. If they anti-commute, then the transpose of a product is the *negative* of the product of the transposes in reverse order. Thus, the corresponding terms in the two expansions above which involve an even matrix will be equal, and the last terms involving two odd matrices will differ by a sign:

$$(A_e \wedge B_e)^T == B_e^T \wedge A_e^T$$

$$(A_e \wedge B_o)^T == B_o^T \wedge A_e^T$$

$$(A_o \wedge B_e)^T == B_e^T \wedge A_o^T$$

$$(A_o \wedge B_o)^T == -(B_o^T \wedge A_o^T)$$

Thus

$$\boxed{\begin{gathered}(A \wedge B)^T == B^T \wedge A^T - 2\,(B_o^T \wedge A_o^T) == \\ B^T \wedge A^T + 2\,(A_o \wedge B_o)^T\end{gathered}} \qquad \mathbf{13.1}$$

$$(A \wedge B)^T == B^T \wedge A^T \quad \Leftrightarrow \quad A_o \wedge B_o == 0 \qquad 13.2$$

Checking the transpose of a product

It is useful to be able to run a check on a theoretically derived relation such as this. To reduce duplication effort in the checking of several cases we devise a test function called `TestTransposeRelation`.

```
TestTransposeRelation[A_,B_]:=
Module[{T},T=Transpose;
𝒢[T[ℳ[A∧B]]]== 𝒢[ℳ[T[B]∧T[A]]-2ℳ[T[OddGrade[B]]∧T[OddGrade[A]]]]]
```

As an example, we test it for two 2×2 matrices in 2-space:

A // MatrixForm

$$\begin{pmatrix} 2+3e_1 & e_1+6e_2 \\ 5-9e_1\wedge e_2 & e_2+e_1\wedge e_2 \end{pmatrix}$$

M // MatrixForm

$$\begin{pmatrix} a_{1,1}+e_1 b_{1,1}+e_2 c_{1,1}+d_{1,1} e_1\wedge e_2 & a_{1,2}+e_1 b_{1,2}+e_2 c_{1,2}+d_{1,2} e_1\wedge e_2 \\ a_{2,1}+e_1 b_{2,1}+e_2 c_{2,1}+d_{2,1} e_1\wedge e_2 & a_{2,2}+e_1 b_{2,2}+e_2 c_{2,2}+d_{2,2} e_1\wedge e_2 \end{pmatrix}$$

TestTransposeRelation[M, A]

True

It should be remarked that for the transpose of a product to be equal to the product of the transposes in reverse order, the evenness of the matrices is a sufficient condition, but not a *necessary* one. All that is required is that the *exterior product of the odd components* of the matrices be zero. This may be achieved without the odd components themselves necessarily being zero.

Conditions on the determinant of a general matrix

Due to the non-commutativity of Grassmann numbers, the notion of determinant does not carry over directly to matrices of Grassmann numbers. To see why, consider the diagonal Grassmann matrix composed of 1-elements **x** and **y**:

$$\begin{pmatrix} \mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{y} \end{pmatrix}$$

The natural definition of the determinant of this matrix would be the product of the diagonal elements. However, which product should it be, $\mathbf{x}\wedge\mathbf{y}$ or $\mathbf{y}\wedge\mathbf{x}$?. The possible non-commutativity of the Grassmann numbers on the diagonal may give rise to different products of the diagonal elements, depending on the ordering of the elements.

On the other hand, if the product of the diagonal elements of a diagonal Grassmann matrix is unique, as would be the case for even elements, the determinant may be defined uniquely as that product.

Determinants of even Grassmann matrices

If a Grassmann matrix is even, its elements commute and its determinant may be defined in the same way as for matrices of scalars.

To calculate the determinant of an even Grassmann matrix, we can use the same basic algorithm as for scalar matrices. In *GrassmannAlgebra* this is implemented with the function `GrassmannDeterminant`. For example:

```
B = EvenGrade[M]; MatrixForm[B]
```

$$\begin{pmatrix} a_{1,1} + d_{1,1}\, e_1 \wedge e_2 & a_{1,2} + d_{1,2}\, e_1 \wedge e_2 \\ a_{2,1} + d_{2,1}\, e_1 \wedge e_2 & a_{2,2} + d_{2,2}\, e_1 \wedge e_2 \end{pmatrix}$$

```
B1 = GrassmannDeterminant[B]
```

$$(a_{1,1} + d_{1,1}\, e_1 \wedge e_2) \wedge (a_{2,2} + d_{2,2}\, e_1 \wedge e_2) - (a_{1,2} + d_{1,2}\, e_1 \wedge e_2) \wedge (a_{2,1} + d_{2,1}\, e_1 \wedge e_2)$$

Note that `GrassmannDeterminant` does not carry out any simplification, but instead leaves the products in raw form to facilitate the interpretation of the process occurring. To simplify the result one needs to use `GrassmannSimplify`.

```
𝒢[B1]
```

$-a_{1,2}\,a_{2,1}+a_{1,1}\,a_{2,2}+(a_{2,2}\,d_{1,1}-a_{2,1}\,d_{1,2}-a_{1,2}\,d_{2,1}+a_{1,1}\,d_{2,2})\,e_1\wedge e_2$

13.6 Matrix Powers

Positive integer powers

Powers of matrices with no body

Positive integer powers of a matrix of Grassmann numbers are simply calculated by taking the exterior product of the matrix with itself the requisite number of times - and then using `GrassmannSimplify` to compute and simplify the result.

We begin in a 4-space. For example, suppose we let

```
𝕍4; A = {{x, y}, {u, v}}; MatrixForm[A]
```

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix}$$

The second power is:

```
ℳ[A ∧ A]
```

$\{\{x\wedge x+y\wedge u,\ x\wedge y+y\wedge v\},\ \{u\wedge x+v\wedge u,\ u\wedge y+v\wedge v\}\}$

When simplified this becomes:

```
A2 = 𝒢[ℳ[A ∧ A]]; MatrixForm[A2]
```

$$\begin{pmatrix} -(u\wedge y) & -(v\wedge y)+x\wedge y \\ -(u\wedge v)+u\wedge x & u\wedge y \end{pmatrix}$$

We can extend this process to form higher powers:

```
A3 = 𝒢[ℳ[A ∧ A2]]; MatrixForm[A3]
```

$$\begin{pmatrix} -(u\wedge v\wedge y)+2\,u\wedge x\wedge y & v\wedge x\wedge y \\ -(u\wedge v\wedge x) & -2\,u\wedge v\wedge y+u\wedge x\wedge y \end{pmatrix}$$

A3 = 𝒢[A ∧ A2]; MatrixForm[A3]

$$\begin{pmatrix} -(u\wedge v\wedge y)+2\,u\wedge x\wedge y & v\wedge x\wedge y \\ -(u\wedge v\wedge x) & -2\,u\wedge v\wedge y+u\wedge x\wedge y \end{pmatrix}$$

Because this matrix has no body, there will eventually be a power which is zero, dependent on the dimension of the space, which in this case is 4. Here we see that it is the fourth power.

A4 = 𝒢[ℳ[A ∧ A3]]; MatrixForm[A4]

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Powers of matrices with a body

By giving the matrix a body, we can see that higher powers may no longer be zero.

B = A + IdentityMatrix[2]; MatrixForm[B]

$$\begin{pmatrix} 1+x & y \\ u & 1+v \end{pmatrix}$$

B4 = 𝒢[ℳ[B ∧ ℳ[B ∧ ℳ[B ∧ B]]]]; MatrixForm[B4]

$$\begin{pmatrix} 1+4\,x-6\,u\wedge y-4\,u\wedge v\wedge y+8\,u\wedge x\wedge y & 4\,y-6\,v\wedge y+6\,x\wedge y+4\,v\wedge x\wedge y \\ 4\,u-6\,u\wedge v+6\,u\wedge x-4\,u\wedge v\wedge x & 1+4\,v+6\,u\wedge y-8\,u\wedge v\wedge y+4\,u\wedge x \end{pmatrix}$$

GrassmannIntegerPower

Although we will see later that the *GrassmannAlgebra* function `GrassmannMatrixPower` will actually deal with more general cases than the positive integer powers discussed here, we can write a simple function `GrassmannIntegerPower` to calculate any integer power. Like `GrassmannMatrixPower` we can also easily store the result for further use in the same *Mathematica* session. This makes subsequent calculations involving the powers already calculated much faster.

```
GrassmannIntegerPower[M_,n_?PositiveIntegerQ]:=
Module[{P},P=NestList[𝒢[ℳ[M∧#]]&,M,n-1];
    GrassmannIntegerPower[M,m_,Dimension]:=First[Take[P,{m}]];
    Last[P]]
```

As an example we take the matrix **B** created above.

```
B
```

```
{{1 + x, y}, {u, 1 + v}}
```

Now we apply `GrassmannIntegerPower` to obtain the fourth power of **B**.

```
GrassmannIntegerPower[B, 4] // MatrixForm
```

$$\begin{pmatrix} 1+4x-6u\wedge y-4u\wedge v\wedge y+8u\wedge x\wedge y & 4y-6v\wedge y+6x\wedge y+4v\wedge x\wedge y \\ 4u-6u\wedge v+6u\wedge x-4u\wedge v\wedge x & 1+4v+6u\wedge y-8u\wedge v\wedge y+4u\wedge x \end{pmatrix}$$

However, on the way to calculating the fourth power, the function `GrassmannIntegerPower` has remembered the values of the all the integer powers up to the fourth, in this case the second, third and fourth. We can get immediate access to these by adding the `Dimension` of the space as a third argument. Since the power of a matrix may take on a different form in spaces of different dimensions (due to some terms being zero because their degree exceeds the dimension of the space), the power is recalled only by including the `Dimension` as the third argument.

```
Dimension
```

```
4
```

```
GrassmannIntegerPower[B, 2, 4] // MatrixForm // Timing
```

$$\left\{0.\ \text{Second}, \begin{pmatrix} 1+2x-u\wedge y & 2y-v\wedge y+x\wedge y \\ 2u-u\wedge v+u\wedge x & 1+2v+u\wedge y \end{pmatrix}\right\}$$

```
GrassmannIntegerPower[B, 3, 4] // MatrixForm // Timing
```

$$\left\{0.\ \text{Second}, \begin{pmatrix} 1+3x-3u\wedge y-u\wedge v\wedge y+2u\wedge x\wedge y & 3y-3v\wedge y+3x\wedge y \\ 3u-3u\wedge v+3u\wedge x-u\wedge v\wedge x & 1+3v+3u\wedge y-2u\wedge v \end{pmatrix}\right.$$

```
GrassmannIntegerPower[B, 4, 4] // MatrixForm // Timing
```

$$\left\{0.\ \text{Second}, \begin{pmatrix} 1+4x-6u\wedge y-4u\wedge v\wedge y+8u\wedge x\wedge y & 4y-6v\wedge y+6x\wedge \\ 4u-6u\wedge v+6u\wedge x-4u\wedge v\wedge x & 1+4v+6u\wedge y-8u\wedge \end{pmatrix}\right.$$

GrassmannMatrixPower

The principal and more general function for calculating matrix powers provided by *GrassmannAlgebra* is `GrassmannMatrixPower`. Here we verify that it gives the same result as our simple `GrassmannIntegerPower` function.

```
GrassmannMatrixPower[B, 4]
```

```
{{1 + 4 x - 6 u ∧ y - 4 u ∧ v ∧ y + 8 u ∧ x ∧ y, 4 y - 6 v ∧ y + 6 x ∧ y + 4 v ∧ x ∧ y},
 {4 u - 6 u ∧ v + 6 u ∧ x - 4 u ∧ v ∧ x, 1 + 4 v + 6 u ∧ y - 8 u ∧ v ∧ y + 4 u ∧ x ∧ y}}
```

Negative integer powers

From this point onwards we will be discussing the application of the general *GrassmannAlgebra* function `GrassmannMatrixPower`.

We take as an example the following simple 2×2 matrix.

```
A = {{1, x}, {y, 1}}; MatrixForm[A]
```

$$\begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}$$

The matrix A^{-4} is calculated by `GrassmannMatrixPower` by taking the fourth power of the inverse.

```
iA4 = GrassmannMatrixPower[A, -4]
```

```
{{1 + 10 x ∧ y, -4 x}, {-4 y, 1 - 10 x ∧ y}}
```

We could of course get the same result by taking the inverse of the fourth power.

```
GrassmannMatrixInverse[GrassmannMatrixPower[A, 4]]
```

```
{{1 + 10 x ∧ y, -4 x}, {-4 y, 1 - 10 x ∧ y}}
```

We check that this is indeed the inverse.

```
𝒢[ℳ[iA4 ∧ GrassmannMatrixPower[A, 4]]]
```

```
{{1, 0}, {0, 1}}
```

Non-integer powers of matrices with distinct eigenvalues

If the matrix has distinct eigenvalues, `GrassmannMatrixPower` will use `GrassmannMatrixFunction` to compute any non-integer power of a matrix of Grassmann numbers.

Let **A** be the 2×2 matrix which differs from the matrix discussed above by having distinct eigenvalues 1 and 2. We will discuss eigenvalues in Section 13.9 below.

```
A = {{1, x}, {y, 2}}; MatrixForm[A]
```

$$\begin{pmatrix} 1 & x \\ y & 2 \end{pmatrix}$$

We compute the square root of **A**.

```
sqrtA = GrassmannMatrixPower[A, 1/2]
```

$$\left\{\left\{1 + \left(-\frac{3}{2} + \sqrt{2}\right) x \wedge y,\ \left(-1 + \sqrt{2}\right) x\right\},\ \left\{\left(-1 + \sqrt{2}\right) y,\ \sqrt{2} + \frac{1}{4}\left(-4 + 3\sqrt{2}\right) x \wedge y\right\}\right\}$$

To verify that this is indeed the square root of **A** we 'square' it.

```
𝒢[ℳ[sqrtA ∧ sqrtA]]
```

More generally, we can extract a symbolic *p*th root. But to have the simplest result we need to ensure that *p* has been declared a scalar first,

```
DeclareExtraScalars[{p}]
```

$$\{a, b, c, d, e, f, g, h, p, \mathbb{k}, (\_ \ominus \_)?\text{InnerProductQ}, \underset{0}{\_}\}$$

```
pthRootA = GrassmannMatrixPower[A, 1/p]
```

$$\left\{\left\{1 + \left(-1 + 2^{\frac{1}{p}} - \frac{1}{p}\right) x \wedge y,\ \left(-1 + 2^{\frac{1}{p}}\right) x\right\},\ \left\{\left(-1 + 2^{\frac{1}{p}}\right) y,\ 2^{\frac{1}{p}} + \left(-1 + 2^{\frac{1}{p}} - \frac{2^{-1+\frac{1}{p}}}{p}\right) x \wedge y\right\}\right\}$$

We can verify that this gives us the previous result when $p = 2$.

```
sqrtA == pthRootA /. p → 2 // Simplify
```

True

If the power required is not integral and the eigenvalues are not distinct, `GrassmannMatrixPower` will return a message and the unevaluated input. Let **B** be a simple 2×2 matrix with equal eigenvalues 1 and 1

```
B = {{1, x}, {y, 1}}; MatrixForm[B]
```

$$\begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}$$

```
GrassmannMatrixPower[B, 1/2]
```

```
Eigenvalues::notDistinct :
 The matrix {{1, x}, {y, 1}} does not have distinct scalar eigenvalues. The
   operation applies only to matrices with distinct scalar eigenvalues.
```

```
GrassmannMatrixPower[{{1, x}, {y, 1}}, 1/2]
```

Integer powers of matrices with distinct eigenvalues

In some circumstances, if a matrix is known to have distinct eigenvalues, it will be more efficient to compute powers using `GrassmannMatrixFunction` for the basic calculation engine. The function of *GrassmannAlgebra* which enables this is `GrassmannDistinctEigenvalueMatrixPower`.

```
? GrassmannDistinctEigenvaluesMatrixPower
```

```
GrassmannDistinctEigenvaluesMatrixPower[A,p]
  calculates the power p of a Grassmann matrix A with
  distinct eigenvalues. p may be either numeric or symbolic.
```

We can check whether a matrix has distinct eigenvalues with `DistinctEigenvaluesQ`.

For example, suppose we wish to calculate the 100th power of the matrix **A** below.

```
A // MatrixForm
```

$$\begin{pmatrix} 1 & x \\ y & 2 \end{pmatrix}$$

```
DistinctEigenvaluesQ[A]
```

```
True
```

```
A

{{1, x}, {y, 2}}

GrassmannDistinctEigenvaluesMatrixPower[A, 100]

{{1 + 1267650600228229401496703205275 x ∧ y,
  1267650600228229401496703205375 x},
 {1267650600228229401496703205375 y,
  1267650600228229401496703205376 -
   62114879411183240673338457063425 x ∧ y}}
```

13.7 Matrix Inverses

A formula for the matrix inverse

We can develop a formula for the inverse of a matrix of Grassmann numbers following the same approach that we used for calculating the inverse of a Grassmann number. Suppose $\mathbf{I}$ is the identity matrix, X_k is a bodiless matrix of Grassmann numbers and X_k^i is its *i*th exterior power. Then we can write:

$$(\mathbf{I} + X_k) \wedge (\mathbf{I} - X_k + X_k^2 - X_k^3 + X_k^4 - \ \ldots \pm X_k^q) == \mathbf{I} \pm X_k^{q+1}$$

Now, since X_k has no body, its highest non-zero power will be equal to the dimension n of the space. That is $X_k^{n+1} == 0$. Thus

$$\boxed{(\mathbf{I} + X_k) \wedge (\mathbf{I} - X_k + X_k^2 - X_k^3 + X_k^4 - \ \ldots \pm X_k^n) == \mathbf{I}} \quad 13.3$$

We have thus shown that for a bodiless X_k, $(\mathbf{I} - X_k + X_k^2 - X_k^3 + X_k^4 - \ \ldots \pm X_k^n)$ is the inverse of $(\mathbf{I} + X_k)$.

If now we have a general Grassmann matrix X, say, we can write X as $X == X_b + X_s$, where X_b is the body of X and X_s is the soul of X, and then take X_b out as a factor to get:

$$X == X_b + X_s == X_b\ (\mathbf{I} + X_b^{-1}\, X_s) == X_b\ (\mathbf{I} + X_k)$$

Pre- and post-multiplying the equation above for the inverse of $(\mathbf{I} + X_k)$ by X_b and X_b^{-1} respectively gives:

$$X_b\ (I + X_k) \wedge (I - X_k + X_k{}^2 - X_k{}^3 + X_k{}^4 - \ \ldots \pm X_k{}^n)\ X_b{}^{-1} == I$$

Since the first factor is simply X, we finally obtain the inverse of X in terms of $X_k == X_b{}^{-1}\ X_s$ as:

$$X^{-1} == (I - X_k + X_k{}^2 - X_k{}^3 + X_k{}^4 - \ \ldots \pm X_k{}^n)\ X_b{}^{-1} \qquad 13.4$$

Or, specifically in terms of the body and soul of X:

$$X^{-1} == \left(I - (X_b{}^{-1}\ X_s) + (X_b{}^{-1}\ X_s)^2 - \ \ldots \pm (X_b{}^{-1}\ X_s)^n\right) X_b{}^{-1} \qquad 13.5$$

GrassmannMatrixInverse

This formula is straightforward to implement as a function. In *GrassmannAlgebra* we can calculate the inverse of a Grassmann matrix X by using `GrassmannMatrixInverse`.

```
? GrassmannMatrixInverse
```

```
GrassmannMatrixInverse[X] computes the
  inverse of the matrix X of Grassmann numbers in a
  space of the currently declared number of dimensions.
```

A definition for `GrassmannMatrixInverse` may be quite straighforwardly developed from formula 13.4.

```
GrassmannMatrixInverse[X_]:=
Module[{B,S,iB,iBS,K},B=Body[X];S=Soul[X];
iB=Inverse[B];iBS=𝒢[ℳ[iB∧S]];
K=Sum[(-1)^i GrassmannMatrixPower[iBS,i],{i,0,Dimension}];
𝒢[ℳ[K∧iB]]]
```

As a first simple example we take the matrix:

```
A = {{1, x}, {y, 1}}; MatrixForm[A]
```

$$\begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}$$

The inverse of **A** is:

```
iA = GrassmannMatrixInverse[A]; MatrixForm[iA]
```

$$\begin{pmatrix} 1 + x \wedge y & -x \\ -y & 1 - x \wedge y \end{pmatrix}$$

We can check that this is indeed the inverse.

```
𝒢[{ℳ[A ∧ iA], ℳ[iA ∧ A]}]
```

{{{1, 0}, {0, 1}}, {{1, 0}, {0, 1}}}

As a second example we take a somewhat more complex matrix.

```
A = {{1 + e1, 2, e3 ∧ e2}, {3, e2 ∧ e4, e4}, {e2, e1, 1 + e4 ∧ e1}};
MatrixForm[A]
```

$$\begin{pmatrix} 1 + e_1 & 2 & e_3 \wedge e_2 \\ 3 & e_2 \wedge e_4 & e_4 \\ e_2 & e_1 & 1 + e_4 \wedge e_1 \end{pmatrix}$$

```
iA = GrassmannMatrixInverse[A] // Simplify
```

$$\Big\{\Big\{\frac{1}{6}\,(-(e_1 \wedge e_4) - e_2 \wedge e_4),\ \frac{1}{18}\,(6 + e_1 \wedge e_4 - e_2 \wedge e_4 + e_1 \wedge e_2 \wedge e_4),\ -\frac{e_4}{3}\Big\},$$
$$\Big\{\frac{1}{12}\,(6 + e_1 \wedge e_4 + e_2 \wedge e_4 - 3\, e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_4),$$
$$\frac{1}{36}\,(-6 - 6\, e_1 - e_1 \wedge e_4 + e_2 \wedge e_4 + 3\, e_1 \wedge e_2 \wedge e_3),$$
$$\frac{1}{6}\,(e_4 + e_1 \wedge e_4 + 3\, e_2 \wedge e_3)\Big\},$$
$$\Big\{\frac{1}{4}\,(-2\, e_1 - e_1 \wedge e_2 \wedge e_4),\ \frac{1}{36}\,(6\, e_1 - 12\, e_2 + 13\, e_1 \wedge e_2 \wedge e_4),$$
$$\frac{1}{6}\,(6 + 5\, e_1 \wedge e_4 + 2\, e_2 \wedge e_4 - 3\, e_1 \wedge e_2 \wedge e_3)\Big\}\Big\}$$

We check that this is indeed the inverse

```
𝒢[{ℳ[A ∧ iA], ℳ[iA ∧ A]}]
```

{{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}},
 {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}}

Finally, if we try to find the inverse of a matrix with a singular body we get the expected type of error messages.

```
A = {{1 + x, 2 + y}, {1 + z, 2 + w}}; MatrixForm[A]
```

$$\begin{pmatrix} 1+x & 2+y \\ 1+z & 2+w \end{pmatrix}$$

```
GrassmannMatrixInverse[A]
```

```
Inverse::sing : Matrix {{1, 2}, {1, 2}} is singular.
```

13.8 Matrix Equations

Two types of matrix equations

If we can recast a system of equations into a matrix form which maintains the correct ordering of the factors, we can use the matrix inverse to obtain the solution. Suppose we have an equation **M∧X==Y** where **M** is an invertible matrix of Grassmann numbers, **X** is a vector of unknown Grassmann numbers, and **Y** is a vector of known Grassmann numbers. Then the solution **X** is given by:

```
𝒢[ℳ[GrassmannMatrixInverse[M] ∧ Y]]
```

Alternatively, if the equations need to be expressed as **X∧M==Y** in which case the solution **X** is given by:

```
𝒢[ℳ[Y ∧ GrassmannMatrixInverse[M]]]
```

GrassmannAlgebra implements these as `GrassmannLeftLinearSolve` and `GrassmannRightLinearSolve` respectively.

```
? GrassmannLeftLinearSolve
```

```
GrassmannLeftLinearSolve[M,Y] calculates a
  matrix or vector X which solves the equation M∧X==Y.
```

Solving matrix equations

Here we take an example and show that `GrassmannLeftLinearSolve` and `GrassmannRightLinearSolve` both gives solutions to a matrix exterior equation.

```
M = {{1 + x, x ∧ y}, {1 - y, 2 + y + x + 5 x ∧ y}}; MatrixForm[M]
```

$$\begin{pmatrix} 1+x & x\wedge y \\ 1-y & 2+x+y+5\,x\wedge y \end{pmatrix}$$

```
Y = {3 + 2 x ∧ y, 7 - 5 x}; MatrixForm[Y]
```

$$\begin{pmatrix} 3+2\,x\wedge y \\ 7-5\,x \end{pmatrix}$$

`GrassmannLeftLinearSolve` gives:

```
X = GrassmannLeftLinearSolve[M, Y]; MatrixForm[X]
```

$$\begin{pmatrix} 3-3\,x \\ 2-2\,x+\frac{y}{2}-\frac{19\,x\wedge y}{4} \end{pmatrix}$$

```
𝒢[ℳ[M ∧ X]] == Y
```

```
True
```

`GrassmannRightLinearSolve` gives:

```
X = GrassmannRightLinearSolve[M, Y]; MatrixForm[X]
```

$$\begin{pmatrix} -\frac{1}{2}+\frac{19\,x}{4}+\frac{21\,y}{4}+\frac{41\,x\wedge y}{4} \\ \frac{7}{2}-\frac{17\,x}{4}-\frac{7\,y}{4}-\frac{29\,x\wedge y}{4} \end{pmatrix}$$

```
𝒢[ℳ[X ∧ M]] == Y
```

```
True
```

13.9 Matrix Eigensystems

Exterior eigensystems of Grassmann matrices

Grassmann matrices have eigensystems just as do real or complex matrices. Eigensystems are useful because they allow us to calculate functions of matrices. One major difference in approach with Grassmann matrices is that we may no longer be able to use the determinant to obtain the characteristic equation, and hence the eigenvalues. We can, however, return to the basic definition of an eigenvalue and eigenvector to obtain our results. We treat only those matrices with distinct eigenvalues. Suppose the matrix **A** is $n\times n$ and has distinct eigenvalues, then we are looking for an $n\times n$ diagonal matrix **L** (the matrix of eigenvalues) and an $n\times n$ matrix **X** (the matrix of eigenvectors) such that:

$$\boxed{A \wedge X == X \wedge L} \quad 13.6$$

The pair **{X,L}** is called the *eigensystem* of **A**. If **X** is invertible we can post-multiply by X^{-1} to get:

$$\boxed{A == X \wedge L \wedge X^{-1}} \quad 13.7$$

and pre-multiply by X^{-1} to get:

$$\boxed{X^{-1} \wedge A \wedge X == L} \quad 13.8$$

It is this decomposition which allows us to define functions of Grassmann matrices. This will be discussed further in Section 13.10.

The number of scalar equations resulting from the matrix equation $\mathbf{A}\wedge\mathbf{X} == \mathbf{X}\wedge\mathbf{L}$ is n^2. The number of unknowns that we are likely to be able to determine is therefore n^2. There are n unknown eigenvalues in $\mathbf{L}$, leaving $n^2 - n$ unknowns to be determined in $\mathbf{X}$. Since $\mathbf{X}$ occurs on both sides of the equation, we need only determine its columns (or rows) to within a scalar multiple. Hence there are really only $n^2 - n$ unknowns to be determined in $\mathbf{X}$, which leads us to hope that a decomposition of this form is indeed possible.

The process we adopt is to assume unknown general Grassmann numbers for the diagonal components of $\mathbf{L}$ and for the components of $\mathbf{X}$. If the dimension of the space is *N*, there will be 2^N scalar coefficients to be determined for each of the n^2 unknowns. We then use *Mathematica*'s powerful `Solve` function to obtain the values of these unknowns. In practice, during the calculations, we assume that the basis of the space is composed of just the non-scalar symbols existing in the matrix $\mathbf{A}$. This enables the reduction of computation should the currently declared basis be of higher dimension than the number of different 1-elements in the matrix, and also allows the eigensystem to be obtained for matrices which are not expressed in terms of basis elements.

For simplicity we also use *Mathematica*'s `JordanDecomposition` function to determine the *scalar eigensystem* of the matrix $\mathbf{A}$ before proceeding on to find the non-scalar components. To see the relationship of the scalar eigensystem to the complete Grassmann eigensystem, rewrite the eigensystem equation $\mathbf{A}\wedge\mathbf{X} == \mathbf{X}\wedge\mathbf{L}$ in terms of body and soul, and extract that part of the equation that is pure body.

$$(\mathbf{A_b} + \mathbf{A_s}) \wedge (\mathbf{X_b} + \mathbf{X_s}) == (\mathbf{X_b} + \mathbf{X_s}) \wedge (\mathbf{L_b} + \mathbf{L_s})$$

Expanding this equation gives

$$\mathbf{A_b} \wedge \mathbf{X_b} + (\mathbf{A_b} \wedge \mathbf{X_s} + \mathbf{A_s} \wedge \mathbf{X_b} + \mathbf{A_s} \wedge \mathbf{X_s}) == \mathbf{X_b} \wedge \mathbf{L_b} + (\mathbf{X_b} \wedge \mathbf{L_s} + \mathbf{X_s} \wedge \mathbf{L_b} + \mathbf{X_s} \wedge \mathbf{L_s})$$

The first term on each side is the body of the equation, and the second terms (in brackets) its soul. The body of the equation is simply the eigensystem equation for the body of $\mathbf{A}$ and shows that the scalar components of the unknown $\mathbf{X}$ and $\mathbf{L}$ matrices are simply the eigensystem of the body of $\mathbf{A}$.

$$\boxed{\mathbf{A_b} \wedge \mathbf{X_b} == \mathbf{X_b} \wedge \mathbf{L_b}} \qquad \mathbf{13.9}$$

By decomposing the soul of the equation into a sequence of equations relating matrices of elements of different grades, the solution for the unknown elements of $\mathbf{X}$ and $\mathbf{L}$ may be obtained sequentially by using the solutions from the equations of lower grade. For example, the equation for the elements of grade one is

$$A_b \wedge \underset{1}{X_s} + \underset{1}{A_s} \wedge X_b == X_b \wedge \underset{1}{L_s} + \underset{1}{X_s} \wedge L_b \qquad 13.10$$

where $\underset{1}{A_s}$, $\underset{1}{X_s}$, and $\underset{1}{L_s}$ are the components of A_s, X_s, and L_s of grade 1. Here we know A_b and $\underset{1}{A_s}$ and have already solved for X_b and L_b, leaving just $\underset{1}{X_s}$ and $\underset{1}{L_s}$ to be determined. However, although this process would be helpful if trying to solve by hand, it is not necessary when using the Solve function in *Mathematica*.

GrassmannMatrixEigensystem

The function implemented in *GrassmannAlgebra* for calculating the eigensystem of a matrix of Grassmann numbers is GrassmannMatrixEigensystem. It is capable of calculating the eigensystem only of matrices whose body has distinct eigenvalues.

```
? GrassmannMatrixEigensystem
```

```
GrassmannMatrixEigensystem[A] calculates a list comprising the
  matrix of eigenvectors and the diagonal matrix of eigenvalues
  for a Grassmann matrix A whose body has distinct eigenvalues.
```

If the matrix does not have distinct eigenvalues, GrassmannMatrixEigensystem will return a message telling us. For example:

```
A = {{2, x}, {y, 2}}; MatrixForm[A]
```

$$\begin{pmatrix} 2 & x \\ y & 2 \end{pmatrix}$$

```
GrassmannMatrixEigensystem[A]
```

```
Eigenvalues::notDistinct :
 The matrix {{2, x}, {y, 2}} does not have distinct scalar eigenvalues. The
   operation applies only to matrices with distinct scalar eigenvalues.
```

```
GrassmannMatrixEigensystem[{{2, x}, {y, 2}}]
```

If the matrix has distinct eigenvalues, a list of two matrices is returned. The first is the matrix of eigenvectors (whose columns have been normalized with an algorithm which tries to produce as simple a form as possible), and the second is the (diagonal) matrix of eigenvalues.

```
A = {{2, 3 + x}, {2 + y, 3}}; MatrixForm[A]
```

$$\begin{pmatrix} 2 & 3+x \\ 2+y & 3 \end{pmatrix}$$

```
{X, L} = GrassmannMatrixEigensystem[A]; MatrixForm /@ {X, L}
```

$$\left\{\begin{pmatrix} -3-\frac{2x}{5}+\frac{9y}{10} & 1 \\ 2 & 1-\frac{x}{5}+\frac{y}{5} \end{pmatrix}, \begin{pmatrix} -\frac{2x}{5}-\frac{3y}{5}+\frac{x\wedge y}{5} & 0 \\ 0 & 5+\frac{2x}{5}+\frac{3y}{5}+\frac{x\wedge y}{5} \end{pmatrix}\right\}$$

To verify that this is indeed the eigensystem for **A** we can evaluate the equation $A \wedge X == X \wedge L$.

$$\mathcal{G}[\mathcal{M}[A \wedge X]] == \mathcal{G}[\mathcal{M}[X \wedge L]]$$

```
True
```

Next we take a slightly more complex example.

$$A = \{\{2 + e_1 + 2\, e_1 \wedge e_2,\ 0,\ 0\},\ \{0,\ 2 + 2\, e_1,\ 5 + e_1 \wedge e_2\},\ \{0,\ -1 + e_1 + e_2,\ -2 - e_2\}\};\ \text{MatrixForm}[A]$$

$$\begin{pmatrix} 2+e_1+2\,e_1\wedge e_2 & 0 & 0 \\ 0 & 2+2\,e_1 & 5+e_1\wedge e_2 \\ 0 & -1+e_1+e_2 & -2-e_2 \end{pmatrix}$$

```
{X, L} = GrassmannMatrixEigensystem[A];
```

```
X
```

$$\left\{\{0, 0, 1\}, \left\{(-2+i) - \left(3+\frac{7i}{2}\right)e_1 - \left(\frac{5}{2}+\frac{5i}{2}\right)e_2 + \left(\frac{17}{4}-\frac{7i}{2}\right)e_1\wedge e_2,\ (-2-i) - \left(3-\frac{7i}{2}\right)e_1 - \left(\frac{5}{2}-\frac{5i}{2}\right)e_2 + \left(\frac{17}{4}+\frac{7i}{2}\right)e_1\wedge e_2,\ 0\right\}, \{1, 1, 0\}\right\}$$

```
L
```

$$\left\{\left\{-i + \left(1+\frac{9i}{2}\right)e_1 - \left(\frac{1}{2}-\frac{7i}{2}\right)e_2 - \left(\frac{15}{4}-\frac{9i}{2}\right)e_1\wedge e_2,\ 0,\ 0\right\}, \left\{0,\ i + \left(1-\frac{9i}{2}\right)e_1 - \left(\frac{1}{2}+\frac{7i}{2}\right)e_2 - \left(\frac{15}{4}+\frac{9i}{2}\right)e_1\wedge e_2,\ 0\right\}, \{0, 0, 2+e_1+2\,e_1\wedge e_2\}\right\}$$

$$\mathcal{G}[\mathcal{M}[A \wedge X]] == \mathcal{G}[\mathcal{M}[X \wedge L]]$$

```
True
```

13.10 Matrix Functions

Distinct eigenvalue matrix functions

As shown above in the section on matrix eigensystems, we can express a matrix with distinct eigenvalues in the form:

$$A == X \wedge L \wedge X^{-1}$$

Hence we can write its exterior square as:

$$A^2 == (X \wedge L \wedge X^{-1}) \wedge (X \wedge L \wedge X^{-1}) == X \wedge L \wedge (X^{-1} \wedge X) \wedge L \wedge X^{-1} == X \wedge L^2 \wedge X^{-1}$$

This result is easily generalized to give an expression for any power of a matrix.

$$\boxed{A^p == X \wedge L^p \wedge X^{-1}} \qquad 13.11$$

Indeed, it is also valid for any linear combination of powers,

$$\sum a_p\ A^p == \sum a_p\ X \wedge L^p \wedge X^{-1} == X \wedge \left(\sum a_p\ L^p\right) \wedge X^{-1}$$

and hence any function **f[A]** defined by a linear combination of powers (series):

$$f[A] == \sum a_p\ A^p$$

$$\boxed{f[A] == X \wedge f[L] \wedge X^{-1}} \qquad 13.12$$

Now, since **L** is a diagonal matrix:

$$L == \mathbf{DiagonalMatrix}[\{\lambda_1, \lambda_2, \ldots, \lambda_m\}]$$

its *p*th power is just the diagonal matrix of the *p*th powers of the elements.

$$L^p == \mathbf{DiagonalMatrix}[\{\lambda_1{}^p, \lambda_2{}^p, \ldots, \lambda_m{}^p\}]$$

Hence:

$$\sum a_p\, L^p == \text{DiagonalMatrix}\left[\left\{\sum a_p\, \lambda_1{}^p,\ \sum a_p\, \lambda_2{}^p,\ \ldots,\ \sum a_p\, \lambda_m{}^p\right\}\right]$$

$$f[L] == \text{DiagonalMatrix}[\{f[\lambda_1],\ f[\lambda_2],\ \ldots,\ f[\lambda_m]\}]$$

Hence, finally we have an expression for the function **f** of the matrix **A** in terms of the eigenvector matrix and a diagonal matrix in which each diagonal element is the function **f** of the corresponding eigenvalue.

$$\boxed{f[A] == X \wedge \text{DiagonalMatrix}[\{f[\lambda_1],\ f[\lambda_2],\ \ldots,\ f[\lambda_m]\}] \wedge X^{-1}} \qquad \textbf{13.13}$$

GrassmannMatrixFunction

We have already seen in Chapter 9 how to calculate a function of a Grassmann number by using `GrassmannFunction`. We use `GrassmannFunction` and the formula above to construct `GrassmannMatrixFunction` for calculating functions of matrices.

```
? GrassmannMatrixFunction
```

```
GrassmannMatrixFunction[f[A]] calculates a function f
  of a Grassmann matrix A with distinct eigenvalues, where
  f is a pure function. GrassmannMatrixFunction[{fx,x},A]
  calculates a function fx of a Grassmann matrix A with
  distinct eigenvalues, where fx is a formula with x as variable.
```

It should be remarked that, just as functions of a Grassmann number will have different forms in spaces of different dimension, so too will functions of a Grassmann matrix.

Exponentials and Logarithms

In what follows we shall explore some examples of functions of Grassmann matrices in 3-space.

```
𝕍3; A = {{1 + x, x ∧ z}, {2, 3 - z}}; MatrixForm[A]
```

$$\begin{pmatrix} 1+x & x \wedge z \\ 2 & 3-z \end{pmatrix}$$

We first calculate the exponential of **A**.

```
expA = GrassmannMatrixFunction[Exp[A]]
```

$$\left\{\left\{e + e\,x + \frac{1}{2}\,e\,(-3 + e^2)\,x \wedge z,\ \frac{1}{2}\,e\,(-1 + e^2)\,x \wedge z\right\},\ \left\{-e + e^3 + \frac{1}{2}\,e\,(-3 + e^2)\,x - \frac{e\,z}{2} - \frac{e^3\,z}{2} + 3\,e\,x \wedge z,\ e^3 - e^3\,z + \frac{1}{2}\,(e + e^3)\,x \wedge z\right\}\right\}$$

We then calculate the logarithm of this exponential and see that it is **A** as we expect.

```
GrassmannMatrixFunction[Log[expA]]
```

$$\{\{1 + x,\ x \wedge z\},\ \{2,\ 3 - z\}\}$$

Trigonometric functions

We compute the square of the sine and the square of the cosine of **A**:

```
s2A = GrassmannMatrixFunction[{Sin[x]^2, x}, A]
```

$$\left\{\left\{2\,x\,\mathrm{Cos}[1]\,\mathrm{Sin}[1] + \mathrm{Sin}[1]^2 + \frac{1}{2}\,\mathrm{Sin}[1]\,(-4\,\mathrm{Cos}[1] + \mathrm{Sin}[3] + \mathrm{Sin}[5])\,x \wedge z,\ \mathrm{Cos}[2]\,\mathrm{Sin}[2]^2\,x \wedge z\right\},\ \left\{-z\,\mathrm{Sin}[2] - 2\,z\,\mathrm{Cos}[4]\,\mathrm{Sin}[2] + \frac{1}{2}\,x\,\mathrm{Sin}[2]\,(-2 + \mathrm{Sin}[4]) + \mathrm{Sin}[2]\,\mathrm{Sin}[4] + \frac{1}{2}\,z\,\mathrm{Sin}[2]\,\mathrm{Sin}[4] + \frac{1}{2}\,\mathrm{Sin}[2]\,(6 + 6\,\mathrm{Cos}[4] - 3\,\mathrm{Sin}[4])\,x \wedge z,\ \mathrm{Sin}[3]^2 - z\,\mathrm{Sin}[6] + \frac{1}{4}\,(-\mathrm{Cos}[2] + \mathrm{Cos}[6] + 4\,\mathrm{Sin}[6])\,x \wedge z\right\}\right\}$$

```
c2A = GrassmannMatrixFunction[{Cos[x]^2, x}, A]
```

$$\left\{\left\{\mathrm{Cos}[1]^2 - 2\,x\,\mathrm{Cos}[1]\,\mathrm{Sin}[1] + \frac{1}{2}\,\mathrm{Cos}[1]\,(-\mathrm{Cos}[3] + \mathrm{Cos}[5] + 4\,\mathrm{Sin}[1])\,x \wedge z,\ -\mathrm{Cos}[2]\,\mathrm{Sin}[2]^2\,x \wedge z\right\},\ \left\{z\,\mathrm{Sin}[2] + 2\,z\,\mathrm{Cos}[4]\,\mathrm{Sin}[2] - \frac{1}{2}\,x\,\mathrm{Sin}[2]\,(-2 + \mathrm{Sin}[4]) - \mathrm{Sin}[2]\,\mathrm{Sin}[4] - \frac{1}{2}\,z\,\mathrm{Sin}[2]\,\mathrm{Sin}[4] - \frac{1}{2}\,\mathrm{Sin}[2]\,(6 + 6\,\mathrm{Cos}[4] - 3\,\mathrm{Sin}[4])\,x \wedge z,\ \mathrm{Cos}[3]^2 + z\,\mathrm{Sin}[6] - \frac{1}{4}\,(-\mathrm{Cos}[2] + \mathrm{Cos}[6] + 4\,\mathrm{Sin}[6])\,x \wedge z\right\}\right\}$$

and show that $\sin^2 x + \cos^2 x = 1$, even for Grassmann matrices:

```
s2A + c2A // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Symbolic matrices

```
B = {{a + x, x ∧ z}, {b, c - z}}; MatrixForm[B]
```

$$\begin{pmatrix} a+x & x\wedge z \\ b & c-z \end{pmatrix}$$

```
GrassmannMatrixFunction[Exp[B]]
```

$$\Big\{\Big\{e^{a}\,(1+x)+\frac{b\,(-e^{a}+a\,e^{a}-c\,e^{a}+e^{c})\,x\wedge z}{(a-c)^{2}},\ \frac{(e^{a}-e^{c})\,x\wedge z}{a-c}\Big\},$$
$$\Big\{\frac{1}{(a-c)^{3}}\,(b\,((a-c)\,(-c\,e^{a}+c\,e^{c}-e^{a}\,x-c\,e^{a}\,x+$$
$$e^{c}\,x-e^{a}\,z+e^{c}\,z-c\,e^{c}\,z+a\,(e^{a}-e^{c}+e^{a}\,x+e^{c}\,z))+$$
$$(1+b)\,(-2\,e^{a}-c\,e^{a}+2\,e^{c}-c\,e^{c}+a\,(e^{a}+e^{c}))\,x\wedge z)),$$
$$-e^{c}\,(-1+z)+\frac{b\,(e^{a}-e^{c}-a\,e^{c}+c\,e^{c})\,x\wedge z}{(a-c)^{2}}\Big\}\Big\}$$

Symbolic functions

Suppose we wish to compute the symbolic function **f** of the matrix **A**.

```
A // MatrixForm
```

$$\begin{pmatrix} 1+x & x\wedge z \\ 2 & 3-z \end{pmatrix}$$

If we try to compute this we get an incorrect result because the derivatives evaluated at various scalar values are not recognized by `GrassmannSimplify` as scalars. Here we print out just three lines of the result.

```
Short[GrassmannMatrixFunction[f[A]], 3]
```

$$\{\{\ll 1\gg\},\ \{-f[1]+f[3]+\frac{1}{2}x\wedge f[1]-\frac{1}{2}x\wedge f[3]+\ll 22\gg+ x\wedge z\left(2f'[1]+\frac{5}{2}x f'[1]+z f'[1]+f'[3]+\frac{1}{2}x f'[3]+2z f'[3]\right),\ f[3]+\ll 9\gg+x\wedge z\left(\frac{1}{2}x f'[1]+f'[3]+\frac{1}{2}z f'[3]\right)\}\}$$

We see from this that the derivatives in question are the zeroth and first, **f[_]**, and **f'[_]**. We therefore declare these *patterns* to be scalars so that the derivatives evaluated at *any* scalar arguments will be considered by `GrassmannSimplify` as scalars.

```
DeclareExtraScalars[{f[_], f'[_]}]
```

$$\{a, b, c, d, e, f, g, h, \Bbbk, f[\_], (\_\ominus\_)?\text{InnerProductQ}, \underset{0}{\_}, f'[\_]\}$$

```
fA = GrassmannMatrixFunction[f[A]]
```

$$\{\{f[1]+x f'[1]-\frac{1}{2}x\wedge z(f[1]-f[3]+2f'[1]),\ -\frac{1}{2}(f[1]-f[3])x\wedge z\},\ \{-f[1]-\frac{1}{2}x f[1]-\frac{1}{2}z f[1]+f[3]+\frac{1}{2}x f[3]+\frac{1}{2}z f[3]- x f'[1]-z f'[3]+\frac{3}{2}x\wedge z(f[1]-f[3]+f'[1]+f'[3]),\ f[3]-z f'[3]+\frac{1}{2}x\wedge z(f[1]-f[3]+2f'[3])\}\}$$

The function **f** may be replaced by any specific function. We replace it here by `Exp` and check that the result is the same as that given above.

```
expA == (fA /. f → Exp) // Simplify
```

```
True
```

13.11 Supermatrices

To be completed.

A Brief Biography of Grassmann

Hermann Günther Grassmann was born in 1809 in Stettin, a town in Pomerania a short distance inland from the Baltic. His father Justus Günther Grassmann taught mathematics and physical science at the Stettin Gymnasium. Hermann was no child prodigy. His father used to say that he would be happy if Hermann became a craftsman or a gardener.

In 1827 Grassmann entered the University of Berlin with the intention of studying theology. As his studies progressed he became more and more interested in studying philosophy. At no time whilst a student in Berlin was he known to attend a mathematics lecture.

Grassmann was however only 23 when he made his first important geometric discovery: a method of adding and multiplying lines. This method was to become the foundation of his *Ausdehnungslehre* (extension theory). His own account of this discovery is given below.

Grassmann was interested ultimately in a university post. In order to improve his academic standing in science and mathematics he composed in 1839 a work (over 200 pages) on the study of tides entitled *Theorie der Ebbe und Flut*. This work contained the first presentation of a system of spacial analysis based on vectors including vector addition and subtraction, vector differentiation, and the elements of the linear vector function, all developed for the first time. His examiners failed to see its importance.

Around Easter of 1842 Grassmann began to turn his full energies to the composition of his first '*Ausdehnungslehre*', and by the autumn of 1843 he had finished writing it. The following is an excerpt from the foreword in which he describes how he made his seminal discovery. The translation is by Lloyd Kannenberg (Grassmann 1844).

> The initial incentive was provided by the consideration of negatives in geometry; I was used to regarding the displacements *AB* and *BA* as opposite magnitudes. From this it follows that if *A*, *B*, *C* are points of a straight line, then *AB* + *BC* = *AC* is always true, whether *AB* and *BC* are directed similarly or *oppositely*, that is even if *C* lies between *A* and *B*. In the latter case *AB* and *BC* are not interpreted merely as lengths, but rather their directions are simultaneously retained as well, according to which they are precisely oppositely oriented. Thus the distinction was drawn between the sum of lengths and the sum of such displacements in which the directions were taken into account. From this there followed the demand to establish this latter concept of a sum, not only for the case that the displacements were similarly or oppositely directed, but also for all other cases. This can most easily be accomplished if the law *AB* + *BC* = *AC* is imposed even when *A*, *B*, *C* do not lie on a single straight line.
>
> Thus the first step was taken toward an analysis that subsequently led to the new branch of mathematics presented here. However, I did not then recognize the rich and fruitful domain I had reached; rather, that result seemed scarcely worthy of note until it was combined with a related idea.
>
> While I was pursuing the concept of product in geometry as it had been established by my father, I concluded that not only rectangles but also parallelograms in general may be regarded as products of an adjacent pair of their sides, provided one again interprets the

> product, not as the product of their lengths, but as that of the two displacements with their directions taken into account. When I combined this concept of the product with that previously established for the sum, the most striking harmony resulted; thus whether I multiplied the sum (in the sense just given) of two displacements by a third displacement lying in the same plane, or the individual terms by the same displacement and added the products with due regard for their positive and negative values, the same result obtained, and must always obtain.
>
> This harmony did indeed enable me to perceive that a completely new domain had thus been disclosed, one that could lead to important results. Yet this idea remained dormant for some time since the demands of my job led me to other tasks; also, I was initially perplexed by the remarkable result that, although the laws of ordinary multiplication, including the relation of multiplication to addition, remained valid for this new type of product, one could only interchange factors if one simultaneously changed the sign (i.e. changed + into – and vice versa).

As with his earlier work on tides, the importance of this work was ignored. Since few copies were sold, most ended by being used as waste paper by the publisher. The failure to find acceptance for Grassmann's ideas was probably due to two main reasons. The first was that Grassmann was just a simple schoolteacher, and had none of the academic charisma that other contemporaries, like Hamilton for example, had. History seems to suggest that the acceptance of radical discoveries often depends more on the discoverer than the discovery.

The second reason is that Grassmann adopted the format and the approach of the modern mathematician. He introduced and developed his mathematical structure axiomatically and abstractly. The abstract nature of the work, initially devoid of geometric or physical significance, was just too new and formal for the mathematicians of the day and they all seemed to find it too difficult. More fully than any earlier mathematician, Grassmann seems to have understood the associative, commutative and distributive laws; yet still, great mathematicians like Möbius found it unreadable, and Hamilton was led to write to De Morgan that to be able to read Grassmann he 'would have to learn to smoke'.

In the year of publication of the *Ausdehnungslehre* (1844) the Jablonowski Society of Leipzig offered a prize for the creation of a mathematical system fulfilling the idea that Leibniz had sketched in 1679. Grassmann entered with '*Die Geometrische Analyse geknüpft und die von Leibniz Characteristik*', and was awarded the prize. Yet as with the *Ausdehnungslehre* it was subsequently received with almost total silence.

However, in the few years following, three of Grassmann's contemporaries were forced to take notice of his work because of priority questions. In 1845 Saint-Venant published a paper in which he developed vector sums and products essentially identical to those already occurring in Grassmann's earlier works (Barré 1845). In 1853 Cauchy published his method of 'algebraic keys' for solving sets of linear equations (Cauchy 1853). Algebraic keys behaved identically to Grassmann's units under the exterior product. In the same year Saint-Venant published an interpretation of the algebraic keys geometrically and in terms of determinants (Barré 1853). Since such were fundamental to Grassmann's already published work he wrote a reply for Crelle's Journal in 1855 entitled '*Sur les différentes genres de multiplication*' in which he claimed priority over Cauchy and Saint-Venant and published some new results (Grassmann 1855).

It was not until 1853 that Hamilton heard of the *Ausdehnungslehre*. He set to reading it and soon after wrote to De Morgan.

> I have recently been reading … more than a hundred pages of Grassmann's *Ausdehnungslehre*, with great admiration and interest …. If I could hope to be put in rivalship with Des Cartes on the one hand and with Grassmann on the other, my scientific ambition would be fulfilled.

During the period 1844 to 1862 Grassmann published seventeen scientific papers, including important papers in physics, and a number of mathematics and language textbooks. He edited a political paper for a time and published materials on the evangelization of China. This, on top of a heavy teaching load and the raising of a large family. However, this same period saw only few mathematicians — Hamilton, Cauchy, Möbius, Saint-Venant, Bellavitis and Cremona — having any acquaintance with, or appreciation of, his work.

In 1862 Grassmann published a completely rewritten *Ausdehnungslehre*: *Die Ausdehnungslehre: Vollständing und in strenger Form*. In the foreword Grassmann discussed the poor reception accorded his earlier work and stated that the content of the new book was presented in 'the strongest mathematical form that is actually known to us; that is Euclidean …'. It was a book of theorems and proofs largely unsupported by physical example.

This apparently was a mistake, for the reception accorded this new work was as quiet as that accorded the first, although it contained many new results including a solution to Pfaff's problem. Friedrich Engel (the editor of Grassmann's collected works) comments: 'As in the first *Ausdehnungslehre* so in the second: matters which Grassmann had published in it were later independently rediscovered by others, and only much later was it realized that Grassmann had published them earlier' (Engel 1896).

Thus Grassmann's works were almost totally neglected for forty-five years after his first discovery. In the second half of the 1860s recognition slowly started to dawn on his contemporaries, among them Hankel, Clebsch, Schlegel, Klein, Noth, Sylvester, Clifford and Gibbs. Gibbs discovered Grassmann's works in 1877 (the year of Grassmann's death), and Clifford discovered them in depth about the same time. Both became quite enthusiastic about Grassmann's new mathematics.

Grassmann's activities after 1862 continued to be many and diverse. His contribution to philology rivals his contribution to mathematics. In 1849 he had begun a study of Sanskrit and in 1870 published his *Wörtebuch zum Rig-Veda*, a work of 1784 pages, and his translation of the Rig-Veda, a work of 1123 pages, both still in use today. In addition he published on mathematics, languages, botany, music and religion. In 1876 he was made a member of the American Oriental Society, and received an honorary doctorate from the University of Tübingen.

On 26 September 1877 Hermann Grassmann died, departing from a world only just beginning to recognize the brilliance of the mathematical creations of one of its most outstanding eclectics.

Notation

To be completed.

Operations

$\wedge$	exterior product operation
$\vee$	regressive product operation
$\ominus$	interior product operation
$\underset{\lambda}{\Delta}$	generalized product operation
$\diamond$	Clifford product operation
$\circ$	hypercomplex product operation
$\overline{\square}$	complement operation
$\overrightarrow{\square}$	vector subspace complement operation
$\times$	cross product operation
$\lvert\square\rvert$	measure
$=$	defined equal to
$==$	equal to
$\equiv$	congruence

Elements

$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$	scalars
$\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$	1-elements or vectors
$\underset{m}{\alpha}, \underset{m}{\beta}, \underset{m}{\gamma}, \ldots$	*m*-elements
$\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$	Grassmann numbers
$\nu_1, \nu_2, \ldots$	position vectors
$\mathbf{P}_1, \mathbf{P}_2, \ldots$	points
$\mathbf{L}_1, \mathbf{L}_2, \ldots$	lines
$\Pi_1, \Pi_2, \ldots$	planes

Declarations

$\mathbb{V}_n$ declare a linear or vector space of *n* dimensions

$\mathbb{P}_n$ declare an *n*-plane comprising an origin point and a vector space of *n* dimensions

$\mathbb{S}$ declare the default list of scalars

$\mathbb{E}$ declare the default Euclidean metric

Special objects

1 unit scalar

$\overline{1}$ unit *n*-element

$\mathbb{O}$ origin point

$\mathbb{k}$ metric factor

$\mathbb{D}$ symbolic dimension

$\mathbb{n}$ dimension of the currently declared space

$\mathbb{N}$ dimension of the current Grassmann algebra

$\mathbb{G}$ list of grades of the current Grassmann algebra

Spaces

$\underset{0}{\Lambda}$ linear space of scalars or field of scalars

$\underset{1}{\Lambda}$ linear space of 1-elements, underlying linear space

$\underset{m}{\Lambda}$ linear space of *m*-elements

$\underset{n}{\Lambda}$ linear space of *n*-elements

$\underset{\mathbb{G}}{\Lambda}$ Grassmann algebra

Basis elements

e_i basis 1-element or covariant basis element

e^i contravariant basis 1-element

$\underset{m}{e_i}$ basis *m*-element or covariant basis *m*-element

$\underset{m}{e^i}$ contravariant basis *m*-element

$\underline{\mathbf{e}_{\underset{m}{i}}}$ cobasis element of $\mathbf{e}_{\underset{m}{i}}$

Glossary

To be completed.

Ausdehnungslehre

The term *Ausdehnungslehre* is variously translated as 'extension theory', 'theory of extension', or 'calculus of extension'. Refers to Grassmann's original work and other early work in the same notational and conceptual tradition.

Bivector

A *bivector* is a sum of simple bivectors.

Bound vector

A *bound vector* $\mathbf{B}$ is the exterior product of a point and a vector. $\mathbf{B} == \mathbf{P} \wedge \mathbf{x}$. It may also always be expressed as the exterior product of two points.

Bound bivector

A *bound bivector* $\mathbf{B}$ is the exterior product of a point $\mathbf{P}$ and a bivector $\mathbf{W}$: $\mathbf{B} == \mathbf{P} \wedge \mathbf{W}$.

Bound simple bivector

A *bound simple bivector* $\mathbf{B}$ is the exterior product of a point $\mathbf{P}$ and a simple bivector $\mathbf{x} \wedge \mathbf{y}$: $\mathbf{B} == \mathbf{P} \wedge \mathbf{x} \wedge \mathbf{y}$. It may also always be expressed as the exterior product of two points and a vector, or the exterior product of three points.

Cofactor

The *cofactor* of a minor M of a matrix A is the signed determinant formed from the rows and columns of A which are not in M.
The sign may be determined from $(-1)^{\Sigma (r_i + c_i)}$, where r_i and c_i are the row and column numbers.

Dimension of a linear space

The *dimension of a linear space* is the maximum number of independent elements in it.

Dimension of an exterior linear space

The *dimension of an exterior linear space* $\underset{m}{\Lambda}$ is $\binom{n}{m}$ where *n* is the dimension of the underlying linear space $\underset{1}{\Lambda}$.

The dimension $\binom{n}{m}$ is equal to the number of combinations of *n* elements taken *m* at a time.

Dimension of a Grassmann algebra

The *dimension of a Grassmann algebra* is the sum of the dimensions of its component exterior linear spaces.
The dimension of a Grassmann algebra is then given by 2^n , where *n* is the dimension of the underlying linear space.

Direction

A *direction* is the space of a vector and is therefore the set of all vectors parallel to a given vector.

Displacement

A *displacement* is a physical interpretation of a vector. It may also be viewed as the difference of two points.

Exterior linear space

An *exterior linear space* of grade *m,* denoted $\underset{m}{\Lambda}$, is the linear space generated by *m*-elements.

Force

A *force* is a physical entity represented by a bound vector. This differs from common usage in which a force is represented by a vector. For reasons discussed in the text, common use does not provide a satisfactory model.

Force vector

A *force vector* is the vector of the bound vector representing the force.

General geometrically interpreted 2-element

A *general geometrically interpreted 2-element* $\mathbf{U}$ is the sum of a bound vector $\mathbf{P} \wedge \mathbf{x}$ and a bivector $\mathbf{W}$. That is, $\mathbf{U} == \mathbf{P} \wedge \mathbf{x} + \mathbf{W}$.

Geometric entities

Points, lines, planes, … are *geometric entities*. Each is defined as the space of a geometrically interpreted element.
A point is a geometric 1-entity.
A line is a geometric 2-entity.
A plane is a geometric 3-entity.

Geometric interpretations

Points, weighted points, vectors, bound vectors, bivectors, … are geometric *interpretations* of *m*-elements.

Geometrically interpreted algebra

A *geometrically interpreted algebra* is a Grassmann algebra with a geometrically interpreted underlying linear space.

Grade

The *grade* of an *m*-element is *m*.
The grade of a simple *m*-element is the number of 1-element factors in it.
The grade of the exterior product of an *m*-element and a *k*-element is $m+k$.
The grade of the regressive product of an *m*-element and a *k*-element is $m+k-n$.
The grade of the complement of an *m*-element is $n-m$.
The grade of the interior product of an *m*-element and a *k*-element is $m-k$.
The grade of a scalar is zero.
(The dimension *n* is the dimension of the underlying linear space.)

GrassmannAlgebra

The concatenated italicized term *GrassmannAlgebra* refers to the *Mathematica* software package which accompanies this book.

***A* Grassmann algebra**

A *Grassmann algebra* is the direct sum of an underlying linear space $\underset{1}{\Lambda}$, its field $\underset{0}{\Lambda}$, and the exterior linear spaces $\underset{m}{\Lambda}$ ($2 \le m \le n$).

$$\underset{0}{\Lambda} \oplus \underset{1}{\Lambda} \oplus \underset{2}{\Lambda} \oplus \cdots \oplus \underset{m}{\Lambda} \oplus \cdots \oplus \underset{n}{\Lambda}$$

***The* Grassmann algebra**

The Grassmann algebra is used to describe that body of algebraic theory and results based on the *Ausdehnungslehre*, but extended to include more recent results and viewpoints.

Intersection

An *intersection* of two simple elements is any of the congruent elements defined by the intersection of their spaces.

Laplace expansion theorem

The *Laplace expansion theorem* states: If any *r* rows are fixed in a determinant, then the value of the determinant may be obtained as the sum of the products of the minors of *r*th order (corresponding to the fixed rows) by their cofactors.

Line

A *line* is the space of a bound vector. Thus a line consists of all the (perhaps weighted) points on it and all the vectors parallel to it.

Linear space

A *linear space* is a mathematical structure defined by a standard set of axioms. It is often referred to simply as a 'space'.

Minor

A *minor* of a matrix $\mathbf{A}$ is the determinant (or sometimes matrix) of degree (or order) k formed from $\mathbf{A}$ by selecting the elements at the intersection of k distinct columns and k distinct rows.

***m*-direction**

An *m*-direction is the space of a simple *m*-vector. It is also therefore the set of all vectors parallel to a given simple *m*-vector.

***m*-element**

An *m*-element is a sum of simple *m*-elements.

***m*-plane**

An *m*-plane is the space of a bound simple *m*-vector. Thus a plane consists of all the (perhaps weighted) points on it and all the vectors parallel to it.

***m*-vector**

An *m*-vector is a sum of simple *m*-vectors.

***n*-algebra**

The term *n*-algebra is an alias for the phrase *Grassmann algebra with an underlying linear space of n dimensions*.

Origin

The *origin* $\mathbb{O}$ is the geometric interpretation of a specific 1-element as a reference point.

Plane

A *plane* is the space of a bound simple bivector. Thus a plane consists of all the (perhaps weighted) points on it and all the vectors parallel to it.

Point

A *point* $\mathbf{P}$ is the sum of the origin $\mathbb{O}$ and a vector $\mathbf{x}$: $\mathbf{P} == \mathbb{O} + \mathbf{x}$.

Point space

A *point space* is a linear space whose basis elements are interpreted as an origin point $\mathbb{O}$ and vectors.

Point mass

A *point mass* is a physical interpretation of a weighted point.

Physical entities

Point masses, displacements, velocities, forces, moments, angular velocities, … are *physical entities*. Each is represented by a geometrically interpreted element.
A point mass, displacement or velocity is a physical 1-entity.
A force, moment or angular velocity is a physical 2-entity.

Physical representations

Points, weighted points, vectors, bound vectors, bivectors, … are geometric *representations* of physical entities such as point masses, displacements, velocities, forces, moments.
Physical entities are *represented* by geometrically interpreted elements.

Scalar

A *scalar* is an element of the field $\underset{0}{\Lambda}$ of the underlying linear space $\underset{1}{\Lambda}$.
A scalar is of grade zero.

Screw

A *screw* is a geometrically interpreted 2-element in a three-dimensional (physical) space (four-dimensional linear space) in which the bivector is orthogonal to the vector of the bound vector.
The bivector is necessarily simple since the vector subspace is three-dimensional.

Simple element

A *simple element* is an element which may be expressed as the exterior product of 1-elements.

Simple bivector

A *simple bivector* $\mathbf{V}$ is the exterior product of two vectors. $\mathbf{V} == \mathbf{x} \wedge \mathbf{y}$.

Simple *m*-element

A *simple m-element* is the exterior product of *m* 1-elements.

Simple *m*-vector

A *simple m-vector* is the exterior product of *m* vectors.

Space of a simple *m*-element

The *space of a simple m-element* $\underset{m}{\alpha}$ is the set of all 1-elements $\mathbf{x}$ such that $\underset{m}{\alpha} \wedge \mathbf{x} == 0$.
The space of a simple *m*-element is a linear space of dimension *m*.

Space of a non-simple *m*-element

The *space of a non-simple m-element* is the union of the spaces of its component simple *m*-elements.

2-direction

A *2-direction* is the space of a simple bivector. It is therefore the set of all vectors parallel to a given simple bivector.

Underlying linear space

The *underlying linear space* of a Grassmann algebra is the linear space $\underset{1}{\Lambda}$ of 1-elements, which together with the exterior product operation, generates the algebra.
The dimension of an underlying linear space is denoted by the symbol *n*.

Underlying point space

An *underlying point space* is an underlying linear space whose basis elements are interpreted as an origin point $\mathbb{O}$ and vectors. It can be shown that from this basis a second basis can be constructed, all of whose basis elements are points.

Union

A *union* of two simple elements is any of the congruent elements which define the union of their spaces.

Vector

A *vector* is a geometric interpretation of a 1-element.

Vector space

A *vector space* is a linear space in which the elements are interpreted as vectors.

Vector subspace of a geometrically interpreted underlying linear space

The *vector subspace of a geometrically interpreted underlying linear space* is the subspace of elements which do not involve the origin.

Weighted point

A *weighted point* is a scalar multiple $\mathbf{a}$ of a point $\mathbf{P}$: $\mathbf{a}\,\mathbf{P}$.

Bibliography

To be completed

Armstrong H L 1959

'On an alternative definition of the vector product in n-dimensional vector analysis'
Matrix and Tensor Quarterly, **IX** no 4, pp 107-110.

The author proposes a definition equivalent to the complement of the exterior product of n–1 vectors.

Ball R S 1900

A Treatise on the Theory of Screws
Cambridge University Press (1900). Reprinted 1998.

A classical work on the theory of screws, containing an annotated bibliography in which Ball refers to the *Ausdehnungslehre* of 1862: 'This remarkable work … contains much that is of instruction and interest in connection with the present theory … . Here we have a very general theory which includes screw coordinates as a special case.' Ball does not use Grassmann's methods in his treatise.

Barton H 1927

'A Modern Presentation of Grassmann's Tensor Analysis'
American Journal of Mathematics, **XLIX**, pp 598-614.

This paper covers similar ground to that of Moore (1926).

Bowen R M and Wang C-C 1976

Introduction to Vectors and Tensors
Plenum Press. Two volumes.

This is the only contemporary text on vectors and tensors I have sighted which relates points and vectors via the explicit introduction of the origin into the calculus (p 254).

Brand L 1947

Vector and Tensor Analysis
Wiley, New York.

Contains a chapter on motor algebra which, according to the author in his preface '… is apparently destined to play an important role in mechanics as well as in line geometry'. There is also a chapter on quaternions.

Buchheim A 1884-1886

'On the Theory of Screws in Elliptic Space'
Proceedings of the London Mathematical Society, **xiv** (1884) pp 83-98, **xvi** (1885) pp 15-27, **xvii** (1886) pp 240-254, **xvii** p 88.

The author writes 'My special object is to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in elliptic space. Clifford was apparently led to construct his theory of biquaternions by the want of such a calculus, but Grassmann's method seems to afford a simpler and more natural means of expression than biquaternions.' (*xiv*, p 90) Later he extends this theory to '… all kinds of space.' (*xvi*, p 15)

Burali-Forti C 1897

Introduction à la Géométrie Différentielle suivant la Méthode de H. Grassmann
Gauthier-Villars, Paris

This work covers both algebra and differential geometry in the tradition of the Peano approach to the *Ausdehnungslehre*.

Burali-Forti C and Marcolongo R 1910

Éléments de Calcul Vectoriel
Hermann, Paris

Mostly a treatise on standard vector analysis, but it does contain an appendix (pp 176-198) on the methods of the *Ausdehnungslehre* and some interesting historical notes on the vector calculi and their notations. The authors use the wedge $\wedge$ to denote Gibbs' cross product $\times$ and use the $\times$, initially introduced by Grassmann to denote the scalar or inner product.

Carvallo M E 1892

'La Méthode de Grassmann'
Nouvelles Annales de Mathématiques, serie 3 **XI**, pp 8-37.

An exposition of some of Grassmann's methods applied to three dimensional geometry following the approach of Peano. It does not treat the interior product.

Chevalley C 1955

The Construction and Study of Certain Important Algebras
Mathematical Society of Japan, Tokyo.

Lectures given at the University of Tokyo on graded, tensor, Clifford and exterior algebras.

Clifford W K 1873

'Preliminary Sketch of Biquaternions'
Proceedings of the London Mathematical Society, **IV**, nos 64 and 65, pp 381-395

This paper includes an interesting discussion of the geometric nature of mechanical quantities. Clifford adopts the term 'rotor' for the bound vector, and 'motor' for the general sum of rotors. By analogy with the quaternion as a quotient of vectors he defines the biquaternion as a quotient of motors.

Clifford W K 1878

'Applications of Grassmann's Extensive Algebra'
American Journal of Mathematics Pure and Applied, **I**, pp 350-358

In this paper Clifford lays the foundations for general Clifford algebras.

Clifford W K 1882

Mathematical Papers
Reprinted by Chelsea Publishing Co, New York (1968).

Of particular interest in addition to his two published papers above are the otherwise unpublished notes:
'Notes on Biquaternions' (~1873)
'Further Note on Biquaternions' (1876)
'On the Classification of Geometric Algebras' (1876)
'On the Theory of Screws in a Space of Constant Positive Curvature' (1876).

Coffin J G 1909

Vector Analysis
Wiley, New York.

This is the second English text in the Gibbs-Heaviside tradition. It contains an appendix comparing the various notations in use at the time, including his view of the Grassmannian notation.

Collins J V 1899-1900

'An elementary Exposition of Grassmann's *Ausdehnungslehre* or Theory of Extension'
American Mathematical Monthly, **6** (1899) pp 193-198, 261-266, 297-301; **7** (1900) pp 31-35, 163-166, 181-187, 207-214, 253-258.

This work follows in summary form the *Ausdehnungslehre* of 1862 as regards general theory but differs in its discussion of applications. It includes applications to geometry and brief applications to linear equations, mechanics and logic.

Coolidge J L 1940

'Grassmann's Calculus of Extension' in
A History of Geometrical Methods
Oxford University Press, pp 252-257

This brief treatment of Grassmann's work is characterized by its lack of clarity. The author variously describes an exterior product as 'essentially a matrix' and as 'a vector perpendicular to the factors' (p 254). And confusion arises between Grassmann's matrix and the division of two exterior products (p 256).

Cox H 1882

'On the application of Quaternions and Grassmann's *Ausdehnungslehre* to different kinds of Uniform Space'
Cambridge Philosophical Transactions, **XIII** part II, pp 69-143.

The author shows that the exterior product is the multiplication required to describe non-metric geometry, for 'it involves no ideas of distance' (p 115). He then discusses exterior, regressive and interior products, applying them to geometry, systems of forces, and linear complexes – using the notation of 1844. In other papers Cox applies the *Ausdehnungslehre* to non-Euclidean geometry (1873) and to the properties of circles (1890).

Crowe M J 1967
A History of Vector Analysis
Notre Dame.

This is the most informative work available on the history of vector analysis from the discovery of the geometric representation of complex numbers to the development of the Gibbs-Heaviside system Crowe's thesis is that the Gibbs-Heaviside system grew mostly out of quaternions rather than from the *Ausdehnungslehre*. His explanation of Grassmannian concepts is particularly accurate in contradistinction to many who supply a more casual reference.

Dibag I 1974
'Factorization in Exterior Algebras'
Journal of Algebra, **30**, pp 259-262

The author develops necessary and sufficient conditions for an *m*-element to have a certain number of 1-element factors. He also shows that an (n–2)-element in an odd dimensional space always has a 1-element factor.

Grassmann H G 1878
'Verwendung der Ausdehnungslehre fur die allgemeine Theorie der Polaren und den Zusammenhang algebraischer Gebilde'
Crelle's Journal, **84**, pp 273–283.

This is Grassmann's last paper. It contains, among other material, his most complete discussion on the notion of 'simplicity'.

Gibbs J W 1886
'On multiple algebra'
Address to the American Association for the Advancement of Science
In *Collected Works*, Gibbs 1928, vol 2.

This paper is probably the most authoritative historical comparison of the different 'vectorial' algebras of the time. Gibbs was obviously very enthusiastic about the *Ausdehnungslehre*, and shows himself here to be one of Grassmann's greatest proponents.

Gibbs J W 1891
'Quaternions and the Ausdehnungslehre'
Nature, **44**, pp 79–82. Also in *Collected Works*, Gibbs 1928.

Gibbs compares Hamilton's Quaternions with Grassmann's *Ausdehnungslehre* and concludes that '... Grassmann's system is of indefinitely greater extension ...'. Here he also concludes that to Grassmann must be attributed the discovery of matrices. Gibbs published a further three papers in *Nature* (also in *Collected Works*, Gibbs 1928) on the relationship between quaternions and vector analysis, providing an enlightening insight into the quaternion–vector analysis controversy of the time.

Gibbs J W 1928

The Collected Works of J. Willard Gibbs Ph.D. LL.D.
Two volumes. Longmans, New York.

In part 2 of Volume 2 is reprinted Gibbs' only personal work on vector analysis: *Elements of Vector Analysis, Arranged for the Use of Students of Physics* (1881–1884). This was not published elsewhere.

To be completed.

A note on sources to Grassmann's work

The best source for Grassmann's contributions to science is his *Collected Works* (Grassmann 1896) which contain in volume 1 both *Die Ausdehnungslehre von 1844* and *Die Ausdehnungslehre von 1862*, as well as *Geometrische Analyse*, his prizewinning essay fulfilling Leibniz's search for an algebra of geometry. Volume 2 contains papers on geometry, analysis, mechanics and physics, while volume 3 contains *Theorie der Ebbe und Flut*.

Die Ausdehnungslehre von 1862, fully titled: *Die Ausdehnungslehre. Vollständig und in strenger Form* is perhaps Grassmann's most important mathematical work. It comprises two main parts: the first devoted basically to the *Ausdehnungslehre* (212 pages) and the second to the theory of functions (155 pages). The Collected Works edition contains 98 pages of notes and comments. The discussion on the *Ausdehnungslehre* includes chapters on addition and subtraction, products in general, progressive and regressive products, interior products, and applications to geometry. A Cartesian metric is assumed.

Both Grassmann's *Ausdehnungslehre* have been translated into English by Lloyd C Kannenberg. The 1844 version is published as *A New Branch of Mathematics: The Ausdehnungslehre of 1844 and Other Works,* Open Court 1995. The translation contains *Die Ausdehnungslehre von 1844*, *Geometrische Analyse*, selected papers on mathematics and physics, a bibliography of Grassmann's principal works, and extensive editorial notes. The 1862 version is published as *Extension Theory*. It contains work on both the theory of extension and the theory of functions. Particularly useful are the editorial and supplementary notes.

Apart from these translations, probably the best and most complete exposition on the *Ausdehnungslehre* in English is in Alfred North Whitehead's *A Treatise on Universal Algebra* (Whitehead 1898). Whitehead saw Grassmann's work as one of the foundation stones on which he hoped to build an algebraic theory which united the several important and new mathematical systems which emerged during the nineteenth century — the algebra of symbolic logic, Grassmann's theory of extension, quaternions, matrices and the general theory of linear algebras.

The second most complete exposition of the *Ausdehnungslehre* is Henry George Forder's *The Theory of Extension* (Forder 1941). Forder's interest is mainly in the geometric applications of the theory of extension.

The only other books on Grassmann in English are those by Edward Wyllys Hyde, *The Directional Calculus* (Hyde 1890) and *Grassmann's Space Analysis* (Hyde 1906). They treat the theory of extension in two and three-dimensional geometric contexts and include some applications to statics. Several topics such as Hyde's treatment of screws are original contributions.

The seminal papers on Clifford algebra are by William Kingdon Clifford and can be found in his collected works *Mathematical Papers* (Clifford 1882), republished in a facsimile edition by Chelsea.

Fortunately for those interested in the evolution of the emerging 'geometric algebras', *The International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics* published a bibliography (Macfarlane 1913) which, together with supplements to 1913, contains about 2500 articles. This therefore most likely contains all the works on the *Ausdehnungslehre* and related subjects up to 1913.

The only other recent text devoted specifically to Grassmann algebra (to the author's knowledge as of 2001) is Arno Zaddach's *Grassmanns Algebra in der Geometrie*, BI-Wissenschaftsverlag, (Zaddach 1994).